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Fractal-Fractional Mathematical Model Addressing the Situation of Corona Virus in Pakistan

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A B S T R A C T

This work is the consideration of a fractal fractional mathematical model on the transmission and control of corona virus (COVID-19), in which the total population of an infected area is divided into susceptible, infected and recovered classes. We consider a fractal-fractional order *SIR* type model for investigation of Covid-19. To realize the transmission and control of corona virus in a much better way, first we study the stability of the corresponding deterministic model using next generation matrix along with basic reproduction number. After this, we study the qualitative analysis using "fixed point theory" approach. Next, we use fractional Adams-Bashforth approach for investigation of approximate solution to the considered model. At the end numerical simulation are been given by matlab to provide the validity of mathematical system having the arbitrary order and fractal dimension.

1. Introduction

Our discussion is about covid-19 which was started firstly from Chines city Wuhan, transmitted throughout the globe very rapidly. This disease of COVID-19 named after the attack of corona virus in Chines city Wuhan at the end of 2019. Due to this disease more than 0.616 million individuals in initial eight months have been died. The pandemic of a terrible and much more spreading virus of recent time is of covid-19 and this is tested in the "Wuhan (Chinese city)" on 31st of December, 2019 [1,2]. This outbreak has affected an about 13.5 millions all over the globe. The discovery of Crona virus was done in "1965", as "Tyrrell" and "Bynoe" have find and passes a virus called "B814" [3], which is situated in human beings "embryonic tracheal organ" grows through respiratory system organs of an aged one [4]. Such kind of bacteria transmits in air through social gathering of infected people to healthy ones by droplets of coughing or sneezing. It is also spreading through keeping hands or fingers on the area or surface of different things touched by the infected ones, which is then transmitted to healthy people by touching nose, mouth and eye. This will affects "respiratory system" and the transmitted peoples will symptoms of high fever, coughing and breathing problem . The infection and the onset of symptoms ranges from one to fourteen days. Infectious person shows symptoms within five to six days. To overcome the spreading of such kind of disease peoples must follow hand washing after every 20 minutes, taking masks, and isolate from gathering in different areas.

Scientists and politicians are trying to stabilize the aforesaid infection from transmission and spreading. The reason of transmission of such kind of pandemic is the traveling of affected persons from one area to another which infect much more community of peoples of different areas and spread the disease. For this various steps on national and international level have been taken so far, as different countries of the globe have stopped traveling and journeys of aeroplanes, trains, busses for fixed time and also closed different economic and business activities in cities for applying some careful ways to minimize large number of loss

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Table 1
Description and numerical values of the parameters.

Parameters	Description	value
S_0	Initial susceptible class	220 millions [68]
I_0	Initial infected class	0.142 million [68]
R_0	Initial value of recovered class	0.0125 million [68]
a	Natural birth rate rate	0.00009
b	Transmission rate	0.001664, 0.001663, 0.0016628
c	Contact rate	0.49
μ	Natural death rate	0.019 [68]
λ	death rate due to virous	0.00134
k	recovery rate	0.001

of lives of peoples. Further each and every government of the world try to minimize gurnets of peoples and to decrease number of infection ones in their government territory [5].

As scholars and analyst are making different experiments and analysis to find cure or vaccine for the afore mentioned pandemic to control and stabilized it. Knowing the transmission of a disease has vital work in stabilizing the pandemic in a community. Accepting of a proper thinking about the disease spreading is also another important task for implementing. Engineering in medical sides aware the people and pointed out the importance of modeling approach of mathematics, which is one of the important formulation for handling and understanding such kind of pandemic. Mathematical formulation like modeling have been applied for various infections in past [6–9]. Mathematical models have a lot of property and aspect to give information to the researchers and scholars of physical and medical sciences about how to control such type of pandemic or epidemic. These models can also be applied for prediction of the expected patients in the incoming days by any controlling policy and to obtained their aims and objectives. Basic research is made by scholars and scientists to formulate viral diseases and was applied by politicians to minimize such outbreak, (for detail see [10–15]). Therefore, the afore mentioned diseases has been analyzed in many journals [16,1,17–24]).

The models of mathematics formulation are generally ordinary (ODEs) or partial differential equations (PDEs), saturated with equations of integrations of natural orders (IDEs). Since the 1990, the arbitrary order (ODEs) and (PDEs) can be applied to model real problems with much better results having accurate result. Next, uses of such type of equations will be available in various fields of physics and medicine, engineering, economical problems, business and in analysis of various diseases. Fractional calculus is the vide range of arbitrary order differential and integral calculus. The scope of applying *FDEs* in formation of ODEs and PDEs of real global issues is because of its well known properties of heredity which are not found in integer order ODEs and PDEs. Inspire of IDEs, which are localized for global problems, the FDEs are delocalized and have the past study of history effects, which is the reason of their superiority then IDEs. Another factor is, in different conditions the coming state of the mathematical formulation not only effected by the recent state but also on the past[25,26,4,27]. These properties make FDEs to model the real world problems having "non-Markovian behavior". Next, the integer order differential equations (IDEs) are not able to give it behavior between any two natural order numbers. Different types of fractal dimensions and arbitrary-order derivatives were presented in Books to solve such limit of natural-order derivatives. Such type of derivatives can be applied to different areas of physical and natural sciences. The most suitable field of applied research in present era is devotion to analysis of epidemiological formulation of infectious pandemic. More analysis about the models of mathematical formulation are developed to discuss predictions by simulation, "stability theory", "existence results" and "optimization", see [28–32].

Because of the recent conditions, many analysis have been done on modeling of terrible pandemic of "COVID-19", see [33–35,5]. In present this field of mathematical formulation for the "COVID-19" infected diseases is an interested field of research. Because of such importance in [36] scientists analyzed the mathematical model of three individuals, namely "healthy or susceptible population" $\mathbb{S}(t)$, the "infected population" $\mathbb{I}(\mathbb{t})$ and the "recovered class" $\mathbb{R}(\mathbb{t})$ at time t as

$$\begin{cases} \dot{\mathbb{S}}(t) = a - \mu\mathbb{S} - bc\mathbb{S}\mathbb{I}, \\ \dot{\mathbb{I}}(\mathbb{t}) = \mathbb{I}(\mathbb{b}\mathbb{c}\mathbb{S} - \mu - \mathbb{k} - \lambda), \\ \dot{\mathbb{R}}(\mathbb{t}) = \mathbb{k}\mathbb{I} - \mu\mathbb{R}, \\ \mathbb{S}(0) = \mathbb{S}_0 \geqslant 0, \quad \mathbb{I}(0) = \mathbb{I}_0 \geqslant 0, \quad \mathbb{R}(0) = \mathbb{R}_0 \geqslant 0, \end{cases} \tag{1}$$

having the rate of new born and migrated individuals is denoted by a, transmission rate from susceptible to infected is denoted by b, contact rate of susceptible with infected by c, μ is naturally death rate or without infection, k is the recovery rate while λ is the death rate of infected class from aforesaid virus.

We are going to study the model given in (1) by including recovered individuals equation for fractal-fractional order derivative with $0 < \omega \leqslant 1$ and $0 < r \leq 1$ as given by

$$\begin{cases} {}^{\mathrm{ABC}}\mathrm{D}_t^{\omega,r}\mathbb{S}(t) = a - \mu\mathbb{S} - bc\mathbb{S}\mathbb{I}, \\ {}^{\mathrm{ABC}}\mathrm{D}_{\mathbb{t}}^{\omega,r}\mathbb{I}(\mathbb{t}) = \mathbb{I}(\mathbb{b}\mathbb{c}\mathbb{S} - \mu - \mathbb{k} - \lambda), \\ {}^{\mathrm{ABC}}\mathrm{D}_{\mathbb{t}}^{\omega,r}\mathbb{R}(\mathbb{t}) = \mathbb{k}\mathbb{I} - \mu\mathbb{R}, \\ \mathfrak{S}(0) = \mathfrak{S}_0 \geqslant 0, \quad \mathbb{I}(0) = \mathbb{I}_0 \geqslant 0, \quad \mathbb{R}(0) = \mathbb{R}_0 \geqslant 0, \quad 0 < \omega, \mathbb{r} \leqslant 1. \end{cases} \tag{2}$$

The Transfer diagram for (2) is given in Fig. 1 which shows the interaction among the compartments and various rates.

