$\langle \mathbb{R}, +, <, 1 \rangle$ Is Decidable in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$

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Abstract. We show that it is decidable whether or not a relation on the reals definable in the structure $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ can be defined in the structure $\langle \mathbb{R}, +, <, 1 \rangle$. This result is achieved by obtaining a topological characterization of $\langle \mathbb{R}, +, <, 1 \rangle$-definable relations in the family of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relations and then by following Muchnik's approach of showing that this characterization can be expressed in the logic of $\langle \mathbb{R}, +, <, 1 \rangle$.

1 Introduction

Consider the structure $\langle \mathbb{R}, +, <, 1 \rangle$ of the additive ordered group of reals along with the constant 1. It is well-known that the subgroup $\mathbb{Z}$ of integers is not first-order-definable. Add the predicate $x \in \mathbb{Z}$ resulting in the structure $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$. Our main result shows that given a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relation it is decidable whether or not it is $\langle \mathbb{R}, +, <, 1 \rangle$-definable.

The structure $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ is a privileged area of application of algorithmic verification of properties of reactive and hybrid systems, where logical formalisms involving reals and arithmetic naturally appear, see e.g [1,4,13]. It admits quantifier elimination and is decidable as proved independently by Miller [16] and Weisfpfenning [20]. The latter's proof uses reduction to the theories of $\langle \mathbb{Z}, +, < \rangle$ and $\langle \mathbb{R}, +, <, 1 \rangle$.

There are many ways to come across the structure $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$, which highlights its significance. One approach is through automata. Cobham considers a fixed base r and represents integers as finite strings of r digits. A subset X of integers is $r-$recognizable if there exists a finite automaton accepting precisely the representations in base r of its elements. Cobham's theorem says that if X is r- and s-recognizable for two multiplicatively independent values r and s (i.e., for all $i, j > 0$ it holds $r^i \neq s^j$) then X is definable in Presburger arithmetic, i.e., in $\langle \mathbb{N}, + \rangle$ [11,18]. Conversely, each Presburger-definable subset of $\mathbb{N}$ is r-recognizable for every r. This result was extended to integer relations of arbitrary arity by Semënov [19].

Consider now recognizability of sets of reals. As early as in 1962 Büchi interprets subsets of integers as characteric functions of reals in their binary representations and shows the decidability of a structure which is essentially an extension of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$, namely $\langle \mathbb{R}_+, <, P, \mathbb{N} \rangle$ where P if the set of positive powers of 2

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and $\mathbb{N}$ the set of natural numbers [9, Thm 4]. Going one step further Boigelot et al. [7] consider reals as infinite strings of digits and use Muller automata to speak of r-recognizable subsets and more generally of r-recognizable relations of reals. In the papers [3,5,6] the equivalence was proved between (1) $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definability, (2) r- and s-recognizability where the two bases have distinct primes in their factorization [6, Thm 5] and (3) r- and s-weakly recognizability for two independently multiplicative bases, [6, Thm 6] (a relation is r-weakly recognizable if it is recognized by some deterministic Muller automaton in which all states in the same strongly connected component are either final or nonfinal). Consequently, as far as reals are concerned, definability in $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$ compared to recognizability or weak recognizability by automata on infinite strings can be seen as the analog of Presburger arithmetic for integers compared to recognizability by automata on finite strings.

A natural issue is to find effective characterizations of subclasses of $r-$recognizable relations. In the case of relations over integers, Muchnik proved that for every base $r \geq 2$ and arity $k \geq 1$, it is decidable whether a r-recognizable relation $X \subseteq \mathbb{N}^k$ is Presburger-definable [17] (see a different approach in [14] which provides a polynomial time algorithm). For relations over reals, up to our knowledge, the only known result is due to Milchior who proved that it is decidable (in linear time) whether a weakly $r-$recognizable subset of $\mathbb{R}$ is definable in $\langle \mathbb{R}, +, <, 1\rangle$ [15]. Our result provides an effective characterization of $\langle \mathbb{R}, +, <, 1\rangle$-definable relations within $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable relations. Our approach is inspired by Muchnik's one, which consists of giving a combinatorical characterization of $\langle \mathbb{N}, +\rangle$-definable relations that can be expressed in $\langle \mathbb{N}, +\rangle$ itself.

