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ORIGINAL PAPER

Modified Asano-Ohya-Khrennikov quantum-like model for decision-making process in a two-player game with nonlinear self- and cross-interaction terms of brain's amygdala and prefrontal-cortex

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Abstract
In this report, we propose a modification on the Asano-Ohya-Khrennikov quantum-like decision-making process model of a two-player game by adding additional nonlinear terms to the related comparison step dynamical equation. The additions are in the form of a self-interaction and cross-interaction of the brain's amygdala and prefrontal cortex. We show that the cross-interaction significantly determines the final decision of a player, whether it becomes a rational or an irrational choice. In contrast, the nonlinear self-interaction term provides a feedback mechanism that speeds up the corresponding decision-making process. We also suggest the form of expectation values of the overall reaction rate coefficients of those nonlinear terms by making an analogy with the original model formulation.

Keywords Asano-Ohya-Khrennikov quantum-like model · Decision-making process · Two-player game · Amygdala · Prefrontal cortex

1 Introduction

The physical mechanism underlying the high complexity of human brain behavior undoubtedly has attracted much attention in recent years [1, 2]. Mathematical and computational models have been proposed to explain some particular brain's behaviors [3–5]. One of those behaviors is the ability to decide on a specific situation, e.g., when two persons play a specific game. Players will take actions that are increasing their payoffs in order to win the game. For the rational players, if there is an appropriate action which can be decided uniquely, they will choose it with a probability of 1, without considering the action of other players. On the other

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hand, for irrational players, this situation does not automatically lead them to decide uniquely or in other words, which actions they will take are probabilistic. Shafir and Tversky [6] have shown from their experiment in two-player prisoner's dilemma (PD) game that the probability of a player to choose action irrationally is 0.37.

In [7], Asano, Ohya, and Khrennikov have developed a model to describe the behavior of such player's rational/irrational choices in a two-player game. Their model formulation borrowed the quantum theory technical description, where each of the players is represented by state vectors, which are called "predictive state" and "mental state," as elements of a 4-dimensional Hilbert space. The basic assumption of their quantum-like model was the independency of each player choices, where a player does not know the choice of the other player and can only make predictions on it. They defined three different states, namely predictive state, alternative state, and mental state, to describe the decision-making process that consists of prediction, comparison, and decision steps. In the comparison step, the dynamical process is governed by a set of differential equations related to the probabilities of players to choose their final decisions. It was assumed that the associated differential equation is adopted from the model of a chemical reaction with a linear interaction term.

In this report, we consider two things. First, we assume that the corresponding decision-making process in a two-player game is mainly due to the cognitive-emotional interaction between the brain's amygdala and prefrontal cortex. It has already known that the amygdala is responsible for the irrational (emotional) decision in which a fear-driven emotional decision plays a significant role [9, 10]. On the other hand, the prefrontal cortex, which is a part of the brain's neocortex, is likely responsible for the rational (cognitive) decision, where it has an executive function [11, 12]. Second, based on the first assumption, we propose a model modification, namely to include the self- and cross-interaction terms of the amygdala and prefrontal cortex in the related dynamical equation of the comparison step. The interaction between the amygdala and prefrontal cortex has been a subject for intense investigation in recent years [13–19]. The assumption to include the interaction between the brain's amygdala and the prefrontal cortex, to some extent, offers flexibility in explaining the dynamics of the decision-making process. To our best knowledge, this modification has never been reported elsewhere.

It should be emphasized from the beginning that our proposal does not alter the basic quantum technical formulation of the model but mainly focus on the modification of the associated dynamical equation on the comparison step, namely by considering the nonlinear interactions between the amygdala and prefrontal cortex. It should be realized that the corresponding Asano-Ohya-Khrennikov model neglects the interaction between players and their environment. Such shortcomings have been remarkably addressed and generalized by Bagarello [23, 24], namely by introducing a quantum-like formulation in terms of fermionic-based creation and annihilation operators in the Heisenberg picture where the dynamics are nicely expressed in terms of the open system full Hamiltonian operator $H = H_0 + H_{\text{int}} + H_I$. The operators H_0, H_{int}, and H_I account for the number operator of players and environment reservoir, interaction between players, and interaction between players and their environment reservoir, respectively. Remarkably, Bagarello's model also introduced the H_{int} interaction term in the full Hamiltonian which further expands the Asano-Ohya-Khrennikov model to account for the interaction between the players. This term is relevant in the discussion of our present manuscript and any quantum technical based-formulation regarding prisoner's dilemma game. Hence, from an open quantum-like system point of view, the model that we propose can be considered as a specific case. Our proposal is likely related to a nonlinear H_{int}

Hamiltonian formulation based on the related fermionic creation and annihilation operators. Unfortunately, the nonlinear addition will require perturbative methods to be solved which is currently beyond the scope of this work but worth for further research.