For the last few decades, it is noted that arbitrary-order equations of differentiations (FDEs) and integrations (FIDEs) can be use for modeling real world problems by much better way than integer order ODEs, PDEs and IDEs. In the 1750s when "Reimann and Liouvilli", "Euler and Fourier" give interesting analytical results in integer order of differential and integral calculus. Due to their work the field of fractal-fractional calculus was also introduced and some best analysis has been done later on. Because of their much more uses of non-integer differential and integral calculus in the field of formulation, in which much more hereditary ideas and memorizing ways cannot be cleared by old or integer order calculus. Due to non-integer order calculus much more error has been reduced present in integer order derivative or anti-derivative. The useful uses of the aforesaid calculus may be seen in [4,25–27,37–42]. Due to these uses scholars and doctors have given more valuable time in studding of arbitrary order calculus. Surely non-integer order derivative is antiderivative of definite type which means the summation of the entire function or spectrum which make it generalized and globalized. As compared to integer order derivative which is a special derivative of the non-integer order. Investigation of various mathematical models for existence and uniqueness, approximation and maximization or minimization, beneficial efforts have been done by scholars, see as [43–49]. This is also notable that arbitrary-order operators of differentiation have been formulated by large number of ways. Definite integration has no kernel of regular type, so, different types of "kernel" are in different lemmas. One such type of formula having currently gained more interest is of "ABC" non-integer derivative defined by "Attangana-Baleanu" and "Caputo" [50] in 2016. This arbitrary order derivative changed the

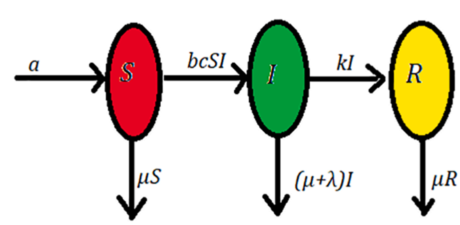

Fig. 1. Dynamical behavior of all the three compartments for the fractal-fractional model (2).

"singular kernel" by "non-singular kernel" and because of this, it is studied on high level [51–57]. Now the question how to solve these problems. In this regards plenty of methods available in literature which has been applied to the old definitions of fractional derivative. For instance, to handle nonlinear problems analytically, famous decomposition and homotopy methods were increasingly used (see [9,58,59]). For numerical purpose in simulation usually Runge Kutta methods were used in large number for dealing of mathematical modeling. Here for numerical simulation we will use fractional *AB* method for numerical simulation. The mentioned method is simple two step technique and more powerful than Euler's, Taylor's and RK methods. The concerned method is powerful as well as rapidly converging and stable, (for detail see [60,61]).

2. Basic Definitions

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Definition 2.1. [33,54,55,62] Let us take the continuous and differentiable mapping $\mho(t)$ in (a,b) with $0<r\leq 1$ order, then the fractal-arbitrary order derivative of $\mho(t)$ in ABC form with fractional order $0<\omega\leq 1$ and the law of power is given as

$$^{\mathrm{ABC}}\mathbb{D}^{\omega,r}(\mho(t))=\frac{\mathrm{ABC}(\omega)}{1-\omega}\frac{d}{dz^r}\int_0^t\mho(z)\kappa_\omega[\frac{-\sigma}{1-\omega}(t-z\omega)^\omega]dz.,$$

We find that if we replace $\kappa_\omega\left[\frac{-\omega}{1-\omega}(\mathbb{t}-\mathbb{z})^\omega\right]$ by $\kappa_1=\exp\left[\frac{-\omega}{1-\omega}(\mathbb{t}-\mathbb{z})\right]$, we will than get the type of derivative known as "Caputo-Fabrizo differential operator". Next it is written that

$$^{\mathrm{ABC}}\mathbb{D}^{\omega,r}[Constant]=0.$$

In this result ABC(ω) is known as "normalization mapping" which is given as ABC(0) = ABC(1) = 1. κ_ω is the well known mapping called "Mittag-Leffler" which is also known as general case of the exponential mapping [37–39].

Definition 2.2. Let us take the continuous and differentiable mapping $\mho(t)$ in (a,b) with $0<r\leq 1$ dimensional order, then the fractal-arbitrary order integral of $\mho(t)$ in ABC form along with arbitrary order $0<\omega\leq 1$ and the law of power is given by":

$$^{\mathrm{ABC}}\mathbb{I}_0^\omega(\mho(t))=\frac{1-\omega}{\mathrm{ABC}(\omega)}t^{r-1}\mho(t)+\frac{r\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\int_0^t(t-z)^{\omega-1}z^{r-1}\mho(z)dz. \tag{3}$$

Lemma 2.1. [63] The solution of the given problem for $0<\omega,r\leq 1$

$$\begin{aligned}^{\mathrm{ABC}}\mathbb{D}_0^\omega\mho(t)&=rt^{r-1}\mathbb{Y}(t,\mho(t)),\ t\in[0,T],\\ \mho(0)&=\mho_0,\quad 0<\omega,r\leq 1,\end{aligned}$$

is provided by

$$\mho(t)=\mho_0+\frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}\mathbb{Y}(t,\mho(t))+\frac{r\omega}{\Gamma(\omega)\mathrm{ABC}(\omega)}\int_0^t(t-z)^{\omega-1}z^{r-1}\mathbb{Y}(z,\mho(z))dz.$$

Note: For finding existence and uniqueness, we take "Banach space"

$$\mathbb{Z}=\mathbb{Y}=\mathbb{F}([0,\mathrm{T}]\times\mathbb{R}^3,\mathbb{R})$$

, where $\mathbb{Y}=F[0,\mathrm{T}]$ having the norm in the space is

$$\|W\|=\|\mho\|=\max_{t\in[0,T]}[|\mathfrak{S}(t)|+|\mathfrak{I}(t)|+|\mathfrak{R}(t)|]$$

.

Here we present a theorem on fixed point which will be utilize to prove our next results.

Theorem 2.1. [64–67] **statement:** Let $\mathbf{A}$ be a subset convexed in space $\mathbf{Z}$along with assumption that $\mathbf{F}_1$ and $\mathbf{F}_2$ are the operators with

1. $\mathbf{F}_1(w)+\mathbf{F}_2(w)\in\mathbf{A}$ for every $w\in\mathbf{A}$;
2. $\mathbf{F}_1$ will satisfy the conditions of contraction;
3. $\mathbf{F}_2$ will satisfy the conditions of continuity and compactness.

Then the operators or functional equations $\mathbf{F}_1w+\mathbf{F}_2w=w$ has one or more than one solution.

3. Feasibility and Stability

.

Lemma 3.1. The roots or zeros of (2) in the feasible region have bounds, as

$$\mathbb{T}=\left\{\left(\$,\mathbb{I},\mathbb{R}\right)\in\mathbb{R}_+^3:\ 0\leqslant\mathbb{N}(\mathbb{t})\leqslant\frac{\mathbb{a}}{\mu}\right\}.$$

Proof. By adding all equations of (2), we have

$$\begin{aligned}\frac{d\mathbb{N}}{dt}=&\ a-\mu\$-bc\$\mathbb{I}+\mathbb{I}(\mathbb{bc}\$-\mu-\mathbb{k}-\lambda)\\ &+\mathbb{kI}-\mu\mathbb{R},\\ =&\ \mathbb{a}-\mu(\$(\mathbb{t})+\mathbb{I}(\mathbb{t})+\mathbb{R})-\lambda\mathbb{I}(\mathbb{t}),\\ \leqslant&\ \mathbb{a}-\mu(\$(\mathbb{t})+\mathbb{I}(\mathbb{t})+\mathbb{R}(\mathbb{t})),\\ \leqslant&\ \mathbb{a}-\mu(\mathbb{N}(\mathbb{t})),\\ \frac{\mathbb{dN}}{\mathbb{dt}}+\mu\mathbb{N}\leqslant&\ \mathbb{a}.\end{aligned} \tag{4}$$

Solving (4), we have

$$\mathbb{N}(\mathbb{t})\leqslant\frac{\mathbb{a}}{\mu}+\mathbb{C}\exp(-\mu\mathbb{t}), \tag{5}$$

if $\mathtt{t}\to\infty$, $\mathbb{N}(\mathtt{t})\leqslant\frac{\mathbb{a}}{\mu}$, the last result proved our required result. Next we to prove some basics results about stability analysis, for this we have to compute free equilibrium point and pandemic equilibrium point of (2) as

$$\begin{aligned}^{\mathrm{ABC}}\mathbb{D}_t^{\omega,r}\$(t)=&\ 0,\\ ^{\mathrm{ABC}}\mathbb{D}_t^{\omega,r}\mathbb{I}(t)=&\ 0,\\ ^{\mathrm{ABC}}\mathbb{D}_t^{\omega,r}\mathbb{R}(t)=&\ 0.\end{aligned}$$

As earlier mentioned that We will compute two equilibria points which are given as: $\mathbb{E}_0=\left(\frac{\mathbb{a}}{\mu},\ 0,\ 0\right)$ is the pandemic free equilibrium point of (2) and the pandemic is $\mathbb{E}^*=(\$^*,\ \mathbb{I}^*,\ \mathbb{R}^*)$, and

$$\$^*=\frac{\mu+k+\lambda}{bc},$$

$$\mathbb{I}^*=\frac{\mathbb{abc}-\mu(\mu+\mathbb{k}+\lambda)}{\mathbb{bc}(\mu+\mathbb{k}+\lambda)}$$

and

$$\mathbb{R}^*=\frac{\mathbb{abck}-\mu\mathbb{k}(\mu+\mathbb{k}+\lambda)}{\mu\mathbb{bc}(\mu+\mathbb{k}+\lambda)}$$