Now we give a short outline of our paper. Section 2 gathers all the basic on the two specific structures $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$ and $\langle \mathbb{R}, +, <, 1\rangle$, taking advantage of the existence of quantifier elimination which allows us to work with simpler formulas. Section 3 introduces topological notions. In particular we say that the neighborhood of a point $x \in \mathbb{R}^n$ relative to a relation $X \subseteq \mathbb{R}^n$ has strata if there exists a direction such that the intersection of all sufficiently small neighborhoods around x with X is the trace of a union of lines parallel to the given direction. This reflects the fact that the relations we work with are defined by finite unions of regions of the spaces delimited by hyperplanes of arbitrary dimension. In Sect. 5 we show that when X is $\langle \mathbb{R}, +, <, 1\rangle$-definable all points (except finitely many which we call singular) have at least one direction which is a stratum. In Sect. 6 we give a necessary and sufficient condition for a $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable relation to be $\langle \mathbb{R}, +, <, 1\rangle$-definable, namely (1) it has finitely many singular points and (2) all intersections of X with arbitrary hyperplanes parallel to $n-1$ axes and having rational components on the remaining axis are $\langle \mathbb{R}, +, <, 1\rangle$-definable. Then we show that these properties are expressible in $\langle \mathbb{R}, +, <, 1, X\rangle$.

2 Preliminaries

Throughout this work we assume the vector space $\mathbb{R}^n$ is provided with the metric L_∞ (i.e., $|x| = \max_{1\leq i\leq n} |x_i|$). The open ball centered at $x \in \mathbb{R}^n$ and of radius

$r > 0$ is denoted by $B(x, r)$. Given $x, y \in \mathbb{R}^n$ we denote by $[x, y]$ (resp. (x, y)) the closed segment (resp. open segment) with extremities x, y. We use also notations such as $[x, y)$ or $(x, y]$ for half-open segments.

Let us specify our logical conventions and notations. We work within first-order predicate calculus with equality. We confuse formal symbols and their interpretations, except in Sect. 6.2 where the distinction is needed. We are mainly concerned with the structures $\langle \mathbb{R}, +, <, 1 \rangle$ and $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$. In the latter structure, $\mathbb{Z}$ should be understood as a unary predicate which is satisfied only by elements of $\mathbb{Z}$ - in other words, we deal only with one-sorted structures. Given a structure $\mathcal{M}$ with domain D and $X \subseteq D^n$, we say that X is definable in $\mathcal{M}$, or $\mathcal{M}$-definable, if there exists a formula $\varphi(x_1, \ldots, x_n)$ in the signature of $\mathcal{M}$ such that $\varphi(a_1, \ldots, a_n)$ holds in $\mathcal{M}$ if and only if $(a_1, \ldots, a_n) \in X$.

The $\langle \mathbb{R}, +, <, 1 \rangle$-theory admits quantifier elimination in the following way, which can be interpreted geometrically as saying that a $\langle \mathbb{R}, +, <, 1 \rangle$-definable relation is a finite union of closed and open polyhedra.

Theorem 1 *[12, Thm 1]. Every formula in $\langle \mathbb{R}, +, <, 1 \rangle$ is equivalent to a Boolean combination of inequalities between linear combinations of variables with coefficients in $\mathbb{Z}$ (or, equivalently, in $\mathbb{Q}$).*

In particular in the unary case, the definable subsets are finite unions of intervals whose endpoints are rational numbers, which shows that $\mathbb{Z}$ is not $\langle \mathbb{R}, +, <, 1 \rangle$-definable.

In the larger structure $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$ it is possible to separate the integer (superscript 'I') and fractional (superscript 'F') parts of the reals as follows.

Theorem 2 *[8],[6, p. 7]. Let $X \subseteq \mathbb{R}^n$ be definable in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$. Then there exists a unique finite union*

$$X = \bigcup_{k=1}^{K} (X_k^{(I)} + X_k^{(F)}) \tag{1}$$

where

- *the relations $X_k^{(I)}$ are pairwise disjoint subsets of $\mathbb{Z}^n$ and are $\langle \mathbb{Z}, +, < \rangle$-definable*
- *the relations $X_k^{(F)}$ are distinct subsets of $[0, 1)^n$ and are $\langle \mathbb{R}, +, <, 1 \rangle$-definable*

There is again a geometric interpretation of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relations as a regular (in a precise technical way) tiling of the space by a finite number of tiles which are themselves finite unions of polyhedra. As a consequence, the restriction of a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relation to a bounded subset is $\langle \mathbb{R}, +, <, 1 \rangle$-definable as stated in the following lemma.

Lemma 1. *For every $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relation $X \subseteq \mathbb{R}^n$, its restriction to a bounded domain $[a_1, b_1] \times \cdots \times [a_n, b_n]$ where the a_i's and the b_i's are rationals, is $\langle \mathbb{R}, +, <, 1 \rangle$-definable.*

By considering the restriction of the $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-relation to a ball containing all possible tiles with their closest neighbors, we get that the neighborhoods of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$- and $\langle \mathbb{R}, +, <, 1 \rangle$-definable relations are indistinguishable.