We organize this report as follows. In section 2, we briefly discuss the definition of a two-player game. Meanwhile the original Asano-Ohya-Khrennikov model for the decision-making process developed by Asano et al. [7, 8] is discussed in section 3, while in section 4, we discuss our proposed perspective and modified model. Finally, a summary is given in the section 5.

2 The two-player game

A two-player game consists of two players, say Alice (A) and Bob (B), in which they have two distinct choices which can be assigned as "0" and " 1." In the corresponding game, these two players are assumed to seek the highest pay-off. Each of them has to decide in order to achieve it. The rule of the two-player game is given in Table 1. We assume that the choice of option "1" will possibly give a player a highest pay-off, while oppositely, the option "0" will possibly give a player a lowest pay-off. Nevertheless, whether those two distinct choices will eventually give a player a highest pay-off is depending on the other player choice as shown in the table, where $a > b > c > d$ determines the corresponding pay-off. Here, we call the option "1" as a rational decisional choice, while the option "0" is called an irrational choice, since if, for instance, the player A chooses " 1," while player B chooses " 0," then A will get a highest pay-off and vice versa. But this is not the case when both choose "1" simultaneously, it will give them relatively low pay-off, so does if both of them choose option "0" simultaneously.

The process to determine the probability of a player, which is not knowing the decision of the other player, to decide which option to be chosen is the main problem to be solved. On the other hand, since the quantum theory provided a powerful technical tool to describe the uncertainty behavior in the microscopic world, it offers a way to develop a model for mimicking the cognitive-emotional decision-making process of the related two-player game. Based on a similar technical formulation of quantum theory, many models have been proposed, e.g., refs. [7, 8, 20–23] to describe this cognitive-emotional process. Discussion regarding the implementation of quantum technical concept of Asano-Ohya-Khrennikov generalized model for a two-player decision-making process is given by Bagarello's open quantum-like decision-making model [23, 24] which has been described early on in the introduction.

3 Asano-Ohya-Khrennikov quantum-like model

As mentioned in the Introduction, our discussion is based on the Asano-Ohya-Khrennikov decision-making process model of the two-player game [7], where the

Table 1 Pay-off table of a two-player game with pay-off values $a > b > c > d$

A/B	"0"	"1"
"0"	(b, b)	(d, a)
"1"	(a, d)	(c, c)

description of this model relies on the quantum technical formulation. We briefly review the model in this section, followed by our proposed modification in the next section. In principle, the model depends on three different states to be discussed below, namely, predictive, alternative, and mental states. Concerning the decision-making process, all these states are associated with the prediction, comparison, and decision steps. As mentioned in the Introduction section, our primary focus in this report is on modifying the dynamical equation in the comparison step.

Let us recall our two players: Alice (A) and Bob (B). Either of these players can choose two distinct states namely irrational option "0" or rational option "1" decision denoted by $|0\rangle$ and $|1\rangle$, respectively. In the mind of player A, the player B is assumed to have both possible choices, namely represented in terms of the following predictive state vector:

$$|\phi_B\rangle = \alpha|0_B\rangle + \beta|1_B\rangle \tag{1}$$

where the complex α and β coefficients are satisfying normalized condition $|\alpha|^2 + |\beta|^2 = 1$. This predictive state vector defines a predictive state with density matrix $\Theta_A = |\phi_B\rangle\langle\phi_B|$, which is mathematically interpreted to contain the possible situations of player A in making a definite judgment on the player's B choices [7].

