Published in final edited form as:

Pac J Math. 2017 ; 287(1): 19–48. doi:10.2140/pjm.2017.287.19.

HOMOLOGY FOR QUANDLES WITH PARTIAL GROUP OPERATIONS

Scott Carter,

Department of Mathematics and Statistics, University of South Alabama, ILB 325, Mobile, AL 36688, United States

Atsushi Ishii,

Institute of Mathematics, University of Tsukuba, 1-1-1 Tennodai, Ibaraki, Tsukuba 305-8571, Japan

Masahico Saito,

Department of Mathematics and Statistics, University of South Florida, 4202 E Fowler Ave., PHY 114, Tampa, FL 33620, United States

Kokoro Tanaka

Department of Mathematics, Tokyo Gakugei University, 4-1-1 Nukuikita-machi, Koganei-shi, Tokyo 184-8501, Japan

Scott Carter: carter@southalabama.edu; Atsushi Ishii: aishii@math.tsukuba.ac.jp; Masahico Saito: saito@usf.edu; Kokoro Tanaka: kotanaka@u-gakugei.ac.jp

Abstract

A quandle is a set that has a binary operation satisfying three conditions corresponding to the Reidemeister moves. Homology theories of quandles have been developed in a way similar to group homology, and have been applied to knots and knotted surfaces. In this paper, a homology theory is defined that unifies group and quandle homology theories. A quandle that is a union of groups with the operation restricting to conjugation on each group component is called a multiple conjugation quandle (MCQ, defined rigorously within). In this definition, compatibilities between the group and quandle operations are imposed which are motivated by considerations on colorings of handlebody-links. The homology theory defined here for MCQs takes into consideration both group and quandle operations, as well as their compatibility. The first homology group is characterized, and the notion of extensions by 2-cocycles is provided. Degenerate subcomplexes are defined in relation to simplicial decompositions of prismatic (products of simplices) complexes and group inverses. Cocycle invariants are also defined for handlebody-links.

Keywords

quandle; homology; handlebody-link

MSC2010: primary 57M15, 57M25, 57M27, 57Q45; secondary 55N99, 18G99.

1. Introduction

In this paper, a homology theory is proposed that contains aspects of both group and quandle homology theories, for algebraic structures that have both operations and certain compatibility conditions between them.

The notion of a quandle [Joyce 1982; Matveev 1982] was introduced in knot theory as a generalization of the fundamental group. Briefly, a quandle is a set with a binary operation that is idempotent and self-distributive, and a bijective corresponding right action. The axioms correspond to the Reidemeister moves, and quandles have been used extensively to construct knot invariants. They have been considered in various other contexts, for example as symmetries of geometric objects [Takasaki 1943], and with different names, such as distributive groupoids [Matveev 1982] and automorphic sets [Brieskorn 1988]. A typical example is a group conjugation $a * b = b^{-1}ab$ which is an expression of the Wirtinger relation for the fundamental group of the knot complement. The same structure but without idempotency is called a rack, and is used in the study of framed links [Fenn and Rourke 1992].

In [Fenn et al. 1995] a chain complex was introduced for racks. The resulting homology theory was modified in [Carter et al. 2003] by defining a quotient complex that reflected the quandle idempotence axiom. The motivation for this homology was to construct the quandle cocycle invariants for links and surface-links. Since then a variety of applications have been found. The quandle cocycle invariants were generalized to handlebody-links in [Ishii and Iwakiri 2012]. When a set has multiple quandle operations that are parametrized by a group, the structure is called a G-family of quandles; this notion, with its associated homology theory, was introduced in [Ishii et al. 2013] and it too was motivated from handlebody-knots. This homology theory is called *IIJO*. In particular, cocycle invariants were introduced that distinguished mirror images of some handlebody-knots. These G-families were further generalized to an algebraic system called a multiple conjugation quandle (MCQ) in [Ishii 2015b] for colorings of handlebody-knots. An MCQ has a quandle operation and partial group operations, all linked by compatibility conditions.

This paper proposes to unify the group and quandle homology theories for MCQs. The definition of an MCQ is recalled in Section 2 as a generalization of a G-family of quandles. A homology theory is defined (in Section 3) that simultaneously encompasses the group and quandle homologies of the interrelated structures. As in the case of [Carter et al. 2003], some subcomplexes are defined in order to compensate for the topological motivation of the theory. The first homology group is characterized, and the notion of extensions by 2-cocycles is provided in Section 4.