Lemma 2. *For every $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relation $X \subseteq \mathbb{R}^n$ there exists a $\langle \mathbb{R}, +, <, 1 \rangle$-definable relation $Y \subseteq \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$ there exists $y \in \mathbb{R}^n$ and a real $r > 0$ such that the translation $u \mapsto u + y - x$ is a one-to-one mapping between $B(x, r) \cap X$ and $B(y, r) \cap Y$.*

3 Strata

The aim is to decide, given $n \geq 1$ and a $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relation $X \subseteq \mathbb{R}^n$, whether X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable. Though the relations defined in the two structures have very specific properties we define properties that make sense in a setting as general as possible. The following clearly defines an equivalence relation.

Definition 1. *Given $x, y \in \mathbb{R}^n$ we write $x \underset{X}{\sim} y$ or simply $x \sim y$ when X is understood, if there exists a real $r > 0$ such that the translation $w \mapsto w + y - x$ is a one-to-one mapping from $B(x, r) \cap X$ onto $B(y, r) \cap X$.*

Example 1. Consider a closed subset of the plane delimited by a square. There are 10 equivalence classes: the set of points interior to the square, the set of points interior to its complement, the four vertices and the four open edges.

Definition 2. *1. Given a non-zero vector $v \in \mathbb{R}^n$ and a point $y \in \mathbb{R}^n$ we denote by $L_v(y)$ the line passing through y in the direction v. More generally, if $X \subseteq \mathbb{R}^n$ we denote by $L_v(X)$ the set $\bigcup_{x \in X} L_v(x)$.*

2. A non-zero vector $v \in \mathbb{R}^n$ is an X-stratum at x (or simply a stratum when X is understood) if there exists a real $r > 0$ such that

$$B(x, r) \cap X = B(x, r) \cap L_v(X) \tag{2}$$

This can be seen as saying that inside the ball $B(x, r)$, the relation X is a union of lines parallel to v.

3. The set of X-strata at x is denoted by $Str_X(x)$, or simply $Str(x)$.

Proposition 1. *For all $X \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$ the set Str(x) is either empty or a (vector) subspace of $\mathbb{R}^n$.*

Definition 3. *The dimension $dim(x)$ of a point $x \in \mathbb{R}^n$ is the dimension of the subspace $Str(x)$ if $Str(x)$ is nonempty or 0 otherwise.*

Definition 4. *Given a relation $X \subseteq \mathbb{R}^n$, a point $x \in \mathbb{R}^n$ is X-singular, or simply singular, if $Str(x)$ is empty, otherwise it is nonsingular.*

Note that non-$\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relations may have no singular points: consider in the plane the collection of vertical lines at abscissa $\frac{1}{n}$ for all positive integers n. In this case any vertical vector is a stratum.

Now it can be shown that all strata at x can be defined by a common value r in expression (2).

Proposition 2. *If $\underline{Str}(x) \neq \emptyset$ then there exists a real $r > 0$ such that for every $v \in \underline{Str(x)}$ we have*

$$B(x, r) \cap X = B(x, r) \cap L_v(X).$$

Definition 5. *A safe radius (for x) is a real $r > 0$ satisfying the condition of Proposition 2. Clearly if r is safe then so are all $0 < s \leq r$. By convention every real is a safe radius if $Str(x) = \emptyset$.*

Example 2 (Example 1 continued). For an element x of the interior of the square or the interior of its complement, let r be the (minimal) distance from x to the edges of the square. Then r is safe for x. If x is a vertex then $\mathrm{Str}(x)$ is empty and every $r > 0$ is safe for x. In all other cases r is the minimal distance of x to a vertex.

Lemma 3. *If $x \sim y$ then $\underline{Str}(x) = \underline{Str}(y)$.*

The converse of Lemma 3 is false in general. Indeed consider e.g. $X = \{(x, y) \mid y \leq 0\} \cup \{(x, y) \mid y = 1\}$ in $\mathbb{R}^2$. The points $(0, 0)$ and $(0, 1)$ have the same subspace of strata, namely that generated by $(1, 0)$, but $x \not\sim y$.

Now we combine the notions of strata and of safe radius.

Lemma 4. *Let $X \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ and r be a safe radius for x. Then for all $y \in B(x, r)$ we have $\underline{Str}(x) \subseteq \underline{Str}(y)$.*

Example 3 (Example 1 continued). Consider a point x on an (open) edge of the square and a safe radius r. For every point y in $B(x, r)$ which is not on the edge we have $\mathrm{Str}(x) \subset \mathrm{Str}(y) = \mathbb{R}^2$. For all other points we have $\mathrm{Str}(x) = \mathrm{Str}(y)$.