Next, we can define the related mental state vector of A is defined as follows:

$$|\Psi_A\rangle = x|\Phi_{0A}\rangle + y|\Phi_{1A}\rangle \tag{2}$$

where $|\Phi_{0\mathrm{A}}\rangle$ and $|\Phi_{1\mathrm{A}}\rangle$ are given as follows:

$$|\Phi_{0A}\rangle = |0_A\rangle \otimes |\phi_B\rangle = \alpha|0_A 0_B\rangle + \beta|0_A 1_B\rangle \tag{3}$$

$$|\Phi_{1A}\rangle = |1_A\rangle \otimes |\phi_B\rangle = \alpha|1_A 0_B\rangle + \beta|1_A 1_B\rangle \tag{4}$$

respectively. It should be emphasized that the state vectors (3) and (4) are considered to define basis vectors, which can be interpreted as the possibility of the player A to consciously choose the alternative $|0_A\rangle$ or $|1_A\rangle$ state, respectively. Meanwhile, $|x|^2 \equiv P_{0A}$ and $|y|^2 \equiv P_{1A}$ in the mental state vector (2) are interpreted as the probabilities of A to choose either of both choices, satisfying the normalization condition

$$P_{0A} + P_{1A} = 1 \tag{5}$$

Based on the above formulation, it is clear that the mental state vector $|\Psi_A\rangle$ given by Eq. (2) describes the four consequences of choices to be experienced by player A. Note that symmetric formulation can be found from B perspective simply by changing $A \to B$ label.

It is useful to note for the sake of a comprehensive review, that from the possible state basis vectors (3) and (4), and from the mental state vector (2), one can also define the so-called alternative state with density matrix $\Lambda_A = |\Phi_{0A,\ 1A}\rangle\langle\Phi_{0A,\ 1A}|$ and mental state with the density matrix $M_A = \Lambda_A \otimes \Theta_A$, respectively, as discussed in [7], where Θ_A is the predictive state density matrix. However, all these density matrices are irrelevant with the aim of our present discussion, such that they will not be further discussed.

In the original Asano-Ohya-Khrennikov model, the dynamics of P_{0A} and P_{1A} probabilities in the comparison step are assumed to be governed by the following coupled dynamical equation:

$$\frac{dP_{0A}}{dt} = -k_{0L}P_{0A} + k_{1L}P_{1A} \tag{6}$$

$$\frac{dP_{1A}}{dt} = -k_{1L}P_{1A} + k_{0L}P_{0A} \tag{7}$$

which are adopted from a chemical equilibrium model of a reaction system [7]. In the chemical reaction perspective, the coefficients k_{0L} and k_{1L} are considered as the overall possible reaction rates from $|0_A\rangle \to |1_A\rangle$ and $|1_A\rangle \to |0_A\rangle$, respectively. Here, in our perspective based on [13–19], both coefficients are considered to represent the influence of brain's prefrontal-cortex, which is related to the rational decision, and brain's amygdala, which is related to the irrational decision, respectively. It is seen in the Eqs. (6) and (7) that the coefficient $k_{0L}(k_{1L})$ plays a role in stabilizing the irrational (rational), and simultaneously increasing the tendency of rational (irrational), decision of player A.

For the sake of simple calculation, we use the condition (5) to reduce coupled Eqs. (6) and (7) into the following single differential equation:

$$\frac{dP_{0A}}{dt} = k_{1L} - (k_{0L} + k_{1L})P_{0A} \tag{8}$$

which is easy to prove that the equilibrium point of Eq. (8) is given as follows:

$$P_{0A}^{E} = \frac{k_{1L}}{k_{0L} + k_{1L}} \tag{9}$$

such that

$$P_{1A}^{E} = \frac{k_{0L}}{k_{0L} + k_{1L}} \tag{10}$$

Linearizing Eq. (7) around the equilibrium point (8) one finds:

$$\frac{dP_{0A}}{dt} = -k_{1L}P_{0A} \tag{11}$$

Clearly, the corresponding eigenvalue of Eq. (8) is nothing but $\lambda = -k_{1L}$ indicating that the corresponding equilibrium point is a stable point for $k_{1L} > 0$. Following the same procedure, one can also find for the P_{1A}^{E} equilibrium point $\lambda = -k_{0L}$, which is also stable for $k_{0L} > 0$.

It was assumed that the overall reaction rates k_{0L} and k_{1L} are determined by the following four possible choices of comparison processes:

$$\begin{array}{ll} |0_A 0_B\rangle \underset{k_{1L}^{(1)}}{\overset{k_{0L}^{(1)}}{\rightleftharpoons}} |1_A 0_B\rangle, & |0_A 1_B\rangle \underset{k_{1L}^{(2)}}{\overset{k_{0L}^{(2)}}{\rightleftharpoons}} |1_A 1_B\rangle \\ |0_A 1_B\rangle \underset{k_{1L}^{(3)}}{\overset{k_{0L}^{(3)}}{\rightleftharpoons}} |1_A 0_B\rangle, & |0_A 0_B\rangle \underset{k_{1L}^{(4)}}{\overset{k_{0L}^{(4)}}{\rightleftharpoons}} |1_A 1_B\rangle \end{array} \tag{12}$$