Theorem 3.1. The basic reproductory number for (2) is computed as

$$R_0=\frac{bca}{\mu(\mu+k+\lambda)}$$

Proof. Let we to prove the reproduction number by taking 2^{nd} equation of (2) as $\mathbb{X} = \mathbb{I}$,

$$\begin{aligned} {}^{ABC}\mathrm{D}_t^{\omega,r}(\mathbb{X}) &= {}^{ABC}\mathrm{D}_{\mathbb{t}}^{\omega,r}(\mathbb{I}) = \mathbb{bcI\$} - \mathbb{I}(\mu + \mathbb{k} + \lambda), \\ {}^{ABC}\mathrm{D}_{\mathbb{t}}^{\omega,r}(\mathbb{X}) &= \mathbb{F} - \mathbb{V}, \end{aligned}$$

where $\mathbb{F} = \mathbb{bcI\$}$, $\mathbb{V} = \mathbb{I}(\mu + \mathbb{k} + \lambda)$, $\mathbb{F}$ is the infected term of non-linearity and $\mathbb{V}$ term of linearity. Further, the next generation matrix is $\mathbb{FV}^{-1}$, and

$$\mathbb{F} = \left[\frac{\partial}{\partial \mathbb{I}}(\mathbb{bcI\$})\right] = [\mathbb{bc\$}],$$

and

$$\mathbb{V} = \left[\frac{\partial}{\partial \mathbb{I}}(\mathbb{I}(\mu + \mathbb{k} + \lambda))\right] = [(\mu + \mathbb{k} + \lambda)], \quad \mathbb{V}^{-1} = \left[\frac{1}{\mu + \mathbb{k} + \lambda}\right],$$

then

$$\mathbb{FV}^{-1} = \left[\frac{\mathbb{bc\$}}{(\mu + \mathbb{k} + \lambda)}\right].$$

So R_0 is the greater eigen value of our considered matrix $\mathbb{FV}^{-1}$ at pandemic free equilibrium point $\mathbb{E}_0 = (\frac{\mathbb{a}}{\mu}, 0, 0)$, given as follows

$$\rho(\mathbb{FV}^{-1})_{\mathbb{E}_0} = \left[\frac{\mathbb{bca}}{\mu(\mu + \mathbb{k} + \lambda)}\right]. \tag{6}$$

Hence basic reproduction number is proved and is given by

$$R_0 = \frac{bca}{\mu(\mu + k + \lambda)}.$$

The last result shows the the required result.

Theorem 3.2. **Statement**The pandemic free of disease equilibrium point of (2) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof. Let matrix of Jacobian of (2) will be written as

$$\mathscr{J} = \begin{bmatrix} \frac{\partial}{\partial \$}(\phi_1(t, \$(t), \mathbb{I}(\mathbb{t}), \mathbb{R}(\mathbb{t}))) & \frac{\partial}{\partial \mathbb{I}}(\phi_1(t, \$(t), \mathbb{I}(\mathbb{t}), \mathbb{R}(\mathbb{t}))) & \frac{\partial}{\partial \mathbb{R}}(\phi_1(t, \$(t), \mathbb{I}(\mathbb{t}), \mathbb{R}(\mathbb{t}))) \\ \frac{\partial}{\partial \$}(\phi_2(t, \$(t), \mathbb{I}(\mathbb{t}), \mathbb{R}(\mathbb{t}))) & \frac{\partial}{\partial \mathbb{I}}(\phi_2(t, \$(t), \mathbb{I}(\mathbb{t}), \mathbb{R}(\mathbb{t}))) & \frac{\partial}{\partial \mathbb{R}}(\phi_2(t, \$(t), \mathbb{I}(\mathbb{t}), \mathbb{R}(\mathbb{t}))) \\ \frac{\partial}{\partial \$}(\phi_3(t, \$(t), \mathbb{I}(\mathbb{t}), \mathbb{R}(\mathbb{t}))) & \frac{\partial}{\partial \mathbb{I}}(\phi_3(t, \$(t), \mathbb{I}(\mathbb{t}), \mathbb{R}(\mathbb{t}))) & \frac{\partial}{\partial \mathbb{R}}(\phi_3(t, \$(t), \mathbb{I}(\mathbb{t}), \mathbb{R}(\mathbb{t}))) \end{bmatrix},$$

or

$$\mathscr{J} = \begin{bmatrix} -\mu - bc\mathbb{I} & -bc\$ & 0 \\ bc\mathbb{I} & bc\$ - (\mu + k + \lambda) & 0 \\ 0 & k & -\mu \end{bmatrix}. \tag{7}$$

Using the values of E_0, we get

$$\mathscr{J} = \begin{bmatrix} -\mu & \frac{-bca}{\mu} & 0 \\ 0 & \frac{bca}{\mu} - (\mu + k + \lambda) & 0 \\ 0 & k & -\mu \end{bmatrix}.$$

Now the characteristics equation can be find as

$$Det(\mathscr{J} - \Lambda I) = \begin{vmatrix} -\mu - \Lambda & \frac{-bca}{\mu} & 0 \\ 0 & \frac{bca}{\mu} - (\mu + k + \lambda) - \Lambda & 0 \\ 0 & k & -\mu - \Lambda \end{vmatrix} = 0.$$

Thus the eigen values are given by

$$\begin{aligned} \Lambda_1 &= -\mu, \\ \Lambda_2 &= \frac{bca}{\mu} - (\mu + k + \lambda), \\ \Lambda_3 &= -\mu \end{aligned}$$

Further, Λ_2 can be written as

$$\Lambda_2 = \frac{bca}{\mu(\mu + k + \lambda)} - 1.$$

Last result shows that

"$\Lambda_2 = R_0 - 1$"

and Λ_2 will be non-positive if "$R_0 < 1$". So all "eigen values" are non-positive , So (2) is locally asymptotically stable" at $\mathbb{E}_0$, and will be unstable otherwise.

Theorem 3.3. **Statement**The pandemic or after infection the equilibrium point $E^* = (\mathfrak{S}^*, \mathfrak{I}^*, \mathfrak{R}^*)$ is locally asymptotically stable if $R_0 > 1$ and globally asymptotically stable if the minors of Routh-Hurwitz matrix are positive.

Proof. Putting the values of $E^* = (\$^*, \mathbb{I}^*, \mathbb{R}^*)$ in (7), we get

$$\mathscr{J} = \begin{bmatrix} -\mu - bc\mathbb{I}^* & -bc\$^* & 0 \\ bc\mathbb{I}^* & bc\$^* - (\mu + k + \lambda) & 0 \\ 0 & k & -\mu \end{bmatrix}. \tag{8}$$

After simplification we get

$$\mathscr{J} = \begin{bmatrix} -\mu - (\frac{abc}{\mu + k + \lambda} - \mu) & -(\mu + k + \lambda) & 0 \\ \frac{abc}{\mu + k + \lambda} - \mu & 0 & 0 \\ 0 & k & -\mu \end{bmatrix}.$$

or

$$\mathscr{J} = \begin{bmatrix} -\frac{abc}{\mu + k + \lambda} & -(\mu + k + \lambda) & 0 \\ \frac{abc}{\mu + k + \lambda} - \mu & 0 & 0 \\ 0 & k & -\mu \end{bmatrix}.$$

The characteristics equation becomes

$$Det(\mathscr{J}-\Lambda I)=\begin{vmatrix} -\dfrac{abc}{\mu+k+\lambda}-\Lambda & -(\mu+k+\lambda) & 0 \\ \dfrac{abc}{\mu+k+\lambda}-\mu & -\Lambda & 0 \\ 0 & k & -\mu-\Lambda \end{vmatrix}=0,$$

or

$$\Lambda^3+(-\mu+\frac{abc}{\mu+k+\lambda})\Lambda^2+(\frac{\mu abc}{\mu+k+\lambda}+abc-\mu(\mu+k+\lambda))\Lambda+\mu(abc)$$
$$-\mu(\mu+k+\lambda)=0,$$

or

$$a_0\Lambda^3+(a_1)\Lambda^2+(a_2)\Lambda+a_3=0,$$

Making Hurwitz matrix, as follows

$$\begin{bmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{bmatrix}.$$

On applying Routh-Hurwitz criteria, all the principle minors be positive than as given below

$$|a_1|>0,$$

this implies that $a_1=-\mu+\frac{abc}{\mu+k+\lambda}$ or $a_1=-1+R_0$ or $a_1>0$ if $R_0>1$. By similar way one can show that the following minors must also be positive.

$$\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}>0.$$

and

$$\begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix}>0.$$

By $R_0>1$ and positivity of all minors achieved the local asymptotical and global stability for the considered system.