We relativize the notion of singularity and strata to an affine subspace $P \subseteq \mathbb{R}^n$. The next definition should come as no surprise.

Definition 6. *Given an affine subspace $P \subseteq \mathbb{R}^n$, a subset $X \subseteq P$ and a point $x \in P$, we say that a vector v parallel to P is an (X, P)-stratum for the point x if for all sufficiently small $r > 0$ it holds*

$$P \cap X \cap B(x, r) = P \cap L_v(X) \cap B(x, r)$$

A point $x \in P$ is (X, P)-singular if it has no (X, P)-stratum. For simplicity when P is the space $\mathbb{R}^n$ we will still stick to the previous terminology and speak of X-strata and X-singular points.

Singularity and nonsingularity do not go through restriction to affine subpaces.

Example 4. In the real plane, let $X = \{(x, y) \mid y < 0\}$ and P be the line $x = 0$. Then the origin is not $X-$singular but it is $(X \cap P, P)-$singular. All other elements of P admit $(0, 1)$ as an $(X \cap P, P)-$stratum thus they are not $(X \cap P, P)-$singular. The opposite situation may occur. In the real plane, let $X = \{(x, y) \mid y < 0\} \cup P$ where $P = \{(x, y) \mid x = 0\}$. Then the origin is $X-$singular but it is not $(X \cap P, P)-$singular.

4 Local Properties

4.1 Local Neighborhoods

In this section we recall that if $X \subseteq \mathbb{R}^n$ is $\langle \mathbb{R}, +, <, 1\rangle$-definable then the equivalence relation $\sim$ (introduced in Definition 1) has finite index. This extends easily to the case where X is $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable.

We modify the usual notion of cones so that it suits better our purposes.

Definition 7. *A <u>cone</u> is an intersection of finitely many halfspaces defined by a condition of the form $u(x) < 0$ or $u(x) \leq 0$ where u is a linear expression having rational coefficients. The origin of the space is thus an <u>apex</u> of the cone.*

In particular a point, the empty set and the whole space are specific cones in our sense (on the real line they can be described respectively by $x \leq 0 \wedge -x \leq 0$, $x < 0 \wedge -x < 0$ and $x \leq 0 \vee -x \leq 0$). By convention, the origin is an apex of the empty set.

By paraphrasing [2, Thm 1] where "face" means "$\sim$-equivalence class" in our terminology we have.

Proposition 3. *Consider an $\langle \mathbb{R}, +, <, 1\rangle$-definable relation X. There exists a finite collection Θ of $\langle \mathbb{R}, +, <, 1\rangle$-formulas defining finite unions of cones such that for all $\xi \in \mathbb{R}^n$ there exist some θ in Θ and some real $s > 0$ such that for all $t \leq s$ we have*

$$\theta(t) \wedge |t| < s \leftrightarrow \phi(\xi + t) \wedge |t| < s \tag{3}$$

Corollary 1. *Let $X \subseteq \mathbb{R}^n$ be $\langle \mathbb{R}, +, <, 1\rangle$-definable.*

1. *The equivalence relation $\sim$ has finite index.*
2. *The set <u>Str</u>(x) is finite when x runs over $\mathbb{R}^n$.*
3. *There exists a fixed finite collection $\mathcal{C}$ of cones (in the sense of Definition 7) such that for each $\sim$-class E there exists a subset $\mathcal{C}' \subseteq \mathcal{C}$ such that for every $x \in E$ there exists $r > 0$ such that*

$$(x + t \in X) \wedge |t| < r \ \leftrightarrow \ \big(t \in \bigcup_{C \in \mathcal{C}'} C\big) \wedge |t| < r$$

Because of Lemma 2 we have

Corollary 2. *The statements of Corollary 1 extend to the case where X is $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable.*

Combining Corollaries 1 and 2 allows us to specify properties of singular points for $\langle \mathbb{R}, +, <, 1\rangle$- and $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable relations.

Proposition 4. *Let $X \subseteq \mathbb{R}^n$. If X is $\langle \mathbb{R}, +, <, 1\rangle$-definable then it has finitely many singular points and their components are rational numbers. If X is $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable then it has a countable number of singular points and their components are rational numbers.*

4.2 Application: Expressing the Singularity of a Point in a $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-Definable Relation

The singularity of a point x is defined as the property that no intersection of X with a ball centered at x is a union of lines parallel with a given direction. This property is not directly expressible within $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$ since the natural way would be to use multiplication on reals, which is not $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable. In order to be able to express the property, we give an alternative characterization of singularity which relies on the assumption that X is $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable.