where the symbol $k_{0(1)L}^{(i)}$, $i = 1, 2, 3, 4$, is representing the reaction rates of each particular choice. Based on the above fact and by taking into account the effect of quantum interferences, one can further assume that the expectation values of the overall reaction rates are given as follows [7]:

$$k_{0L} = \left| |\alpha|^2 \left[k_{0L}^{(1)}\right]^{1/2} + |\beta|^2 \left[k_{0L}^{(2)}\right]^{1/2} + \alpha\beta^* \left[k_{0L}^{(3)}\right]^{1/2} + \alpha^*\beta \left[k_{0L}^{(4)}\right]^{1/2} \right|^2 \tag{13}$$

$$k_{1L} = \left| |\alpha|^2 \left[k_{1L}^{(1)}\right]^{1/2} + |\beta|^2 \left[k_{1L}^{(2)}\right]^{1/2} + \alpha^*\beta \left[k_{1L}^{(3)}\right]^{1/2} + \alpha\beta^* \left[k_{1L}^{(4)}\right]^{1/2} \right|^2 \tag{14}$$

Depicted in Fig. 1 is an example of the P_{0A} and P_{1A} governed by Eq. (6) and (7). It is shown that for an initial condition with $P_{0A} < P_{1A}$ under a specific set of parameter values, the A player tends to choose an irrational decision with final probability $P_{0A} = P_{0A}^E = 2/3 > P_{1A} = 1/3$. It should be admitted that this model offered relatively limited flexibility, since it only involves two free parameters in the model, namely k_{0L} and k_{1L}, which only describe a linear interaction between the brain's prefrontal-cortex and amygdala in the corresponding decision-making process.

4 Modified Asano-Ohya-Khrennikov quantum-like model

We propose a modification on the Eqs. (6) and (7), which are related to the comparison step of the decision-making process, to include the nonlinear self- and cross-interaction terms of the brain's amygdala and prefrontal cortex as follows:

$$\frac{dP_{0A}}{dt} = -k_{0L}P_{0A} - k_{0S}P_{0A}^2 \pm k_{C,\pm}P_{0A}P_{1A} + k_{1L}P_{1A} + k_{1S}P_{1A}^2 \tag{15}$$

$$\frac{dP_{1A}}{dt} = -k_{1L}P_{1A} - k_{1S}P_{1A}^2 \mp k_{C,\pm}P_{0A}P_{1A} + k_{0L}P_{0A} + k_{0S}P_{0A}^2 \tag{16}$$

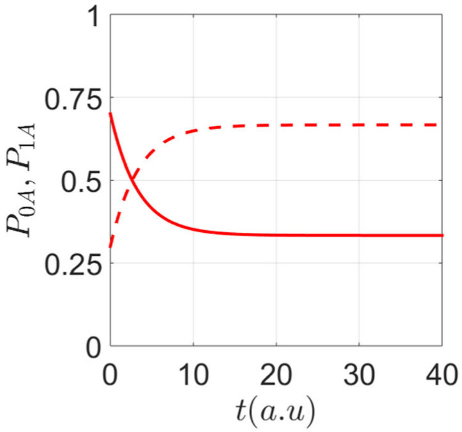


Fig. 1 The dynamics of probability functions P_{0A} (red dash curve) and P_{1A} (red solid curve) for $k_{0L} = 0.1$ and $k_{1L} = 0.2$ with $P_{0A}(0) = 0.3$ and $P_{1A}(0) = 0.7$

where in terms of the chemical reaction system, the Eqs. (15) and (16) represent a nonlinear chemical reaction process.

As mentioned in the Introduction section, we should emphasize that we consider our modification based on the observed amygdala and prefrontal cortex nonlinear interactions [13–19], as well as the specific functions of both brain's part [9–12]. Similar nonlinear interactions can also be found in other physical systems, for instance, in the nonlinear optical systems with two or more different interacting modes (see, e.g., [25–27]). In our perspective, the coefficient $k_{0(1)S}$ corresponds to the self-interaction term of prefrontal-cortex (amygdala) for stabilizing the irrational (rational), and simultaneously increasing the rational (irrational) decision of player A, while $k_{C,\pm}$ denotes the cross-interaction term between the amygdala and prefrontal cortex. Here, the values of all coefficients are positive. It will be shown later that $k_{C,\pm}$ significantly determines the dynamical characteristics of the decision-making process.