4. Existence and uniqueness of model (2)

It is of great importance to ask weather a dynamical problem we investigate exist really or not. This is the basic question and will answered by the theory of fixed points. Here we analyze the concerned need for our considered problem (2) in this part of the paper. Regarding to the aforesaid need as the integral is differentiable, we can write the right sides of model (2) as

$$\begin{cases} {}^{ABC}\mathbb{D}^{\omega}(\$(t))=rt^{r-1}\mathbf{G}_1(\$(t),\mathbb{I}(\mathbb{t}),\mathbb{R}(\mathbb{t})), \\ {}^{ABC}\mathbb{D}^{\omega}(\mathbb{I}(\mathbb{t}))=\mathbb{r}\mathbb{t}^{\mathbb{r}-1}\mathbf{G}_2(\$(\mathbb{t}),\mathbb{I}(\mathbb{t}),\mathbb{R}(\mathbb{t})), \\ {}^{ABC}\mathbb{D}^{\omega}(\mathbb{R}(\mathbb{t}))=\mathbb{r}\mathbb{t}^{\mathbb{r}-1}\mathbf{G}_3(\$(\mathbb{t}),\mathbb{I}(\mathbb{t}),\mathbb{R}(\mathbb{t})), \\ \$(0)=\$_0,\quad \mathbb{I}(0)=\mathbb{I}_0,\quad \mathbb{R}(0)=\mathbb{R}_0. \end{cases} \tag{9}$$

where

$$\begin{cases} \mathbf{G}_1(\$(t),\mathbb{I}(\mathbb{t}),\mathbb{R}(\mathbb{t}))=\mathbb{a}-\mu\$-\mathbb{bc}\$\mathbb{I}, \\ \mathbf{G}_2(\$(\mathbb{t}),\mathbb{I}(\mathbb{t}),\mathbb{R}(\mathbb{t}))=\mathbb{I}(\mathbb{bc}\$-\mu-\mathbb{k}-\lambda), \\ \mathbf{G}_3(\$(\mathbb{t}),\mathbb{I}(\mathbb{t}),\mathbb{R}(\mathbb{t}))=\mathbb{k}\mathbb{I}-\mu\mathbb{R}. \end{cases} \tag{10}$$

With the help of (9) and for $t\in\mathbb{g}$, the (10) follows as

$$\begin{aligned} {}^{ABC}\mathbb{D}_0^{\omega}\mho(t)= &\ rt^{r-1}\mathbb{Y}(t,\mho(t)),\ t\in[0,T], \\ \mho(0)= &\ \mho_0,\quad 0<\omega,r\le 1, \end{aligned} \tag{11}$$

with solution

$$\begin{aligned} \mho(t)=&\mho_0+\frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}\mathbb{Y}(t,\mho(t)) \\ &+\frac{r\omega}{\Gamma(\omega)\mathrm{ABC}(\omega)}\int_0^t(t-z)^{\omega-1}z^{r-1}\mathbb{Y}(z,\mho(z))dz, \end{aligned} \tag{12}$$

where

$$\mho(t)=\begin{cases} \$(t) \\ \mathbb{I}(t) \\ \mathbb{R}(t) \end{cases}\quad \mho_0(t)=\begin{cases} \$_0 \\ \mathbb{I}_0 \\ \mathbb{R}_0 \end{cases},\quad \mathbb{Y}(t,\mho(t))=\begin{cases} \mathbf{G}_1(\$(t),\mathbb{I}(t),\mathbb{R}(\mathbb{t})\mathbb{t}) \\ \mathbf{G}_2(\$(\mathbb{t}),\mathbb{I}(\mathbb{t}),\mathbb{R}(\mathbb{t}),\mathbb{t}) \\ \mathbf{G}_3(\$(\mathbb{t}),\mathbb{I}(\mathbb{t}),\mathbb{R}(\mathbb{t}),\mathbb{t}). \end{cases} \tag{13}$$

Now, transform the (2) into the fixed point problem. Define mapping $T:V\to V$ given as:

$$T\mho(t)=\mho_0+\frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}\mathbb{Y}(t,\mho(t))+\frac{r\omega}{\Gamma(\omega)\mathrm{ABC}(\omega)}\int_0^t(t-z)^{\omega-1}z^{r-1}\mathbb{Y}(z,\mho(z))dz. \tag{14}$$

Assume

$$T=F+G,$$

where

$$\begin{aligned} F(\mho)= &\ \mho_0(t)+\frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}[\mathbb{Y}(t,\mho(t))], \\ G(\mathbb{Z})= &\ \frac{r\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\int_0^t(t-z)^{\omega-1}z^{r-1}\mathbb{Y}(z,\mho(z))dz. \end{aligned} \tag{15}$$

take growth cognition and Lipschitzian assumption for existence and uniqueness as:

(C1) There will be a constants $\mathscr{L}_{\mathbb{Y}},\mathscr{M}_{\mathbb{Y}}$, such that

$$|\mathbb{Y}(t,\mho(t))|\leqslant\mathscr{L}_{\mathbb{Y}}|\mho|+\mathscr{M}_{\mathbb{Y}}.$$

(C2) There exists constants $\mathbb{L}_{\mathbb{Y}}>0$ such that for each $\mho,\ \bar{\mathbb{f}}\in\mho$ such that

$$|\mathbb{Y}(t,\mho)-\mathbb{Y}(t,\bar{\mathbb{f}})|\leqslant\mathbb{L}_{\mathbb{Y}}[|\mho|-\bar{\mathbb{f}}|];$$

Theorem 4.1. "Applying hypothesis $(C1,C2)$, the Integral equation (12) has at least one solution which consequently means that the considered system (2) has the same number of solution if $\frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}\mathbb{L}_{\mathbb{Y}}<1$".

Proof. We prove the theorem in two step as bellow:**Step I**: Let $\bar{\mathbb{f}}\in\mathbf{A}$, where $\mathbf{A}=\{\mho\in\mho:\ \|\mho\|\leqslant\phi,\phi>0\}$ is closed convex set. Then using the definition of F in (15), one has

$$\begin{aligned} \|F(\mho)-F(\bar{\mathbb{f}})\|= &\ \frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}\max_{t\in[0,\tau]}|\mathbb{Y}(t,\mho(t))-\mathbb{Y}(t,\bar{\mathbb{f}}(t))| \\ \leqslant &\ \frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}\mathbb{L}_{\mathbb{Y}}\|\mho-\bar{\mathbb{f}}\|. \end{aligned} \tag{16}$$

Hence F will obey the property of contraction.**Step-II**: To prove that G is relatively compact, we have to prove that G is bounded, and equicontinuous. As G is continuous, $\mathbb{Y}$ is also continuous and for any $\mho\in\mathbf{A}$, we have

$$\begin{aligned}\|G(\mho)\| &= \max_{t\in[0,\tau]}\|\frac{r\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\int_0^t(\tau-z)^{\omega-1}z^{r-1}\mathbb{Y}(z,\mho(z))dz|\\ &\leqslant \frac{r\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\int_0^t(s)^{\omega-1}(1-s)^{r-1}|\mathbb{Y}(s,\mho(s))|ds\\ &\leqslant \frac{r[\mathscr{L}_{\mathbb{Y}}|\mho|+\mathscr{M}_{\mathbb{Y}}T^{\omega+r-1}]}{\mathrm{ABC}(s)\Gamma(s)}[\mathbb{B}(\omega,r)].\end{aligned} \tag{17}$$

Hence (17) shows that $\mathbb{G}$ is bounded. Next for "equi-continuity" let $t_1 > t_2 \in [0,\tau]$, we have

$$\begin{aligned}|G(\mho(t_2))-G(\mho(t_1))| &= \frac{r\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}|\int_0^{t_2}(t_2-y)^{\omega-1}y^{r-1}\mathbb{Y}(y,\mho(y))dy-\int_0^{t_1}(t_1-y)^{\omega-1}y^{r-1}\mathbb{Y}(y,\mho(y))dy|\\ &\leqslant \frac{r[\mathscr{L}_{\mathbb{Y}}|\mho|+\mathscr{M}_{\mathbb{Y}}T^{\omega+r-1}]\mathbb{B}(\omega,r)}{\mathrm{ABC}(\omega)\Gamma(\omega)}[t_2^{\omega}-t_1^{\omega}].\end{aligned} \tag{18}$$

Right side in (17) becomes zero at $t_2 \to t_1$. Since G is continues and so

$$|G(\mho(t_2))-G(\mho(t_1))| \to 0, \ \text{as } t_2 \to t_1.$$

Therefore we have as $\mathbb{G}$ is bounded operator and continuous so one has

$$\|G(\mho(t_2))-G(\mho(t_1))\| \to 0, \ \text{as } t_2 \to t_1.$$

So $\mathbb{G}$ is uniformly continuous and bounded. Thus by Arzelá-Ascoli theorem G is relatively compact and hence completely continuous. Thus by Theorem 4.1, the equation (12) has one or more than one solution and therefore, the (2) has one or more than one solution. For uniqueness we give the next result.

Theorem 4.2. Using assumption $(C2)$, (12) has one solution which gives the information that the system (2) has one solution if $\left[\frac{(1-\omega)t^{r-1}\mathbb{L}_{\mathbb{Y}}}{\mathrm{ABC}(\omega)}+\frac{r[\mathbb{L}_{\mathbb{Y}}T^{\omega+r-1}]\mathbb{B}(\omega,r)}{\mathrm{ABC}(\omega)\Gamma(\omega)}\right]<1$".