Lemma 5. *Given an $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable relation $X \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$ the following two conditions are equivalent:*

1. *x is singular.*
2. *for all $r > 0$, there exists $s > 0$ such that for all vectors v of norm less than s, there exist two points $y, z \in B(x, r)$ such that $y = z + v$ and $y \in X \Leftrightarrow z \notin X$.*

Observe that when X is not $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable, then the two assertions are no longer equivalent. E.g., $\mathbb{Q}$ has only singular points but condition 2 holds for no point in $\mathbb{R}$.

5 Relations Between Neighborhoods

We illustrate the purpose of this section with a very simple example. We start with a cube sitting in the horizontal plane with only one face visible. The rules of the game is that we are given a finite collection of vectors such that for all 6 faces and all 12 edges it is possible to choose vectors that generate the vectorial subspace of the smallest affine subspace in which they live. Let the point at the center of the upper face move towards the observer (assuming that this direction belongs to the initial collection). It will eventually hit the upper edge of the visible face. Now let the point move to the left along the edge (this direction necessarily exists because of the assumption on the collection). The point will hit the upper left vertex. Consequently, in the trajectory the point visits three different $\sim$-classes: that of the points on the open upper face, that of the points on the open edge and that of the upper left vertex. Here we investigate the adjacency of such equivalence classes having decreasing dimensions. Observe that another finite collection of vectors may have moved the point from the center of the upper face directly to the upper left vertex.

Since two $\sim$-equivalent points either have no stratum or the same subspace of strata, given a $\sim$-class E it makes sense to denote by $\mathrm{Str}(E)$ the empty set in the first case and the common subspace of all points in E in the latter case. Similarly, $\dim(E)$ is the common dimension of the points in E.

5.1 Compatibility

The above explanation should help the reader understand the following definition by considering the backwards trajectory: the point passes from an $\sim$-equivalence class of low dimension into an $\sim$-equivalence class of higher dimension along a direction that is proper to this latter class. This leads to the notion of compatibility. For technical reasons we allow a class to be compatible with itself.

Definition 8. *Let E be a nonsingular $\sim$-class and let v be one of its strata. Given a $\sim$-class F, a point $y \in F$ is $v-$compatible with E if there exists $\epsilon > 0$ such that for all $0 < \alpha \le \epsilon$ we have $y + \alpha v \in E$.*

A $\sim$-class F is v-compatible with E if there exists a point $y \in F$ which is v-compatible with E.

Lemma 6. *Given a $\sim$-class F and a vector $v \in \mathbb{R}^n$ there exists at most one $\sim$-class E such that F is v-compatible with E. If F is v-compatible with E, all elements of F are v-compatible with E.*

Observe that for any nonsingular $\sim$-class E and one of its strata v there always exists a $\sim$-class v-compatible with E, namely E itself, but also that conversely there might be different classes v-compatible with E.

Example 5. Let X be the union of the two axes of the 2-dimensional plane and $v = (1, 1)$. The different classes are: the complement of X, the origin $\{0\}$ which is a singular point, the horizontal axis deprived of the origin, and the vertical axis deprived of the origin. The two latter $\sim$-classes are both $v-$compatible with the class $\mathbb{R}^2 \setminus X$.

5.2 Intersection of a Line and Equivalence Classes

In this section we describe the intersection of a $\sim$-class E with a line parallel to some $v \in \mathrm{Str}(E)$.

With the above example of the cube, a line passing through a point x on the upper face along any of the directions of $\mathrm{Str}(x)$ of dimension 2 intersects an open edge or a vertex at point y. In the former case $\dim(y) = 1$ and in the latter $\dim(y) = 0$, and in both cases $\mathrm{Str}(y) \subset \mathrm{Str}(x)$.

Lemma 7. *Let $X \subseteq \mathbb{R}^n$, F, G be two $\sim$-classes, and $v \in \underline{Str}(F)$. Let y be an element of G which is adherent to $L_y(v) \cap F$. Then $\underline{Str}(G) \subseteq \underline{Str}(F)$.*

If F, G are different, then $\underline{Str}(G) \subseteq \underline{Str}(F) \setminus \{v\}$ and therefore $dim(G) < dim(F)$.

With the above example of the cube, every point x of a face (which is an open subset on the delimiting affine space supporting the face) is interior to some open segment passing through x, parallel to any direction of the subspace $\mathrm{Str}(x)$ and included in the face. The same observation holds for a point on an open edge of the cube.

Lemma 8. *Let $X \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ a nonsingular point and $v \in \underline{Str}(x)$. There exist $y, z \in L_v(x)$ such that $x \in (y, z)$ and every element w of (y, z) satisfies $w \sim x$.*

Consequently, via Lemmas 7 and 8 we get the following.