The homology theory for MCQs is well suited for handlebody-links such that each toroidal component has its core circle oriented, as defined in Section 5. When considering colorings for unoriented handlebody-links, we also need to take into consideration issues about the inverse elements in the group (Section 6). Prismatic sets (products of simplices) are decomposed into subsimplices that are higher-dimensional duals of graph moves; Section 7 defines a subcomplex that compensates for these subdivisions. In Sections 8 and 9, we relate this homology theory with group and quandle homology theories. Finally, in Section 10, we discuss approaches to finding new 2-cocycles of our homology theory.

2. Multiple conjugation quandles

First, recall a *quandle* [Joyce 1982; Matveev 1982] is a nonempty set X with a binary operation $*: X \times X \to X$ satisfying the following axioms:

1. For any $a \in X$, we have $a * a = a$.
2. For any $a \in X$, the map $S_a: X \to X$ defined by $S_a(x) = x * a$ is a bijection.
3. For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

Definition 1 [Ishii 2015b]—A *multiple conjugation quandle (MCQ)* X is the disjoint union of groups G_λ, where λ is an element of an index set Λ, with a binary operation $*: X \times X \to X$ satisfying the following axioms:

1. For any $a, b \in G_\lambda$, we have $a * b = b^{-1}ab$.
2. For any $x \in X$ and $a, b \in G_\lambda$, we have $x * e_\lambda = x$ and $x * (ab) = (x * a) * b$, where e_λ is the identity element of G_λ.
3. For any $x, y, z \in X$, we have $(x * y) * z = (x * z) * (y * z)$.
4. For any $x \in X$ and $a, b \in G_\lambda$, we have $(ab) * x = (a * x)(b * x)$ in some group G_μ.

We call the group G_λ a *component* of the MCQ. An MCQ is a type of quandle that can be decomposed as a union of groups, and the quandle operation in each component is given by conjugation. Moreover, there are compatibilities, (2) and (4), between the group and quandle operations.

Note that the quandle axiom $a * a = a$ follows immediately since the operation in any component is given by conjugation. The second quandle axiom also follows, since for the map $S_a: X \to X$ defined by $S_a(x) = x * a$, the inverse map is given by $S_{a^{-1}}$. The second axiom of MCQs implies that the map $\phi: G_\lambda \to \mathrm{Aut}_{\mathrm{Qnd}}\, X$ defined by $\phi(a) = S_a$ is a group homomorphism, where $\mathrm{Aut}_{\mathrm{Qnd}}\, X$ is the set of quandle automorphisms of X and is the group with the multiplication defined by $S_a\, S_b := S_b \circ S_a$. The last axiom (4) may be replaced by the following:

(4′) For any $x \in X$ and $\lambda \in \Lambda$, there is a unique element $\mu \in \Lambda$ such that $S_x(G_\lambda) = G_\mu$ and that $S_x: G_\lambda \to G_\mu$ is a group isomorphism.

The axiom (4) immediately follows from (4′). Conversely, (4′) follows from (4): the condition (4) contains the condition that for any $a, b \in G_\lambda$ and $x \in X$, there exists a unique $\mu \in \Lambda$ such that $a * x, b * x \in G_\mu$. Hence we have $S_x(G_\lambda) \subset G_\mu$, which implies that $S_x: G_\lambda \to G_\mu$ is a well-defined group homomorphism by the condition $(ab) * x = (a * x)(b * x)$. The homomorphism $S_x: G_\lambda \to G_\mu$ is a group isomorphism, since $S_{x^{-1}}: G_\mu \to G_\lambda$ gives its inverse.

A multiple conjugation quandle can be obtained from a G-family of quandles as follows.

Example 2—Let G be a group with identity element e, let $(M, \{*^g\}_{g \in G})$ be a G-family of quandles [Ishii et al. 2013]; i.e., a nonempty set M with a family of binary operations $*^g$: $M \times M \to M$ $(g \in G)$ satisfying

$$x *^g x = x, \quad x *^{gh} y = (x *^g y) *^h y, \quad x *^e y = x,$$
$$(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z)$$

for $x, y, z \in M$ and $g, h \in G$. Then $\coprod_{x \in M}\{x\} \times G$ is a multiple conjugation quandle with

$$(x, g) * (y, h) = (x *^h y, h^{-1}gh), \quad (x, g)(x, h) = (x, gh).$$

The following are specific examples of G-families of quandles.

1. Let M be a group, and G be a subgroup of Aut M. Then for $x, y \in M$ and $g \in G$, $x * y = (xy^{-1})^g y$ gives a G-family of quandles. Here x^g denotes g acting on x. The fact that this is a G-family was pointed out in [Przytycki 2011]; however, that any specific automorphism g yields a quandle was earlier observed in [Joyce 1982; Matveev 1982]. When M is abelian and an element $g \in G$ is fixed, the resulting quandle is called an Alexander quandle.