Proof. Let the operator $T:\mho\to\mho$ defined by

$$\begin{aligned}T\mho(t) = &\ \mho_0(t)+[\mathbb{Y}(t,\mho(t))-\mathbb{Y}_0(t)]\frac{(1-\omega)t^{r-1}}{\mathrm{ABC}(\omega)}\\ &+\frac{r\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\int_0^t(t-y)^{\omega-1}t^{r-1}\mathbb{Y}(y,\mho(y))dy, \ t\in[0,\tau].\end{aligned} \tag{19}$$

As $\mho, \bar{\mathfrak{f}} \in \mho$, so we can take

$$\begin{aligned}\|T\mho-T\bar{\mathfrak{f}}\| \leqslant &\ \frac{(1-\omega)t^{r-1}}{\mathrm{ABC}(\omega)}\max_{t\in[0,\tau]}|\mathbb{Y}(t,\mho(t))-\mathbb{Y}(t,\bar{\mathfrak{f}}(t))|\\ &+\frac{r\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\max_{t\in[0,\tau]}|\int_0^t(t-y)^{\omega-1}t^{r-1}\mathbb{Y}(y,\mho(y))dy\\ &-\int_0^t(t-y)^{\omega-1}t^{r-1}\mathbb{Y}(y,\bar{\mathfrak{f}}(y))dy| \leqslant \Theta\|\mho-\bar{\mathfrak{f}}\|,\end{aligned} \tag{20}$$

and

$$\Theta = \left[\frac{(1-\omega)t^{r-1}\mathbb{L}_{\mathbb{Y}}}{\mathrm{ABC}(\omega)}+\frac{r[\mathbb{L}_{\mathbb{Y}}T^{\omega+r-1}]\mathbb{B}(\omega,r)\mathbb{L}_{\mathbb{Y}}}{\mathrm{ABC}(\omega)\Gamma(\omega)}\right]. \tag{21}$$

Thus T is contraction from (20). So the equation (12) has one solution. Hence (2) has one solution.

5. Ulam-Hyer Stability

Here, we define and give well-known results on stability analysis of (2), we take $\Phi(t)$ as perturbed parameter which depends on the solution having condition of $\Phi(0)=0$ as

- $|\Psi(t)| \leqslant \epsilon$ for $\epsilon > 0$;
- ${}^{\mathrm{ABC}}\mathbb{D}_t^{(}\omega,r)\mho(t) = \mathbb{Y}(t,\mho(t))+\Psi(t)$.

Lemma 5.1. The solution of the perturbed problem

$$\begin{aligned}{}^{\mathrm{ABC}}\mathbb{D}_t^{\omega,r}\mho(t) &= \mathbb{Y}(t,\mho(t))+\Phi(t),\\ \mho(0) &= \mho_0,\end{aligned} \tag{22}$$

satisfies the given relation

$$\begin{aligned}&|\mho(t)-\left(\mho_0(t)+[\mathbb{Y}(t,\mho(t))-\Phi_0(t)]\frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}+\frac{r\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\int_0^t(t-y)^{\omega-1}y^{r-1}\mathbb{Y}(y,\mho(y))dy\right)\Bigg|,\\ &\leqslant \frac{\Gamma(\omega)t^{r-1}+rT^{\omega+r-1}}{\mathrm{ABC}(\omega)\Gamma(\omega)}\mathbb{B}(\omega,r)\varepsilon = \omega_{\omega,r}\varepsilon.\end{aligned} \tag{23}$$

Theorem 5.1. Using assumption $(C2)$ and (23), the solution of the (12) is "Ulam-Hyers" stable and therefore, the analytical results of the system are" Ulam-Hyers" stable if $\Theta < 1$, where Θ is given in (21).

Proof. Take $\mho \in \mho$ be the solution and $\bar{\mathfrak{f}} \in \mho$ be at most solution of (12), then

$$\begin{aligned}
|\mho(t)-\bar{\mathfrak{f}}(t)| = & \ \left|\mho(t)-\left(\mho_0(t)+\left[\mathbb{Y}(t,\bar{\mathfrak{f}}(t))-\mathbb{Y}_0(t)\right]\frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}+\frac{r\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\int_0^t(t-y)^{\omega-1}y^{r-1}\mathbb{Y}(y,\bar{\mathfrak{f}}(y))dy\right)\right|, \\
& \leqslant\left|\mho(t)-\left(\mho_0(t)+[\mathbb{Y}(t,\mho(t))-\mathbb{Y}_0(t)]\frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}+\frac{r\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\int_0^t(t-y)^{\omega-1}y^{r-1}\mathbb{Y}(y,\mho(y))dy\right)\right|, \\
+ & \ \left|\left(\mho_0(t)+[\mathbb{Y}(t,\mho(t))-\mathbb{Y}_0(t)]\frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}+\frac{r\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\int_0^t(t-y)^{\omega-1}y^{r-1}\mathbb{Y}(y,\mho(y))dy\right)\right. \\
- & \ \left.\left(\mho_0(t)+\left[(t,\bar{\mathfrak{f}}(t))-\mathbb{Y}_0(t)\right]\frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}+\frac{r\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\int_0^t(t-y)^{\omega-1}y^{r-1}\mathbb{Y}(y,\bar{\mathfrak{f}}(y))dy\right)\right|, \\
\leqslant & \ \Omega_{\omega,r}+\frac{(1-\omega)\mathbb{L}_{\mathbb{Y}}}{\mathrm{ABC}(\omega)}t^{r-1}\|\mho-\bar{\mathfrak{f}}\|+\frac{rT^{\omega+r-1}\mathbb{L}_{\mathbb{Y}}}{\mathrm{ABC}(\omega)\Gamma(\omega)}\mathbb{B}(\omega,r)\|\mho-\bar{\mathfrak{f}}\|, \\
\leqslant & \ \Omega_{\omega,r}+\Theta\|\mho-\bar{\mathfrak{f}}\|.
\end{aligned} \tag{24}$$

From (24), we can write as

$$\|\mho-\bar{\mathfrak{f}}\|\leqslant\frac{\Omega_{\omega,r}}{1-\Theta}\|\mho-\bar{\mathfrak{f}}\|. \tag{25}$$

Hence the results about the required stability is received.

6. Numerical Solution

In this part of the paper, we are going to find numerical solutions of fractal-arbitrary order model (2) using ABC derivative by fractal-fractional "Adams-Bashforth method". The the approximate solution are obtained by the aforesaid iterative scheme. For such objective, we uses the fractal-fractional *AB* techniques [38] to provide an approximate way for the graphing of the system (2). For this to prove an approximate techniques, we go further with (9) can be noted as :

$$\begin{cases}
{}^{\mathrm{ABC}}\mathrm{D}^{\omega}(\$(t))=rt^{r-1}\mathbf{G}_1(\$(t),\mathbb{I}(\mathbb{t}),\mathbb{R}(\mathbb{t}),\mathbb{t}), \\
{}^{\mathrm{ABC}}\mathrm{D}^{\omega}(\mathbb{I}(\mathbb{t}))=\mathbb{r}\mathbb{t}^{\mathbb{r}-1}\mathbf{G}_2(\$(\mathbb{t}),\mathbb{I}(\mathbb{t}),\mathbb{R}(\mathbb{t}),\mathbb{t}), \\
{}^{\mathrm{ABC}}\mathrm{D}^{\omega}(\mathbb{R}(\mathbb{t}))=\mathbb{r}\mathbb{t}^{\mathbb{r}-1}\mathbf{G}_3(\$(\mathbb{t}),\mathbb{I}(\mathbb{t}),\mathbb{R}(\mathbb{t}),\mathbb{t}), \\
\$(0)=\$_0, \quad \mathbb{I}(0)=\mathbb{I}_0, \quad \mathbb{R}(0)=\mathbb{R}_0.
\end{cases} \tag{26}$$

where $\mathbf{G}_1, \mathbf{G}_2$ and $\mathbf{G}_3$ are defined in (10) By applying antiderivative of fractional order and fractal dimension to the 1^{st} equation of (9) using ABC form, we get

$$\$(t)-\$_(0)=\frac{(1-\omega)}{\mathrm{ABC}(\omega)}t^{r-1}[\mathbf{G}_1(\$(t),\mathbb{I}(\mathbb{t}),\mathbb{R}(\mathbb{t}),\mathbb{t})]+\frac{\mathbb{r}\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}$$

$$\int_0^{\mathbb{t}}(\mathbb{t}-\mathbb{y})^{\omega-1}\mathbb{y}^{\mathbb{r}-1}\mathbf{G}_1(\$(\mathbb{y}),\mathbb{I}(\mathbb{y}),\mathbb{y}(\mathbb{t}),\mathbb{y})d\mathbb{y}.$$

Set $t=t_{n+1}$ for $i=0,1,2\cdots,$

$$\begin{aligned}
\$(t_{n+1})-\$_(0)= & \ \frac{(1-\omega)}{\mathrm{ABC}(\omega)}(t_{n+1}^{r-1})[\mathbf{G}_1(\$(t_n),\mathbb{I}(\mathbb{t}_n),\mathbb{R}(\mathbb{t}_n))] \\
+ & \ \frac{\mathbb{r}\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\int_0^{\mathbb{t}_{n+1}}(\mathbb{t}_{n+1}-\mathbb{y})^{\omega-1}\mathbb{y}^{\mathbb{r}-1}\mathbf{G}_1(\$(\mathbb{y}),\mathbb{I}(\mathbb{y}),\mathbb{R}(\mathbb{y}))d\mathbb{y}. \\
= & \ \frac{(1-\omega)}{\mathrm{ABC}(\omega)}(\mathbb{t}_{n+1}^{\mathbb{r}-1})[\mathbf{G}_1(\$(\mathbb{t}_n),\mathbb{I}(\mathbb{t}_n),\mathbb{R}(\mathbb{t}_n))] \\
+ & \ \frac{\mathbb{r}\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\sum_{q=0}^{n}\int_q^{\mathbb{t}_{q+1}}(\mathbb{t}_{n+1}-\mathbb{y})^{\omega-1}\mathbb{y}^{\mathbb{r}-1}\mathbf{G}_1(\$(\mathbb{y}),\mathbb{I}(\mathbb{y}),\mathbb{R}(\mathbb{y}))d\mathbb{y}.
\end{aligned}$$