Corollary 3. *Let $X \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$, E its $\sim$-class and let $v \in Str(x)$. The set $L_v(x) \cap E$ is a union of disjoint open segments (possibly infinite in one or two directions) of $L_v(x)$, i.e., of the form $(y - \alpha v, y + \beta v)$ with $0 < \alpha, \beta \leq \infty$ and $y \in E$.*

If $\alpha < \infty$ (resp. $\beta < \infty$) then the point $y - \alpha v$ (resp. $y + \beta v$) belongs to a $\sim$-class $F \neq E$ where F is v-compatible (resp. $(-v)$-compatible) with E, and $dim(F) < dim(E)$.

Corollary 4. *Given a nonsingular $\sim$-class E, a point $x \in E$ and $v \in \underline{Str}(x)$, the intersection of E with the line $L_v(x)$ is a union of open segments whose endpoints have dimension (cf. Definition 3) less than that of E.*

6 Characterization and Effectivity

6.1 Characterization of $\langle \mathbb{R}, +, <, 1 \rangle$ in $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$

In this section we give the characterization of $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable relations which are $\langle \mathbb{R}, +, <, 1 \rangle$-definable. A rational section of a relation $X \subseteq \mathbb{R}^n$ is a relation of the form

$$X_c^{(i)} = X \cap (\mathbb{R}^i \times \{c\} \times \mathbb{R}^{n-i-1}) \quad \text{for some } c \in \mathbb{Q},\ 0 \leq i < n$$

Theorem 3. *Let $n \geq 1$ and let $X \subseteq \mathbb{R}^n$ be $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable. Then X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable if and only if the following two conditions hold*

1. *There exist finitely many $X-$singular points.*
2. *Every rational section of X is $\langle \mathbb{R}, +, <, 1 \rangle$-definable.*

Observe that both conditions (1) and (2) are needed. Indeed, the relation $X = \mathbb{R} \times \mathbb{Z}$ is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable. It has no singular point thus it satisfies condition (1), but does not satisfy (2) since, e.g., the rational section $X_0^{(0)} = \{0\} \times \mathbb{Z}$ is not $\langle \mathbb{R}, +, <, 1 \rangle$-definable. Now, consider the relation $X = \{(x, x) \mid x \in \mathbb{Z}\}$ which is $\langle \mathbb{R}, +, <, \mathbb{Z} \rangle$-definable. It does not satisfy condition (1) since every element of X is singular, but it satisfies (2) because every rational section of X is either empty or equal to the singleton $\{(x, x)\}$ for some $x \in \mathbb{Z}$, thus is $\langle \mathbb{R}, +, <, 1 \rangle$-definable.

Now we give an idea of the proof since it cannot fit in the space allowed. The necessity of point 1 follows from Proposition 4. That of point 2 results from the fact that all rational constants are $\langle \mathbb{R}, +, <, 1\rangle$-definable by Theorem 1, and moreover that $\langle \mathbb{R}, +, <, 1\rangle$-definable relations are closed under direct product and intersection.

Now the sufficiency. Corollary 4 suggests that we proceed by induction on the dimension of the $\sim$-classes. There are finitely many classes of dimension 0 since there are finitely many singular points so the base of the induction is guaranteed. Now the intersection of a nonsingular class E with a line passing through a point x in the class and parallel to a direction of the class is a finite union of open segments, see Lemma 6. If the segment containing x is closed or half-closed then one of its adherent point belongs to a class F of lower dimension and we can define E relatively to F via the notion of compatibility. However the line may not intersect any other equivalence class. So we consider the canonical subspaces, see below, since every line has an intersection with one of these.

$$\begin{aligned} &H_i = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = 0\} \quad i \in \{1, \ldots, n\} \\ &Q_I = \bigcap_{i \in I} H_i, \quad Q'_I = (Q_I \setminus \bigcup_{i \in \{1,\ldots,n\}\setminus I} H_i) \text{ for all } \emptyset \subset I \subseteq \{1, \ldots, n\} \end{aligned} \tag{4}$$

In particular $Q_{\{1,\ldots,n\}} = \{0\}$ and by convention $Q_\emptyset = \mathbb{R}^n$. The Q_i's are the canonical subspaces. The Q'_i's are not vectorial subspaces but with some abuse of language we will write $\dim(Q'_I)$ to mean $\dim(Q_I) = n - |I|$. Observe that point 2 of the theorem implies that for every I the intersection $X \cap Q_I$ (resp. $X \cap Q'_I$) is $\langle \mathbb{R}, +, <, 1\rangle$-definable.