2. Let $(X, *)$ be a quandle. We denote $S_b^n(a)$ by $a *^n b$. Put $Z: = \mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$, where $m: = \min\{i > 0 \mid x *^i y = x$ for any $x, y \in X\}$. Then $(X, \{*^n\}_{n \in Z})$ is a Z-family of quandles.

For a multiple conjugation quandle $X = \coprod_{\lambda \in \Lambda} G_\lambda$, an X-*set* is a nonempty set Y with a map $*$: $Y \times X \to Y$ satisfying the following axioms, where we use the same symbol * as the binary operation of X.

- For any $y \in Y$ and $a, b \in G_\lambda$, we have $y * e_\lambda = y$ and $y * (ab) = (y * a) * b$, where e_λ is the identity of G_λ.
- For any $y \in Y$ and $a, b \in X$, we have $(y * a) * b = (y * b) * (a * b)$.

Any multiple conjugation quandle X itself is an X-set with its binary operation. Any singleton set $\{y_0\}$ is also an X-set with the map * defined by $y_0 * x = y_0$ for $x \in X$, which is called a trivial X-set. The index set Λ is an X-set with the map * defined by $\lambda * x = \mu$ when $S_x(G_\lambda) = G_\mu$ for $\lambda, \mu \in \Lambda$ and $x \in X$.

3. Homology theory

In this section, we define a chain complex for MCQs that contains aspects of both group and quandle homology theories. A subcomplex is also defined that corresponds to a Reidemeister move for handlebody-links.

Let $X = \coprod_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. In what follows, we denote a sequence of elements of X by a bold symbol such as $\boldsymbol{a}$, and denote by $|\boldsymbol{a}|$ the length of a sequence $\boldsymbol{a}$. For example, $(\boldsymbol{a})$, $\langle \boldsymbol{a} \rangle$, $(y; \boldsymbol{a}; \boldsymbol{b})$ respectively denote

$$(a_1, \ldots, a_{|\boldsymbol{a}|}), \quad \langle a_1, \ldots, a_{|\boldsymbol{a}|} \rangle, \quad (y; a_1, \ldots, a_{|\boldsymbol{a}|}; b_1, \ldots, b_{|\boldsymbol{b}|}).$$

Let $P_n(X)_Y$ be the free abelian group generated by the elements

$$(y; a_{1,1}, \ldots, a_{1,n_1}; \ldots; a_{k,1}, \ldots, a_{k,n_k}) \in \bigcup_{n_1 + \cdots + n_k = n} Y \times \prod_{i=1}^{k} \bigcup_{\lambda \in \Lambda} G_\lambda^{n_i}$$

if $n \geq 0$, and let $P_n(X)_Y = 0$ otherwise. The elements of $P_n(X)_Y$ are called *prismatic chains* and $P_n(X)_Y$ is called the *prismatic chain group*. Note that for each j, the elements $a_{j,1}, \ldots,$ $a_{j,n}$_{j} belong to one of the G_λ. For example, $P_3(X)_Y$ is generated by the elements $(y; a; b; c)$, $(y; a; e, f)$, $(y; d, e; c)$ and $(y; d, e, f)$ (where $a, b, c \in X$, $d, e, f \in G_\lambda$, $y \in Y$). Here a, b, c may or may not belong to the same G_μ $(\mu \in \Lambda)$, but d, e, f belong to the same G_λ. All may belong to the same G_λ.

We represent $(y; \boldsymbol{a}_1; \ldots; \boldsymbol{a}_k)$ using the noncommutative multiplication form

$$\langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \langle \boldsymbol{a}_k \rangle .$$

We define $\langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \langle \boldsymbol{a}_k \rangle * b := \langle y * b \rangle \langle \boldsymbol{a}_1 * b \rangle \cdots \langle \boldsymbol{a}_k * b \rangle$, where $\langle \boldsymbol{a} * b \rangle$ denotes $\langle a_1 * b,$ $\ldots, a_{|\boldsymbol{a}|} * b \rangle$. We set $|\langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \langle \boldsymbol{a}_k \rangle| := |\boldsymbol{a}_1| + \cdots + |\boldsymbol{a}_k|$.