It is interesting to note that the quantum-like generalization of this modified comparison step might be related to a nonlinear modification of H_{int} Hamiltonian term of Bagarello's open quantum-like decision-making model [23, 24]. This issue deserves further investigation.

Similar to Eqs. (6) and (7), we can also reduce Eqs. (15) and (16) into the following form based on condition (5):

$$\frac{dP_{0A}}{dt} = (k_{1L} + k_{1S}) + \left(\pm k_{C,\pm} - k_{0L} - k_{1L} - 2k_{1S}\right)P_{0A} + \left(k_{1S} - k_{0S} \mp k_{C,\pm}\right)P_{0A}^2 \tag{17}$$

By a straightforward algebraic manipulation, we found the following relevant equilibrium point of Eq. (17):

$$P_{0A}^E = \frac{-k_{0L} - k_{1L} - 2k_{1S} \pm k_{C,\pm} + \sqrt{\Gamma}}{2\left(k_{0S} - k_{1S} \pm k_{C,\pm}\right)} \tag{18}$$

with $\Gamma = (\pm k_{C,\pm} - k_{0L} - k_{1L} - 2k_{1S})^2 + 4(\pm k_{C,\pm} + k_{0S} - k_{1S})(k_{1L} + k_{1S})$, such that:

$$P_{1A}^E = \frac{k_{0L} + k_{1L} + 2k_{1S} \pm k_{C,\pm} - \sqrt{\Gamma}}{2\left(k_{0S} - k_{1S} \pm k_{C,\pm}\right)} \tag{19}$$

Linearizing Eq. (17) around equilibrium point (18) yields the following equation:

$$\frac{dP_{0A}}{dt} = -\sqrt{\Gamma}P_{0A} \tag{20}$$

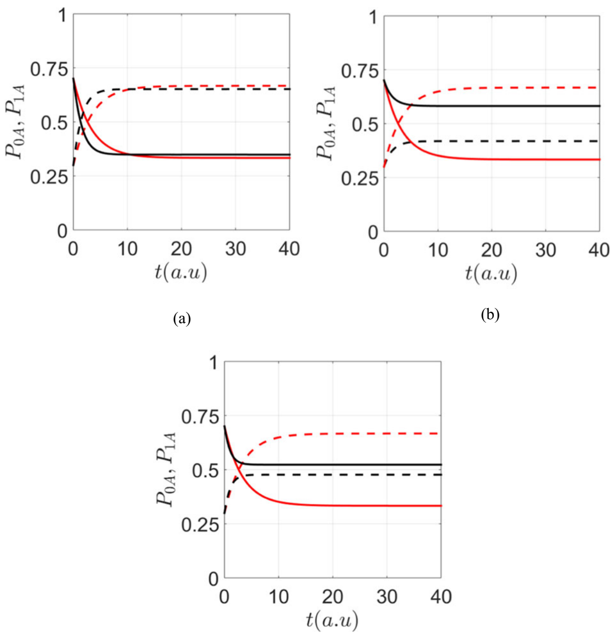
It is easy to prove that for the equilibrium point (19), the related linearized equation is given by:

$$\frac{dP_{1A}}{dt} = -\sqrt{\Gamma}P_{1A} \tag{21}$$

Obviously, from the linearized Eqs. (20) and (21), it is indicated that the corresponding equilibrium points are nothing but stable points under a condition $\Gamma > 0$. This condition leads to the real equilibrium point. On the other hand, we have to rule out the condition $\Gamma < 0$, since it leads to an unphysical condition with complex equilibrium point. Therefore, in our model, this Γ condition restricts the related parameter space. It is also important to note that the replacement of $\sqrt{\Gamma} \to -\sqrt{\Gamma}$ in Eqs. (18) and (19) also gives another equilibrium point, but it is readily seen from Eqs. (20) and (21), it leads to unstable points, which are irrelevant to the model.

Depicted in Fig. 2, examples of the P_{0A} and $P_{1A} = 1 - P_{0A}$ dynamics by solving Eq. (17) numerically based on the Runge-Kutta method. It is demonstrated that the cross-interaction term with coefficient $k_{C,\pm}$ can switch the final decision of player A. Similar to the example in Fig. 1, as shown in Fig. 2a for $k_{C,+}$, that the player A with initial rational decision, i.e., $P_{1A}(t_i) = 0.70$ tends to achieve an irrational final decision with $P_{1A}(t_f) = 0.35$. On the other hand, as shown in Fig. 2b, an opposite decision is found when the corresponding cross-interaction switch from $k_{C,+}$ to $k_{C,-}$, where the player A tends to keep rational with $P_{1A}(t_f) = 0.58$. This feature indicates that the cross-interaction term can play a role as a bifurcation parameter of the system. In other words, the related nonlinear cross-interaction between the amygdala and prefrontal cortex significantly determines the final decision of a player.