Now, we approximate the function $\mathbf{G}_1$ on the interval $[t_q,t_{q+1}]$ through the interpolation polynomial as follows

$$\mathbf{G}_1 \cong \frac{\mathbf{G}_1}{\Delta}(t-t_{q-1})-\frac{\mathbf{R}_1}{\Delta}(t-t_q)$$

which implies that

$$\begin{aligned}
\$(t_{n+1})= & \ \$_(0)+\frac{(1-\omega)}{\mathrm{ABC}(\omega)}(t_{n+1}^{r-1})[\mathbf{G}_1(\$(t_n),\mathbb{I}(\mathbb{t}_n),\mathbb{R}(\mathbb{t}_n))] \\
+ & \ \frac{\mathbb{r}\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\sum_{q=0}^{n}\left(\frac{\mathbf{G}_1(\$(\mathbb{t}_n),\mathbb{I}(\mathbb{t}_n),\mathbb{R}(\mathbb{t}_n))}{\Delta}\int_q^{\mathbb{t}_{q+1}}(\mathbb{t}-\mathbb{t}_{q-1})(\mathbb{t}_{q+1}-\mathbb{t})^{\omega-1}\mathbb{t}_q^{\mathbb{r}-1}\right)d\mathbb{t} \\
& -\frac{\mathbf{G}_1(\$(\mathbb{t}_n),\mathbb{I}(\mathbb{t}_n),\mathbb{R}(\mathbb{t}_n))}{\Delta}\int_q^{\mathbb{t}_{q+1}}(\mathbb{t}-\mathbb{t}_q)(\mathbb{t}_{n+1}-\mathbb{t})^{\omega-1}\mathbb{t}_q^{\mathbb{r}-1}d\mathbb{t}\Bigg).
\end{aligned}$$

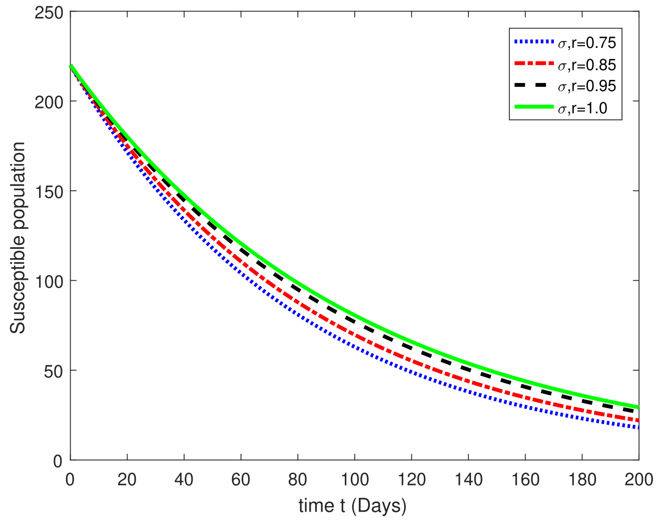

Fig. 2. Dynamics of susceptible population of the fractal-fractional model (2) at various arbitrary order and fractal dimension.

$$
\begin{aligned}
\$(t_{n+1}) = \;& \$_(0) + \frac{(1-\omega)}{\mathrm{ABC}(\omega)}(t_{n+1}^{r-1})[\mathbf{G}_1(\$(t_n), \mathbb{I}(\mathbb{t}_{\mathbb{n}}), \mathbb{R}(\mathbb{t}_{\mathbb{n}}))] \\
+ \;& \frac{\mathbb{r}\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)} \sum_{\mathbb{q}=0}^{\mathbb{n}} \left(\frac{\mathbb{t}_{\mathbb{q}}^{\mathbb{r}-1}\mathbf{G}_1\left(\$(\mathbb{t}_{\mathbb{j}}), \mathbb{I}(\mathbb{t}_{\mathbb{q}}), \mathbb{R}(\mathbb{t}_{\mathbb{q}})\right)}{\Delta} \mathbb{I}_{\mathbb{q}-1,\omega} \right. \\
& \left. - \frac{\mathbb{t}_{\mathbb{q}-1}^{\mathbb{r}-1}\mathbf{G}_1\left(\$(\mathbb{t}_{\mathbb{q}-1}), \mathbb{I}(\mathbb{t}_{\mathbb{q}-1}), \mathbb{R}(\mathbb{t}_{\mathbb{q}-1})\right)}{\Delta} \mathbb{I}_{\mathbb{q},\omega} \right).
\end{aligned}
\tag{27}
$$

Calculating $I_{q-1,\omega}$ and $I_{q,\omega}$ we get

$$
\begin{aligned}
I_{q-1,\omega} = \;& \int_q^{t_{q+1}} (t - t_{q-1})(t_{n+1} - t)^{\omega-1} dt \\
= \;& -\frac{1}{\omega}\left[(t_{q+1} - t_{q-1})(t_{n+1} - t_{q+1})^{\sigma} - (t_q - t_{q-1})(t_{n+1} - t_q)^{\sigma}\right] \\
& -\frac{1}{\omega(\omega-1)}\left[(t_{n+1} - t_{q+1})^{\omega+1} - (t_{n+1} - t_q)^{\omega+1}\right],
\end{aligned}
$$

and

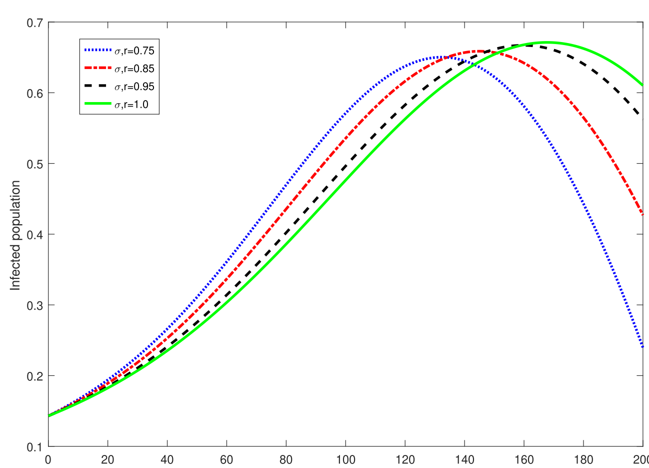

Fig. 3. Dynamics of infected population of the fractal-fractional model (2) at various arbitrary order and fractal dimension.

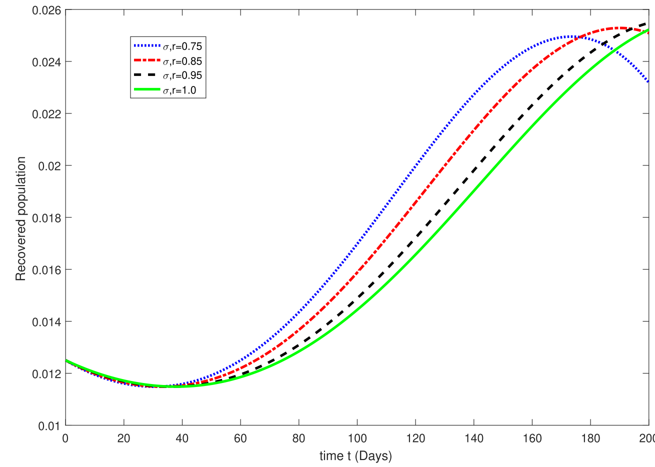

Fig. 4. Dynamics of recovered population of the fractal-fractional model (2) at various arbitrary order and fractal dimension.

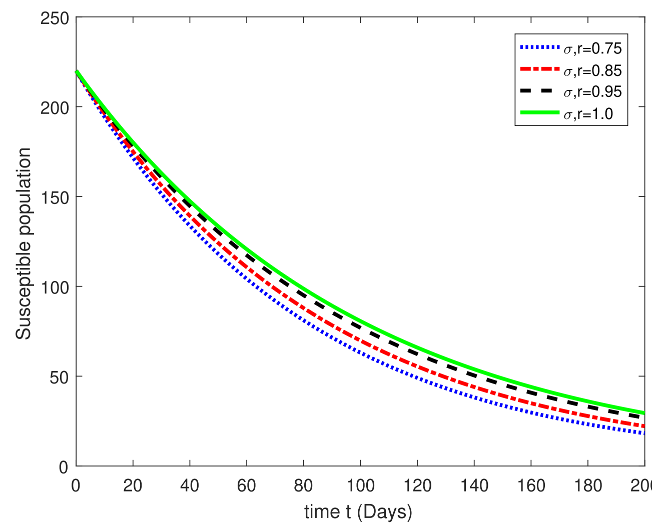

Fig. 5. Dynamics of "susceptible population" of the fractal-fractional model (2) at various arbitrary order and fractal dimension.