We define a boundary homomorphism $\partial_n : P_n(X)_Y \to P_{n-1}(X)_Y$ by

$$\partial(\langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \langle \boldsymbol{a}_k \rangle) = \sum_{i=1}^{k} (-1)^{|\langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \langle \boldsymbol{a}_{i-1} \rangle|} \langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \partial \langle \boldsymbol{a}_i \rangle \cdots \langle \boldsymbol{a}_k \rangle,$$

where

$$\partial \langle a_1, \ldots, a_m \rangle = * a_1 \langle a_2, \ldots, a_m \rangle + \sum_{i=1}^{m-1} (-1)^i \langle a_1, \ldots, a_i a_{i+1}, \ldots, a_m \rangle + (-1)^m \langle a_1, \ldots, a_{m-1} \rangle .$$

The resulting terms $\partial(\langle a \rangle) = *a\langle\ \rangle - \langle\ \rangle$ for $m = 1$ in the above expression mean that the formal symbol $\langle\ \rangle$ is deleted. For $n = 0$, we define $\partial \langle y \rangle = 0$.

Example 3—The boundary maps in two and three dimensions are computed as follows.

$$\begin{aligned}
\partial_2(\langle y\rangle\langle a\rangle\langle b\rangle) &= \langle y*a\rangle\langle b\rangle - \langle y\rangle\langle b\rangle - \langle y*b\rangle\langle a*b\rangle + \langle y\rangle\langle a\rangle,\\
\partial_2(\langle y\rangle\langle a,b\rangle) &= \langle y*a\rangle\langle b\rangle - \langle y\rangle\langle ab\rangle + \langle y\rangle\langle a\rangle,\\
\partial_3(\langle y\rangle\langle a\rangle\langle b\rangle\langle c\rangle) &= \langle y*a\rangle\langle b\rangle\langle c\rangle - \langle y\rangle\langle b\rangle\langle c\rangle - \langle y*b\rangle\langle a*b\rangle\langle c\rangle + \langle y\rangle\langle a\rangle\langle c\rangle + \langle y*c\rangle\langle a*c\rangle\langle b*c\rangle - \langle y\rangle\langle a\rangle\langle b\rangle,\\
\partial_3(\langle y\rangle\langle a\rangle\langle b,c\rangle) &= \langle y*a\rangle\langle b,c\rangle - \langle y\rangle\langle b,c\rangle - \langle y*b\rangle\langle a*b\rangle\langle c\rangle + \langle y\rangle\langle a\rangle\langle bc\rangle - \langle y\rangle\langle a\rangle\langle b\rangle,\\
\partial_3(\langle y\rangle\langle a,b\rangle\langle c\rangle) &= \langle y*a\rangle\langle b\rangle\langle c\rangle - \langle y\rangle\langle ab\rangle\langle c\rangle + \langle y\rangle\langle a\rangle\langle c\rangle + \langle y*c\rangle\langle a*c,b*c\rangle - \langle y\rangle\langle a,b\rangle,\\
\partial_3(\langle y\rangle\langle a,b,c\rangle) &= \langle y*a\rangle\langle b,c\rangle - \langle y\rangle\langle ab,c\rangle + \langle y\rangle\langle a,bc\rangle - \langle y\rangle\langle a,b\rangle\,.
\end{aligned}$$

Proposition 4— *$P_*(X)_Y = (P_n(X)_Y, \partial_n)$ is a chain complex.*

Proof: The Leibniz rule

$$\partial(\sigma\tau) = (\partial\sigma)\tau + (-1)^{|\sigma|}\sigma(\partial\tau)$$

is a restatement of the definition when $k = 2$. In fact, the general definition follows from this by induction. Also $\partial(\sigma * a) = (\partial\sigma) * a$, and $\partial \circ \partial = 0$ follows from these two facts.

We will later define a degeneracy subcomplex that is analogous (albeit more complicated) to the subcomplex of degeneracies for quandle homology. Before its definition, we give a description of simplicial decompositions of products of simplices for motivation. We identify an n-simplex Δ^n with the set

$$\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : 0 \le x_1 \le x_2 \le \cdots \le x_n \le 1\},$$

called the *right n-simplex*. Then the n-cube $[0, 1]^n$ can be decomposed into $n!$ sets each of which is congruent to this right n-simplex that has n edges of length 1, and has $(n - k + 1)$ edges of length $\sqrt{k}$ for $k = 1, \ldots, n$. More specifically, for $\vec{x} \in [0, 1]^n$ consider the permutation $\sigma \in \Sigma_n$ such that $0 \le x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(n)} \le 1$. If the coordinates of $\vec{x}$ are all distinct, then there is a unique such σ and an n-simplex Δ^n_σ congruent to the right n-simplex such that $\vec{x}$ lies in the interior of Δ^n_σ. Otherwise $\vec{x}$ lies in the boundary of more than one such simplex. Now consider the product of right simplices