Meanwhile, compared to the original Asano-Ohya-Khrennikov model, the addition of self-interaction terms with $k_{1(0)S}$ coefficient affects the rapidity of the decision as shown in Fig. 2a. This situation is likely explaining that the related self-interaction term provides a feedback mechanism that speeds up the player's decision-making process. Note that this self-interaction term can also change the final decision player, similar to the cross-interaction term. As shown in Fig. 2c, it is demonstrated that the increase of k_{0S} leads to a similar condition given by Fig. 2b. This feature further



(a) (b) (c)

Fig. 2 The dynamics of probability functions P_{0A} (black dash curve) and P_{1A} (black solid curve) for $k_{0L} = 0.1$, $k_{1L} = 0.2$, $k_{0S} = 0.2$, $k_{1S} = 0.1$, and (**a**) $k_{C,+} = 0.3$, (**b**) $k_{C,-} = 0.3$,and (**c**) $k_{C,+} = 0.3$ with $k_{0S} = 0.7$. We set $P_{0A}(0) = 0.3$ and $P_{1A}(0) = 0.7$. The red curves are similar to Fig. 1

indicates that the corresponding k_{0S} coefficient actually leads to a negative feedback mechanism, while the k_{1S} coefficient plays the opposite role.

At this point, similar to the previous model, to determine the next steps of the decision-making process, we have to define each of k_{0L}, k_{1L}, k_{0S}, k_{1S}, and $k_{C,\pm}$ coefficients in Eqs. (15) and (16) based on the following four possible choices:

$$\begin{array}{ll} |0_A 0_B\rangle \underset{k_{1L}^{(1)},k_{1S}^{(1)},k_{C,\mp}^{(1)}}{\overset{k_{0L}^{(1)},k_{0S}^{(1)},k_{C,\pm}^{(1)}}{\rightleftharpoons}} |1_A 0_B\rangle, & |0_A 1_B\rangle \underset{k_{1L}^{(2)},k_{1S}^{(2)},k_{C,\mp}^{(2)}}{\overset{k_{0L}^{(2)},k_{0S}^{(2)},k_{C,\pm}^{(2)}}{\rightleftharpoons}} |1_A 1_B\rangle \\ |0_A 1_B\rangle \underset{k_{1L}^{(3)},k_{1S}^{(3)},k_{C,\mp}^{(3)}}{\overset{k_{0L}^{(3)},k_{0S}^{(3)},k_{C,\pm}^{(3)}}{\rightleftharpoons}} |1_A 0_B\rangle, & |0_A 0_B\rangle \underset{k_{1L}^{(4)},k_{1S}^{(4)},k_{C,\mp}^{(4)}}{\overset{k_{0L}^{(4)},k_{0S}^{(4)},k_{C,\pm}^{(4)}}{\rightleftharpoons}} |1_A 1_B\rangle \end{array} \tag{23}$$

By making analogy to the previous original model, it is likely reasonable to propose conjectures for the expectation values of the associated coefficients in the following forms:

$$k_{0S} = \left| |\alpha|^2 \left[k_{0S}^{(1)}\right]^{1/4} + |\beta|^2 \left[k_{0S}^{(2)}\right]^{1/4} + \alpha^*\beta \left[k_{0S}^{(3)}\right]^{1/4} + \alpha\beta^* \left[k_{0S}^{(4)}\right]^{1/4} \right|^2 \tag{24}$$

$$k_{1S} = \left| |\alpha|^2 \left[k_{1S}^{(1)}\right]^{1/4} + |\beta|^2 \left[k_{1S}^{(2)}\right]^{1/4} + \alpha\beta^* \left[k_{1S}^{(3)}\right]^{1/4} + \alpha^*\beta \left[k_{1S}^{(4)}\right]^{1/4} \right|^2 \tag{25}$$

$$k_{C,+} = \left| |\alpha|^2 \left[k_{C,+}^{(1)}\right]^{1/4} + |\beta|^2 \left[k_{C,+}^{(2)}\right]^{1/4} + \alpha\beta^* \left[k_{C,+}^{(3)}\right]^{1/4} + \alpha^*\beta \left[k_{C,+}^{(4)}\right]^{1/4} \right|^2 \tag{26}$$