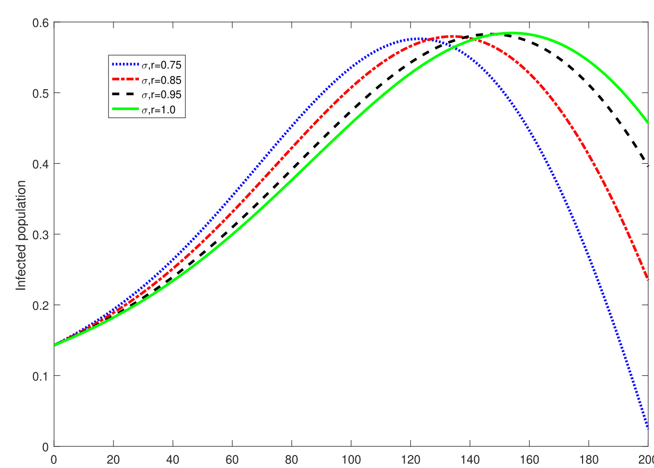

Fig. 6. Dynamics of "Infected population" of the fractal-fractional model (2) at various arbitrary order and fractal dimension.

$$
\begin{aligned}
I_{q,\omega} &= \int_{q}^{t_{q+1}} (t-t_q)(t_{n+1}-t)^{\omega-1}dt \\
&= -\frac{1}{\omega}\left[(t_{q+1}-t_q)(t_{n+1}-t_{q+1})^{\omega}\right] \\
&\quad -\frac{1}{\omega(\omega-1)}\left[(t_{n+1}-t_{q+1})^{\omega+1}-(t_{n+1}-t_q)^{\sigma+1}\right],
\end{aligned}
$$

put $t_q = q\Delta$, we get

$$
\begin{aligned}
I_{q-1,\omega} =& -\frac{\Delta^{\omega+1}}{\omega}[(q+1-(q-1))(n+1-(q+1))^{\omega}-(q-(q-1))(n+1-q^{\omega}] \\
&-\frac{\Delta^{\omega+1}}{\omega(\omega-1)}\left[(n+1-(q+1))^{\omega+1}\right. \\
&\left.-(n+1-q)^{\omega+1}\right], \\
&=\frac{\Delta^{\omega+1}}{\omega(\omega-1)}\left[-2(\omega+1)(n-q)^{\omega}+(\omega+1)(n+1-q)^{\omega}-(n-q)^{\omega+1}\right. \\
&\left.+(n+1-q)^{\omega+1}\right], \\
&=\frac{\Delta^{\omega+1}}{\omega(\omega-1)}[(n-q)^{\omega}(-2(r+1)-(n-q))+(n+1-q)^{\omega}(\omega+1+n+1-q)], \\
&=\frac{\Delta^{\omega+1}}{\omega(\omega-1)}[(n+1-q)^{\omega}(n-q+2+\omega)-(n-q)^{\omega}(n-q+2+2\omega)],
\end{aligned} \tag{28}
$$

and

$$
\begin{aligned}
I_{q,\omega} =& -\frac{\Delta^{\omega+1}}{\omega}[(q+1-q)(n+1-(q+1))^{\omega}]-\frac{\Delta^{\omega+1}}{\omega(\omega-1)}\left[(n+1-(q+1))^{\omega+1}\right. \\
&\left.-(n+1-q)^{\omega+1}\right], =\frac{\Delta^{\omega+1}}{\omega(\omega-1)}\left[-(\omega+1)(n-q)^{\omega}-(n-q)^{\omega+1}\right. \\
&\left.+(n+1-q)^{\omega+1}\right], \\
&=\frac{\Delta^{\omega+1}}{\omega(\omega-1)}\left[(n-q)^{\omega}(-(q+1)-(n-q))+(n+1-q)^{\omega+1}\right], \\
&=\frac{\Delta^{\omega+1}}{\omega(\omega-1)}\left[(n+1-q)^{\omega+1}-(n-q)^{\omega}(n-q+1+\omega)\right],
\end{aligned} \tag{29}
$$

substituting the values of (28) and (29) in (27), we get

$$
\$(t_{n+1}) = \begin{cases}
\$_(0)+\dfrac{(1-\omega)}{\mathrm{ABC}(\omega)}(t_{n+1}^{r-1})[\mathbf{G}_1(\$(t_n),\mathbb{I}(\mathbb{t}_n),\mathbb{R}(\mathbb{t}_n))] \\
+\dfrac{\mathbb{r}\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\displaystyle\sum_{\mathbb{q}=0}^{\mathbb{n}}\left(\dfrac{\mathbb{t}_{\mathbb{q}}^{\mathbb{r}-1}\mathbf{G}_1(\$(\mathbb{t}_{\mathbb{q}}),\mathbb{I}(\mathbb{t}_{\mathbb{q}}),\mathbb{R}(\mathbb{t}_{\mathbb{q}}))}{\Delta}\right. \\
\times\left[\dfrac{\Delta^{\omega+1}}{\omega(\omega-1)}[(\mathbb{n}+1-\mathbb{q})^{\omega}(\mathbb{n}-\mathbb{q}+2+\omega)-(\mathbb{n}-\mathbb{q})^{\omega}(\mathbb{n}-\mathbb{q}+2+2\omega)]\right] \\
\left.-\dfrac{\mathbb{t}_{\mathbb{q}-1}^{\mathbb{r}-1}\mathbf{G}_1(\$(\mathbb{t}_{\mathbb{q}-1}),\mathbb{I}(\mathbb{t}_{\mathbb{q}-1}),\mathbb{R}(\mathbb{t}_{\mathbb{q}-1}))}{\Delta}\left[\dfrac{\Delta^{\omega+1}}{\omega(\omega-1)}\left[(\mathbb{n}+1-\mathbb{q})^{\omega+1}-(\mathbb{n}-\mathbb{q})^{\omega}(\mathbb{n}-\mathbb{q}+1+\omega)\right]\right]\right).
\end{cases} \tag{30}
$$

Similarly for the other two compartments $\mathbb{I}$ and $\mathbb{R}$ we can find the same numerical scheme as

$$
\mathbb{I}(t_{n+1}) = \begin{cases}
\mathbb{I}_(0)+\dfrac{(1-\omega)}{\mathrm{ABC}(\omega)}(t_{n+1}^{r-1})[\mathbf{G}_2(\$(t_n),\mathbb{I}(\mathbb{t}_n),\mathbb{R}(\mathbb{t}_n))] \\
+\dfrac{\mathbb{r}\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\displaystyle\sum_{\mathbb{q}=0}^{\mathbb{n}}\left(\dfrac{\mathbb{t}_{\mathbb{q}}^{\mathbb{r}-1}\mathbf{G}_2(\$(\mathbb{t}_{\mathbb{q}}),\mathbb{I}(\mathbb{t}_{\mathbb{q}}),\mathbb{R}(\mathbb{t}_{\mathbb{q}}))}{\Delta}\right. \\
\times\left[\dfrac{\Delta^{\omega+1}}{\omega(\omega-1)}[(\mathbb{n}+1-\mathbb{q})^{\omega}(\mathbb{n}-\mathbb{q}+2+\omega)-(\mathbb{n}-\mathbb{q})^{\omega}(\mathbb{n}-\mathbb{q}+2+2\omega)]\right] \\
\left.-\dfrac{\mathbb{t}_{\mathbb{q}-1}^{\mathbb{r}-1}\mathbf{G}_1(\$(\mathbb{t}_{\mathbb{q}-1}),\mathbb{I}(\mathbb{t}_{\mathbb{q}-1}),\mathbb{R}(\mathbb{t}_{\mathbb{q}-1}))}{\Delta}\left[\dfrac{\Delta^{\omega+1}}{\omega(\omega-1)}\left[(\mathbb{n}+1-\mathbb{q})^{\omega+1}-(\mathbb{n}-\mathbb{q})^{\omega}(\mathbb{n}-\mathbb{q}+1+\omega)\right]\right]\right).
\end{cases} \tag{31}
$$

$$
\mathbb{R}(t_{n+1}) = \begin{cases}
\mathbb{R}_(0)+\dfrac{(1-\omega)}{\mathrm{ABC}(\omega)}(t_{n+1}^{r-1})[\mathbf{G}_3(\$(t_n),\mathbb{I}(\mathbb{t}_n),\mathbb{R}(\mathbb{t}_n))] \\
+\dfrac{\mathbb{r}\omega}{\mathrm{ABC}(\omega)\Gamma(\omega)}\displaystyle\sum_{\mathbb{q}=0}^{\mathbb{n}}\left(\dfrac{\mathbb{t}_{\mathbb{q}}^{\mathbb{r}-1}\mathbf{G}_3(\$(\mathbb{t}_{\mathbb{q}}),\mathbb{I}(\mathbb{t}_{\mathbb{q}}),\mathbb{R}(\mathbb{t}_{\mathbb{q}}))}{\Delta}\right. \\
\times\left[\dfrac{\Delta^{\omega+1}}{\omega(\omega-1)}[(\mathbb{n}+1-\mathbb{q})^{\omega}(\mathbb{n}-\mathbb{q}+2+\omega)-(\mathbb{n}-\mathbb{q})^{\omega}(\mathbb{n}-\mathbb{q}+2+2\omega)]\right] \\
\left.-\dfrac{\mathbb{t}_{\mathbb{q}-1}^{\mathbb{r}-1}\mathbf{G}_1(\$(\mathbb{t}_{\mathbb{q}-1}),\mathbb{I}(\mathbb{t}_{\mathbb{q}-1}),\mathbb{R}(\mathbb{t}_{\mathbb{q}-1}))}{\Delta}\left[\dfrac{\Delta^{\omega+1}}{\omega(\omega-1)}\left[(\mathbb{n}+1-\mathbb{q})^{\omega+1}-(\mathbb{n}-\mathbb{q})^{\omega}(\mathbb{n}-\mathbb{q}+1+\omega)\right]\right]\right).
\end{cases} \tag{32}
$$