We consider the finite decomposition of the space consisting of all subsets $E \cap Q'_I$ where E is a $\sim$ $-$class and Q'_I is as in 4. We associate to each subset $E \cap Q'_I$ the pair of integers $(\dim(\mathrm{Str}(E) \cap Q_I), \dim(Q'_I))$ equipped with the product ordering, and we proceed by induction. The result follows from the fact that X is a union of finitely many $\sim$-classes, since if $x \sim y$ then both x and y belong to X or both belong to its complement.

The proof can be seen as describing a trajectory starting from a point x in a $\sim$-class E, traveling along a stratum of E until it reaches a class of lower dimension F (by Corollary 4) or some canonical subspace. In the first case it resumes the journey from the new class F on. In the second case it is trapped in the canonical subspace: it resumes the journey by choosing one direction of the subspace until it reaches a new $\sim$-class or a point belonging to a proper canonical subspace. Along the journey, either the dimension of the new class or the dimension of the canonical subspace decreases. The journey stops when the point reaches a $(X, Q_I)-$singular point, or the origin which is the least canonical subspace.

6.2 Decidability

So far we did not distinguish between formal symbols and their interpretations but here we must do it if we want to avoid any confusion. Let $X_n \subseteq \mathbb{R}^n$ be a relation defined by a $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-formula ϕ. In order to express that X_n is actually $\langle \mathbb{R}, +, <, 1\rangle$-definable we proceed as follows. Let $\{\mathcal{X}_n(x_1, \ldots, x_n) \mid n \geq 1\}$ be a collection of relational symbols. We construct a $\{+, <, 1, \mathcal{X}_n\}-$sentence $\psi_n(\mathcal{X}_n)$ such that $\psi_n(X_n)$ holds if and only X_n is $\langle \mathbb{R}, +, <, 1\rangle$-definable.

Proposition 5. *Let $\{\mathcal{X}_n(x_1, \ldots, x_n) \mid n \geq 1\}$ denote a set of relational symbols. For every $n \geq 1$ there exists a $\{+, <, 1, \mathcal{X}_n\}-$sentence ψ_n such that for every $\{+, <, 1, \mathcal{X}_n\}-$ structure $\mathcal{M} = (\mathbb{R}, +, <, 1, X_n)$, if X_n is $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable then we have $\mathcal{M} \models \psi_n$ if and only if X_n is $\langle \mathbb{R}, +, <, 1\rangle$-definable.*

Sketch. The formula is of the form

$$\sigma_n(\mathcal{X}_n) \wedge \bigwedge_{1 \leq i \leq n} \forall y \; \psi^{(i)}_{n-1}(y, \mathcal{X}_{n-1}) \tag{5}$$

where each $\psi^{(i)}_{n-1}(y, \mathcal{X}_{n-1})$ is obtained from $\psi_{n-1}(\mathcal{X}_{n-1})$ by inserting y at position i in the sequence of variables of the interpretation X_n. The conjunct $\sigma_n(\mathcal{X}_n)$ expresses the fact that X_n has finitely many singular points (point 1 of Theorem 3) and each conjunct $\psi^{(i)}_{n-1}(y, \mathcal{X}_{n-1})$ expresses the fact that, interpreting y as a parameter, the section is $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable (point 2 of Theorem 3). As an example $\sigma_1(\mathcal{X}_1)$ is as follows (the formula is correct only when $\mathcal{X}_1$ is interpreted as a $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable relation)

$$\begin{aligned}&\exists r \forall x \in \mathbb{R} \; (\forall t > 0 \\ &((\exists y \in \mathcal{X}_1 \wedge |y - x| < t) \wedge (\exists y \notin \mathcal{X}_1 \wedge |y - x| < t))) \rightarrow |x| \leq r)\end{aligned}$$

Theorem 4. *For every $n \geq 1$ and every $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable relation $X \subseteq \mathbb{R}^n$, it is decidable whether X is $\langle \mathbb{R}, +, <, 1\rangle$-definable.*

Proof. In Proposition 5, if we substitute the predicate $\phi(x)$ for every occurrence of $x \in \mathcal{X}_n$ in ψ_n, then ψ_n can be interpreted in the structure $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$ and the decidability of its truth value results from the decidability of $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$ [20].

7 Conclusion

We discuss some extensions and open problems. Is it possible to remove our assumption that X is $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$-definable in Theorem 3? We believe that the answer is positive and it can be formally proven in dimension 2. Note that even if one proves such a result, the question of providing an effective characterization is more complex. Indeed the sentence ψ_n of Proposition 5 expresses a variant of the criterion of Theorem 3, and we use heavily the fact that we work within $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$ to ensure that this variant is actually equivalent to the criterion. In particular the construction of ψ_n relies on Lemma 5 to express that a point is $X-$singular. However if we consider e.g. $X = \mathbb{Q}$ then every element x of X is singular while no element x of X satisfies the condition stated in Lemma 5.