$$k_{C,-} = \left| |\alpha|^2 \left[k_{C,-}^{(1)}\right]^{1/4} + |\beta|^2 \left[k_{C,-}^{(2)}\right]^{1/4} + \alpha^*\beta \left[k_{C,-}^{(3)}\right]^{1/4} + \alpha\beta^* \left[k_{C,-}^{(4)}\right]^{1/4} \right|^2 \tag{27}$$

while k_{0L} and k_{1L} are given by Eqs. (13) and (14). Here, we have assumed that in our formulation the order of $k_{0(1)L}^{(i)} P_{0(1)A} \sim k_{0(1)S}^{(i)} P_{0(1)A}^2 \sim k_{C,-(+)}^{(i)} P_{0A} P_{1A}$, with $i = 1, 2, 3, 4$. One should realize that the expectation values of reaction rates (13), (14), (24)–(27) arise from the fact that there are four possible comparison processes experienced by the player A as given by (23).

To this end, we have to re-emphasize that this report only focused on the dynamics of the comparison steps. However, all the results, including the proposed conjectures (24)–(27), can be used to determine the whole decision-making process of a specific two-player game, such as the two-player prisoner's dilemma game [6–8], which is beyond the scope of our current discussion. It is also important to note that the proposed dynamics, supplemented by additional nonlinear terms representing the very reasonable nonlinear self-interactions of the brain's amygdala and prefrontal-cortex and their nonlinear cross-interactions. The results change the possible outcomes in nuanced and interesting ways. Therefore, further experimental observation based on, for instance, optical imaging methods [28, 29], is encouraged to validate the model assumption. We also would like to put out further challenges regarding the incorporation of nonlinear interaction in the prisoner's dilemma into a rigid quantum technical description, i.e., in the case of Bagarello's open quantum-like decision-making model [23, 24]. Such an interaction can be build based on a nonlinear term in the corresponding Hamiltonian expansion and solved using perturbation theory. All these issues might be an exciting topic for further studies.

5 Summary

We have discussed in this report our proposed modified Asano-Ohya-Khrennikov quantum-like model for the cognitive-emotional decision-making process in a two-player game. In our perspective, the model should be considered describing the connection between the brain's amygdala and prefrontal-cortex in order to understand the related decision-making process especially in the comparison step. To modify the model, we added two nonlinear interaction terms in the associated comparison step dynamical equation, in the form of a self-interaction and cross-interaction terms. These terms describe the nonlinear relationship between the amygdala and prefrontal cortex. Our results show that the self-interaction term provides a feedback mechanism that can speeds up the decision-making process, while the cross-interaction term can significantly affect the final decision of a player. We also suggested the explicit form of expectation values of reaction rates related coefficients involved in the associated comparison step coupled nonlinear dynamical equation.

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest.

References

1. Van de Ville, D.: Brain dynamics: global pulse and brain state switching. Curr. Biol. **29**, R690 (2019)
2. McIntosh, A.R., Jirsa, V.K.: The hidden repertoire of brain dynamics and dysfunction. Network Neuroscience **3**, 994–1008 (2019)
3. Lopez-Rincon, A., Cantu, C., Etcheverry, G., Soto, R., Shimoda, S.: Function based brain modeling and simulation of an ischemic region in post-stroke patients using the bidomain. J. Neurosci. Methods **331**, 108464 (2020)
4. Donnelly-Kehoe, P., Saenger, V.M., Lisofsky, N., Kühn, S., Kringelbach, M.L., Schwarzbach, J., Lindenberger, U., Deco, G.: Reliable local dynamics in the brain across sessions are revealed by whole-brain modeling of resting state activity. Hum. Brain Mapp. **40**, 2967–2980 (2019)
5. Koyama, K., Niwase, K.: A quantum brain model of decision-making process incorporated with social psychology. NeuroQuantology **17**, 72–76 (2019)
6. Shafir, E., Tversky, A.: Thinking through uncertainty: nonconsequential reasoning and choice. Cogn. Psychol. **24**, 449–474 (1992)
7. Asano, M., Ohya, M., Khrennikov, A.: Quantum-like model for decision making process in two-player game: a non-Kolmogorovian model. Foundation of Physics **41**, 538–548 (2011)
8. Asano, M., Basieva, I., Khrennikov, A., Ohya, M., Tanaka, Y.: Quantum-like dynamics of decision-making. Physica A **391**, 2083–2099 (2012)
9. Nasehi, M., Davoudi, K., Ebrahimi-Ghiri, M., Zarrindast, M.R.: Interplay between serotonin and cannabinoid function in the amygdala in fear conditioning. Brain Res. **1636**, 142–151 (2016)
10. Fadok, J.P., Markovic, M., Tovote, P., Lüthi, A.: New perspectives on central amygdala function. Curr. Opin. Neurobiol. **49**, 141–147 (2018)