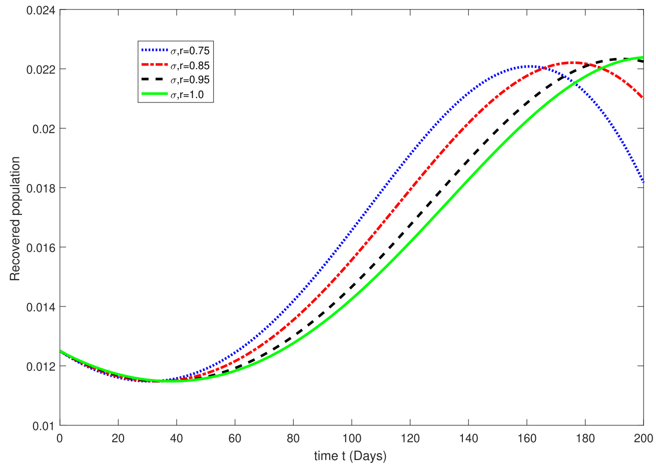

Fig. 7. Dynamics of "recovered population" of the fractal-fractional model (2) at various arbitrary order and fractal dimension.

7. Approximate solution by using values of different parameters with initial conditions

We now take the values for the considered system (2) in the Table 1. The data have been taken for Pakistan. The total susceptible cases of the given country is about $N = 220.0.142$ millions.

7.1. Case-A, when $b = 0.001664$

Using data of 1, we can calculate R_0 as

$$R_0 = \frac{bca}{\mu(\mu + k + \lambda)} = \frac{(0.00009)(0.0001664)(0.49)}{0.019(0.019 + 0.001 + 0.00134)} = 8.242 \times 10^{-9} < 1.$$

Similarly for the remaining two cases we can find that $R_0 < 1$ as for case-A. Otherwise if $R_0 > 1$ as in [69], $R_0 = 5.7$ then our considered system will be unstable and the infection will be on the top. Hence our system is stable and we achieved the the fractional order model 2 by applying the given *AB* techniques in (30).

From 2, we observed that in future 12 weeks the susceptible population will decrease with very high rate i.e in short time. The seen

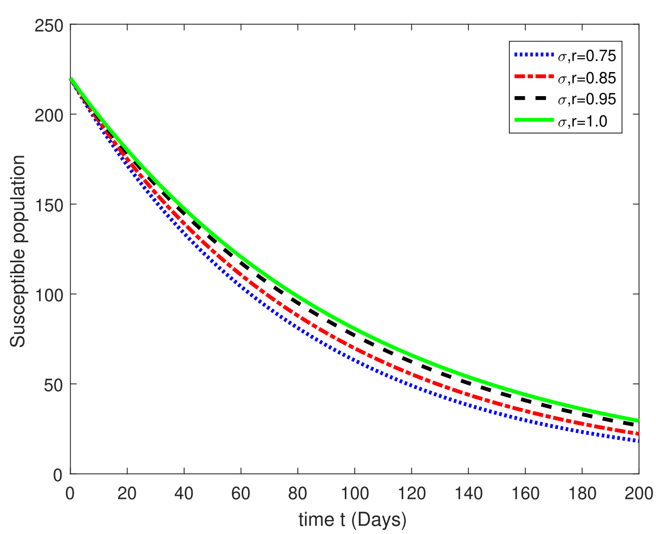

Fig. 8. Dynamics of "susceptible population" of the fractal-fractional model (2) at various arbitrary order and fractal dimension.

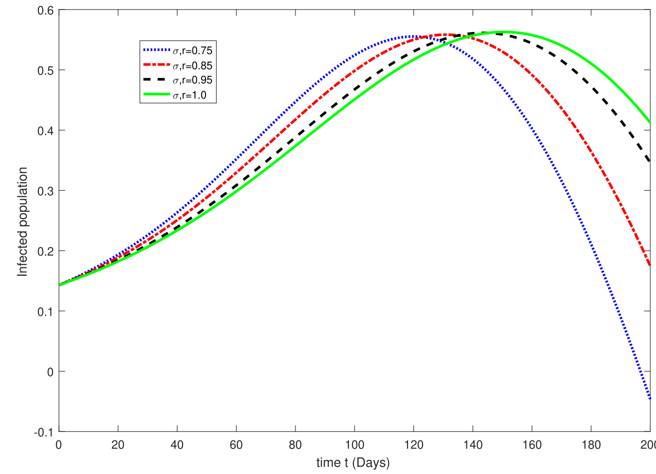

Fig. 9. Dynamics of "infected population" of the fractal-fractional model (2) at various arbitrary order and fractal dimension.

decrease will be rapid at smaller non-integer order and will be slow at larger fractional order and fractal dimension and predicts that in the beginning susceptible class will go towards infected class. Fig. 3 provide a result that on the available data in future few months the infected cases will go up to the maximum peak value 0.8 million in "Pakistan" if precautionary measures are not applied. The increase is high at low arbitrary order with fractal dimensions and as the order raises the rate of infected class goes slow and slow. Similarly Fig. 4 shows that the recovered cases which may also increases by precautionary measures and isolation and the increase occur at smaller fractional order and fractal dimension. All the three figures shows stability and convergency. see Fig. 5–7

7.2. Case-B, when $b = 0.0016630$

Now we take the transmission rate as 0.0016630 and get the result through iteration method as shown in (5) to (7). We observe that as the susceptible class is decaying, then the infection population also decreases by decreasing the transmission rate through social gathering of the people. As the transmission rate decreased the peak value also decreased to 0.6 million. Therefore, we say that in future four or five months increasing transmission rate, the maximum infection cases may have nearly 0.6 million. The number here is less than as compare to the preceding case which shows the effect of lock-down or implementation of the precaution among the society. The figures of case-B also implies

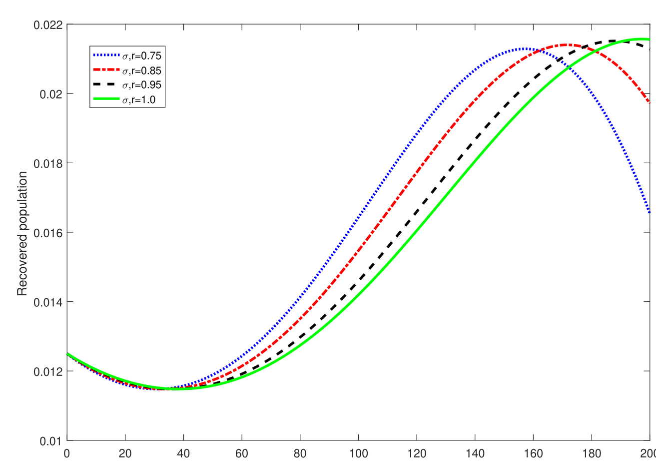

Fig. 10. Dynamics of "recovered population" of the fractal-fractional model (2) at various arbitrary order and fractal dimension.

stability and convergency which can also be showed by plugging values in the formula of $R0$ as in case-A.

7.3. Case-C, when $b = 0.0016628$

Now we notify the same procedure for $b = 0.0016628$, the model behaves decrease in the population of infections class as compared to the previous one, and the peak value decreased attained it in less time. which means that in future it will decrease the number of infected cases addressing the COVID-19. So our numerical solutions provide the best prediction that by decreasing the transmission rate will decrease the infected cases and vice versa in all over the country with other precautionary measure as described earlier would be implemented. The dynamical system has been shown for different compartments in Figs. 8–10 respectively.

8. Conclusion

In our discussion we have investigated the *SIR* fractal-fractional model for the future prediction of COVID-19 in Pakistan and its process using ABC fractal-arbitrary order derivatives. The global and local stability for the considered model have been found by techniques of equilibrium points along with the method of next generation matrix and "Routh-Hurwitz criteria". Next the positivity along with boundedness has been shown by applying non-linear techniques. Few "fixed point results" for the existence of one or more than one solution and "Hyers-Ulam" stability results have been provided for the system (2). Using "Adams-Bashforth method", we have provided an approximate solution for the considered model. By using real data given for "Pakistan", we have graphed the solution and its behavior under the changing of the transmission parameter for various arbitrary order and fractal dimension. On decreasing the transmission rate and implementing the rules and regulation for precaution will give best beneficial effect on the controlling or slowing the spread of the Covid-19. This is also seen that for minimizing the contact with others peoples, the taken system give good output to overcome of the terrible infection.

9. Competing Interest

There exist no competing interest regarding this manuscript.

10. Author statement

Kamal Shah: Conceptualization, Methodology, Design **Muhammad Arfan**: Data curation, Writing- Original draft preparation **Ibrahim Mahariq and Ali Ahmadian**: Supervision, Validation of Data-Reviewing and Original draft **Ali Ahmadian, Soheil Salahshour and Massimiliano Ferrara:**Reviewing and Editing the final version and validation of data.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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