$$\Delta^s \times \Delta^t = \left\{ (\vec{x}, \vec{y}) \in [0,1]^{s+t} \,\middle|\, \begin{matrix} 0 \le x_1 \le x_2 \le \cdots \le x_s \le 1 \\ 0 \le y_1 \le y_2 \le \cdots \le y_t \le 1 \end{matrix} \right\},$$

where the notation $(\vec{x}, \vec{y})$ represents $(x_1, \ldots, x_s, y_1, \ldots, y_t)$. This can be decomposed as a union of simplices of the form given above. For

$$\vec{z} = (\vec{x}, \vec{y}) \in \Delta^s \times \Delta^t \subset [0,1]^n,$$

where $n = s + t$, there is an associated simplex Δ_σ^n that contains the point $(\vec{x}, \vec{y})$. Suppose all coordinates of $\vec{z}$ are distinct, and let $\sigma \in \Sigma_n$ be a permutation such that $0 < z_{\sigma(1)} < \cdots < z_{\sigma(n)}$. Then the subset $\{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, s + t\}$ with $i_1 < i_2 < \cdots < i_s$ is determined from the positions of coordinates of $\vec{x}$, so that $z_{i_k} = x_k$ for $k = 1, \ldots, s$. Thus a given subset $\{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, s + t\}$ where $i_1 < i_2 < \cdots < i_s$ determines an n-simplex in the decomposition of $\Delta^s \times \Delta^t$. We proceed to the definition of the degeneracy subcomplex.

For an expression of the form $\langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle$ in a chain in $P_n(X)_Y$, where $\langle \boldsymbol{a} \rangle = \langle a_1, \ldots, a_s \rangle$ and $\langle \boldsymbol{b} \rangle = \langle b_1, \ldots, b_t \rangle$ satisfy $a_i, b_j \in G_\lambda$ for all $i = 1, \ldots, s$ and $j = 1, \ldots, t$, let the notation $\langle\langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle\rangle_{i_1,\ldots,i_s}$ represent $(-1)^{\sum_{k=1}^{s}(i_k - k)} \langle c_1, \ldots, c_{s+t} \rangle$, where $1 \le i_1 < \cdots < i_k < \cdots < i_s \le s + t$, and

$$c_i = \begin{cases} a_k * (b_1 \cdots b_{i-k}) & \text{if } i = i_k, \\ b_{i-k} & \text{if } i_k < i < i_{k+1}. \end{cases}$$

If $i = k$ in the first case, then we regard $(b_1 \cdots b_{i-k})$ to be empty. For example, $\langle\langle a \rangle \langle b \rangle\rangle_1 = \langle a, b \rangle$, $\langle\langle a \rangle \langle b \rangle\rangle_2 = -\langle b, a * b \rangle$, and $\langle\langle a, b \rangle \langle c \rangle\rangle_{1,3} = -\langle a, c, b * c \rangle$. We also define the notation $\langle\langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle\rangle$ by

$$\langle\langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle\rangle := \sum_{1 \le i_1 < \cdots < i_s \le s+t} \langle\langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle\rangle_{i_1,\ldots,i_s}.$$

Define $D_n(X)_Y$ to be the subgroup of $P_n(X)_Y$ generated by the elements of the form

$$\langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle \cdots \langle \boldsymbol{a}_k \rangle - \langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \langle\langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle\rangle \cdots \langle \boldsymbol{a}_k \rangle,$$

where we implicitly assume the linearity of the notations $\langle\langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle\rangle_{i_1,\ldots,i_s}$ and $\langle\langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle\rangle$, that is,

$$\langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \langle\langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle\rangle \cdots \langle \boldsymbol{a}_k \rangle = \sum_{1 \le i_1 < \cdots < i_{|\boldsymbol{a}|} \le |\langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle|} \langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \langle\langle \boldsymbol{a} \rangle \langle \boldsymbol{b} \rangle\rangle_{i_1,\ldots,i_{|\boldsymbol{a}|}} \cdots \langle \boldsymbol{a}_k \rangle.$$

The chain group $D_n(X)_Y$ is called the group of *decomposition degeneracies*. We will see that $D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $P_*(X)_Y$ in Section 7.

We remark that the elements of the form

$$\langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \langle a \rangle \langle a \rangle \cdots \langle \boldsymbol{a}_k \rangle$$

belong to $D_n(X)_Y$.