11. Funahashi, S., Andreau, J.M.: Prefrontal cortex and neural mechanisms of executive function. Journal of Physiology-Paris **107**, 471–482 (2013)
12. Liang, H., Wang, H.: Top-down anticipatory control in prefrontal cortex. Theor. Biosci. **122**, 70–86 (2003)
13. Ledoux, J.E.: Cognitive-emotional interactions in the brain. Cognit. Emot. **3**, 267–289 (1989)
14. Kale, E.H., Üstün, S., Çiçek, M.: Amygdala–prefrontal cortex connectivity increased during face discrimination but not time perception. Eur. J. Neurosci. **50**, 3873–3888 (2019)
15. Burgos-Robles, A., Kimchi, E., Izadmehr, E.M., et al.: Amygdala inputs to prefrontal cortex guide behavior amid conflicting cues of reward and punishment. Nat. Neurosci. **20**, 824–835 (2017)
16. Fermin, A., Sakagami, M., Kiyonari, T., Li, Y., Matsumoto, Y., Yamagishi, T.: Representation of economic preferences in the structure and function of the amygdala and prefrontal cortex. Sci. Rep. **6**, 20982 (2016)
17. Sato, W., Kochiyama, T., Uono, S., Yoshikawa, S., Toichi, M.: Direction of amygdala–neocortex interaction during dynamic facial expression processing. Cereb. Cortex **27**, 1878–1890 (2017)
18. Oliva, V., Cartoni, E., Latagliata, E.C., Puglisi-Allegra, S., Baldassarre, G.: Interplay of prefrontal cortex and amygdala during extinction of drug seeking. Brain Struct. Funct. **223**, 1071–1089 (2018)
19. Cherniak, C., Rodriguez-Esteban, R.: Information processing limits on generating neuroanatomy: global optimization of rat olfactory cortex and amygdala. J. Biol. Phys. **36**, 45–52 (2010)
20. Busemeyer, J.R., Wang, Z., Townsend, J.T.: Quantum dynamics of human decision making. J. Math. Psychol. **50**, 220–241 (2006)
21. Pothos, E.M., Busemeyer, J.R.: A quantum probability explanation for violation of rational decision theory. Proc. Roy. Soc. B **276**, 2171–2178 (2009)
22. Cheon, T., Takahashi, T.: Interference and inequality in quantum decision theory. Phys. Lett. A **375**, 100–104 (2010)
23. Bagarello, F.: A quantum-like view to a generalized two players game. Int. J. Theor. Phys. **54**, 3612–3627 (2015)
24. Bagarello, F.: Quantum concepts in the social, ecological and biological sciences. Cambridge University Press (2019)
25. Alatas, H., Iskandar, A.A., Tjia, M.-O.: Tailoring spatial soliton characteristics and its dynamical behaviors in nonlinear reflection gratings. Journal of the Optical Society of America B: Optical Physics **27**, 238–245 (2010)
26. Rajan, M.S.M., Bhuvaneshwari, B.V.: Controllable soliton interaction in three mode nonlinear optical fiber. Optik **175**, 39–48 (2018)
27. Yang, C., Liu, W., Zhou, Q., Mihalache, D., Malomed, B.: One-soliton shaping and two-soliton interaction in the fifth-order variable-coefficient nonlinear Schrödinger equation. Nonlinear Dynamics **95**, 369–380 (2019)
28. Villringer, A., Chance, B.: Non-invasive optical spectroscopy and imaging of human brain function. Trends Neurosci. **20**, 435–442 (1997)
29. Adachi, K., Fujita, S., Yoshida, A., Sakagami, H., Koshikawa, N., Kobayashi, M.: Anatomical and electrophysiological mechanisms for asymmetrical excitatory propagation in the rat insular cortex: in vivo optical imaging and whole-cell patch-clamp studies. J. Comp. Neurol. **521**, 1598–1613 (2013)