Another question is the following. In Presburger arithmetic it is decidable whether or not a formula is equivalent to a formula in the structure without $<$, cf. [10]. What about the case where the structure is $\langle \mathbb{R}, +, <, \mathbb{Z}\rangle$?

References

1. Becker, B., Dax, C., Eisinger, J., Klaedtke, F.: LIRA: handling constraints of linear arithmetics over the integers and the reals. In: Damm, W., Hermanns, H. (eds.) CAV 2007. LNCS, vol. 4590, pp. 307–310. Springer, Heidelberg (2007). https://doi.org/10.1007/978-3-540-73368-3_36
2. Bieri, H., Nef, W.: Elementary set operations with d-dimensional polyhedra. In: Noltemeier, H. (ed.) CG 1988. LNCS, vol. 333, pp. 97–112. Springer, Heidelberg (1988). https://doi.org/10.1007/3-540-50335-8_28
3. Boigelot, B., Brusten, J., Bruyère., V.: On the sets of real numbers recognized by finite automata in multiple bases. LMCS **6**(1), 1–17 (2010)
4. Boigelot, B.: The Liege automata-based symbolic handler (LASH). http://www.montefiore.ulg.ac.be/boigelot/research/lash/
5. Boigelot, B., Brusten, J.: A generalization of Cobham's theorem to automata over real numbers. Theor. Comput. Sci. **410**(18), 1694–1703 (2009)
6. Boigelot, B., Brusten, J., Leroux, J.: A generalization of Semenov's theorem to automata over real numbers. In: Schmidt, R.A. (ed.) CADE 2009. LNCS (LNAI), vol. 5663, pp. 469–484. Springer, Heidelberg (2009). https://doi.org/10.1007/978-3-642-02959-2_34
7. Boigelot, B., Rassart, S., Wolper, P.: On the expressiveness of real and integer arithmetic automata. In: Larsen, K.G., Skyum, S., Winskel, G. (eds.) ICALP 1998. LNCS, vol. 1443, pp. 152–163. Springer, Heidelberg (1998). https://doi.org/10.1007/BFb0055049
8. Bouchy, F., Finkel, A., Leroux, J.: Decomposition of decidable first-order logics over integers and reals. In: 2008 15th International Symposium on Temporal Representation and Reasoning, pp. 147–155. IEEE (2008)
9. Büchi, J.R.: On a decision method in the restricted second-order arithmetic. In: Proceedings International Congress Logic, Methodology and Philosophy of Science, Berkeley 1960, pp. 1–11. Stanford University Press (1962)
10. Choffrut, C., Frigeri, A.: Deciding whether the ordering is necessary in a Presburger formula. DMTCS **12**(1), 20–38 (2010)
11. Cobham, A.: On the base-dependence of sets of numbers recognizable by finite automata. Math. Syst. Theor. **3**(2), 186–192 (1969)
12. Ferrante, J., Rackoff, C.: A decision procedure for the first order theory of real addition with order. SIAM J. Comput. **4**(1), 69–76 (1975)
13. Fränzle, M., Quaas, K., Shirmohammadi, M., Worrell, J.: Effective definability of the reachability relation in timed automata. Inf. Proc. Lett. **153**, 105871 (2020)
14. Leroux, J.: A polynomial time Presburger criterion and synthesis for number decision diagrams. In: Proceedings of LICS 2005, pp. 147–156. IEEE (2005)
15. Milchior, A.: Büchi automata recognizing sets of reals definable in first-order logic with addition and order. In: Gopal, T.V., Jäger, G., Steila, S. (eds.) TAMC 2017. LNCS, vol. 10185, pp. 440–454. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-55911-7_32
16. Miller, C.: Expansions of dense linear orders with the intermediate value property. J. Symb. Logic **66**(4), 1783–1790 (2001)
17. Muchnik, A.A.: The definable criterion for definability in Presburger arithmetic and its applications. Theor. Comput. Sci. **290**(3), 1433–1444 (2003)
18. Presburger, M.: Uber die vollstandigkeit eines gewissen systems der arithmetic ganzer zahlen, in welchem die addition als einzige operation hervortritt. In: du Premier Congrès des Mathématiciens des Pays Slaves, Warsaw, vol. 395, pp. 92–101 (1927)

19. Semenov, A.L.: Presburgerness of predicates regular in two number systems. Siberian Math. J. **18**(2), 289–300 (1977)
20. Weispfenning, V.: Mixed real-integer linear quantifier elimination. In: Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation, ISSAC 1999, pp. 129–136. ACM, New York (1999)