For example, $D_2(X)_Y$ is generated by the elements of the form

$$\langle y\rangle\langle a\rangle\langle b\rangle-\langle y\rangle\langle a,b\rangle+\langle y\rangle\langle b,a*b\rangle,$$

and $D_3(X)_Y$ is generated by the elements of the form

$$\begin{aligned}&\langle y\rangle\langle a\rangle\langle b\rangle\langle x\rangle-\langle y\rangle\langle a,b\rangle\langle x\rangle+\langle y\rangle\langle b,a*b\rangle\langle x\rangle,\\&\langle y\rangle\langle x\rangle\langle b\rangle\langle c\rangle-\langle y\rangle\langle x\rangle\langle b,c\rangle+\langle y\rangle\langle x\rangle\langle c,b*c\rangle,\\&\langle y\rangle\langle a,b\rangle\langle c\rangle-\langle y\rangle\langle a,b,c\rangle+\langle y\rangle\langle a,c,b*c\rangle-\langle y\rangle\langle c,a*c,b*c\rangle,\\&\langle y\rangle\langle a\rangle\langle b,c\rangle-\langle y\rangle\langle a,b,c\rangle+\langle y\rangle\langle b,a*b,c\rangle-\langle y\rangle\langle b,c,a*(bc)\rangle\end{aligned}$$

for $a, b, c \in G_\lambda$, $x \in X$.

Definition 5—The quotient complex of $P_*(X)_Y$ modulo decomposition degeneracies $D_*(X)_Y$ is denoted by $C_*(X)_Y = (C_n(X)_Y, \partial_n)$, where $C_n(X)_Y = P_n(X)_Y/D_n(X)_Y$. For an abelian group A, define the cochain complex $C^*(X; A)_Y = \mathrm{Hom}(C_*(X)_Y, A)$. Denote by $H_n(X)_Y$ the n-th homology group of $C_*(X)_Y$.

4. Algebraic aspects of the homology

In this section we study algebraic aspects of the homology theory we defined. Specifically, we characterize the first homology group, and show that a 2-cocycle defines an extension. For simplicity we consider the case $Y = \{y_0\}$ is a singleton, and we suppress the symbols $\langle y_0\rangle$ whenever possible.

Let X be a multiple conjugation quandle, and $Y = \{y_0\}$ be a singleton. Then $P_0(X)_Y$ is infinite cyclic, generated by $\langle y_0\rangle$, and $\partial_1(\langle y_0\rangle\langle a\rangle) = \langle y_0 * a\rangle - \langle y_0\rangle$ for $a \in X$. Hence $H_0(X)_Y = \mathbb{Z}$. If X is a multiple conjugation quandle consisting of a single group, $H_1(X)_Y \cong X^{\mathrm{ab}}$, since $P_1(X)_Y$ is the free abelian group generated by the elements $\langle y_0\rangle\langle a\rangle$ ($a \in X$), and

$$\begin{aligned}\partial_2(\langle y_0\rangle\langle a,b\rangle)&=\langle y_0\rangle\langle b\rangle-\langle y_0\rangle\langle ab\rangle+\langle y_0\rangle\langle a\rangle,\\\partial_2(\langle y_0\rangle\langle a\rangle\langle b\rangle)&=-\langle y_0\rangle\langle a*b\rangle+\langle y_0\rangle\langle a\rangle=\partial_2(\langle y_0\rangle\langle a,b\rangle)-\partial_2(\langle y_0\rangle\langle b,b^{-1}ab\rangle).\end{aligned}$$

Proposition 6—*Let $X = \coprod_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle, let $Y = \{y_0\}$ be a singleton, and A an abelian group. A map $\phi: P_2(X)_Y \to A$ is a 2-cocycle of $C^*(X)_Y$ if and only if $X \times A = \coprod_{\lambda \in \Lambda}(G_\lambda \times A)$ with*

$$\begin{aligned}(a,s)*(b,t)&:=(a*b,s+\phi(\langle a\rangle\langle b\rangle))\ for\ (a,s),(b,t)\in X\times A,\\(a,s)(b,t)&:=(ab,\ s+t+\phi(\langle a,b\rangle))\ for\ (a,s),(b,t)\in G_\lambda\times A\end{aligned}$$

is a multiple conjugation quandle, where $\phi(\langle y_0\rangle\langle a\rangle\langle b\rangle)$ and $\phi(\langle y_0\rangle\langle a, b\rangle)$ are respectively denoted by $\phi(\langle a\rangle\langle b\rangle)$ and $\phi(\langle a, b\rangle)$ for short. Further, $(e_\lambda, -\phi(\langle e_\lambda, e_\lambda\rangle))$ is the identity of the group $G_\lambda \times A$, and $(a^{-1}, -s - \phi(\langle a, a^{-1}\rangle) - \phi(\langle e_\lambda, e_\lambda\rangle))$ is the inverse of $(a, s) \in G_\lambda \times A$.

Proof: We show correspondences between cocycle conditions and MCQ conditions for the extension.

(1) *The correspondence between the cocycle condition* $\phi(\partial_3(\langle a, b, c\rangle)) = 0$ *and the associativity of a group.*

For $(a, s), (b, t), (c, u) \in G_\lambda \times A$, $\phi(\langle a, b\rangle) + \phi(\langle ab, c\rangle) = \phi(\langle b, c\rangle) + \phi(\langle a, bc\rangle)$ if and only if $((a, s)(b, t))(c, u) = (a, s)((b, t)(c, u))$, since

$$\begin{aligned}((a,s)(b,t))(c,u) &= (abc, s+t+u+\phi(\langle a,b\rangle)+\phi(\langle ab,c\rangle)),\\ (a,s)((b,t)(c,u)) &= (abc,\ s+t+u+\phi(\langle b,c\rangle)+\phi(\langle a,bc\rangle)).\end{aligned}$$

We note that $\phi(\langle a, b\rangle) + \phi(\langle ab, c\rangle) = \phi(\langle b, c\rangle) + \phi(\langle a, bc\rangle)$, or equivalently $((a, s)(b, t))(c, u) = (a, s)((b, t)(c, u))$ implies that $\phi(\langle a, e_\lambda\rangle) = \phi(\langle e_\lambda, c\rangle)$ and that $\phi(\langle b^{-1}, b\rangle) = \phi(\langle b, b^{-1}\rangle)$. These equalities respectively imply

$$(a,s) = (a,s)(e_\lambda,\ -\phi(\langle e_\lambda, e_\lambda\rangle)) = (e_\lambda,\ -\phi(\langle e_\lambda, e_\lambda\rangle))(a,s)$$

and

$$\begin{aligned}(e_\lambda,\ -\phi(\langle e_\lambda, e_\lambda\rangle)) &= (a,s)\left(a^{-1},\ -s-\phi(\langle a,a^{-1}\rangle)-\phi(\langle e_\lambda,e_\lambda\rangle)\right)\\ &= \left(a^{-1},\ -s-\phi(\langle a,a^{-1}\rangle)-\phi(\langle e_\lambda,e_\lambda\rangle)\right)(a,s).\end{aligned}$$

It follows that $(e_\lambda, -\phi(\langle e_\lambda, e_\lambda\rangle))$ is the identity of the group $G_\lambda \times A$, and that $(a^{-1}, -s - \phi(\langle a, a^{-1}\rangle) - \phi(\langle e_\lambda, e_\lambda\rangle))$ is the inverse of $(a, s) \in G_\lambda \times A$.

(2) *The correspondence between the degeneracy of* ϕ *on* $D_2(X)_Y$ *and the first axiom of MCQs.*

For $(a, s), (b, t) \in G_\lambda \times A$, $\phi(\langle a\rangle\langle b\rangle) + \phi(\langle b, a * b\rangle) = \phi(\langle a, b\rangle)$ if and only if $(b, t)((a, s) * (b, t)) = (a, s)(b, t)$, since

$$\begin{aligned}(b,t)((a,s)*(b,t)) &= (b(a*b), s+t+\phi(\langle a\rangle\langle b\rangle)+\phi(\langle b, a*b\rangle)),\\ (a,s)(b,t) &= (ab, s+t+\phi(\langle a,b\rangle)).\end{aligned}$$

(3) *The correspondence between the cocycle condition* $\phi(\partial_3(\langle x\rangle\langle a, b\rangle)) = 0$ *and the second axiom of MCQs.*

For $(x, r) \in X \times A$ and $(a, s), (b, t) \in G_\lambda \times A$,

$$\phi(\langle x\rangle\langle ab\rangle) = \phi(\langle x\rangle\langle a\rangle) + \phi(\langle x*a\rangle\langle b\rangle)$$

if and only if $(x, r) * ((a, s)(b, t)) = ((x, r) * (a, s))* (b, t)$, since

$$\begin{aligned}(x, r) * ((a, s)(b, t)) &= (x * (ab), r + \phi(\langle x \rangle \langle ab \rangle)),\\ ((x, r) * (a, s)) * (b, t) &= ((x * a) * b, r + \phi(\langle x \rangle \langle a \rangle) + \phi(\langle x * a \rangle \langle b \rangle)).\end{aligned}$$

Note $\phi(\langle x \rangle \langle ab \rangle) = \phi(\langle x \rangle \langle a \rangle) + \phi(\langle x * a \rangle \langle b \rangle)$, or equivalently $(x, r)*((a, s)(b, t)) = ((x, r) * (a, s)) * (b, t)$, implies that $\phi(\langle x \rangle \langle e_\lambda \rangle) = 0$. Then we have

$$(a, s) * (e_\lambda, -\phi(\langle e_\lambda, e_\lambda \rangle)) = (a, s).$$

(4) *The correspondence between the cocycle condition* $\phi(\partial_3(\langle a \rangle \langle b \rangle \langle c \rangle)) = 0$ *and the third axiom of MCQs.*

For $(a, s), (b, t), (c, u) \in X \times A$,

$$\phi(\langle a \rangle \langle b \rangle) + \phi(\langle a * b \rangle \langle c \rangle) = \phi(\langle a \rangle \langle c \rangle) + \phi(\langle a * c \rangle \langle b * c \rangle)$$

if and only if $((a, s) * (b, t))* (c, u) = ((a, s) * (c, u))* ((b, t) * (c, u))$, since

$$\begin{aligned}((a, s) * (b, t) * (c, u)) &= ((a * b) * c, s + \phi(\langle a \rangle \langle b \rangle) + \phi(\langle a * b \rangle \langle c \rangle)),\\ ((a, s) * (c, u)) * ((b, t) * (c, u)) &= ((a * c) * (b * c), s + \phi(\langle a \rangle \langle c \rangle) + \phi(\langle a * c \rangle \langle b * c \rangle)).\end{aligned}$$

(5) *The correspondence between the cocycle condition* $\phi(\partial_3(\langle a, b \rangle \langle x \rangle)) = 0$ *and the last axiom of MCQs.*

For $(x, r) \in X \times A$ and $(a, s), (b, t) \in G_\lambda \times A$,

$$\phi(\langle a, b \rangle) + \phi(\langle ab \rangle \langle x \rangle) = \phi(\langle a \rangle \langle x \rangle) + \phi(\langle b \rangle \langle x \rangle) + \phi(\langle a * x, b * x \rangle)$$

if and only if $((a, s)(b, t)) * (x, r) = ((a, s) * (x, r))((b, t) * (x, r))$, since

$$\begin{aligned}((a, s)(b, t)) * (x, r) &= ((ab) * x, s + t + \phi(\langle a, b \rangle) + \phi(\langle ab \rangle \langle x \rangle)),\\ ((a, s) * (x, r))((b, t) * (x, r)) &= ((a * x)(b * x), s + t + \phi(\langle a \rangle \langle x \rangle) + \phi(\langle b \rangle \langle x \rangle) + \phi(\langle a * x, b * x \rangle)).\end{aligned}$$

Therefore ϕ is a 2-cocycle if and only if $X \times A$ is a multiple conjugation quandle.

5. Quandle cocycle invariants for handlebody-links

The definition of a multiple conjugation quandle is motivated from handlebody-links and their colorings [Ishii 2015b]. A *handlebody-link* is a disjoint union of handlebodies embedded in the 3-sphere S^3. A *handlebody-knot* is a one component handlebody-link. Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of

S^3 which sends one to the other. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph whose regular neighborhood is the handlebody-link, where a spatial trivalent graph is a finite trivalent graph embedded in S^3. In this paper, a trivalent graph may contain circle components. Two handlebody-links are equivalent if and only if their diagrams are related by a finite sequence of R1–R6 moves depicted in Figure 1 [Ishii 2008].

An S^1*-orientation* of a handlebody-link is an orientation of all genus 1 components of the handlebody-link, where an orientation of a solid torus is an orientation of its core S^1. Two S^1-oriented handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other preserving the S^1-orientation. A *Y-orientation* of a spatial trivalent graph is an orientation of the graph without sources and sinks with respect to the orientation (see Figure 2). We note that the term Y-orientation is a symbolic convention, and has no relation to an X-set Y. A *diagram* of an S^1-oriented handlebody-link is a diagram of a Y-oriented spatial trivalent graph whose regular neighborhood is the S^1-oriented handlebody-link where the S^1-orientation is induced from the Y-orientation by forgetting the orientations except on circle components of the Y-oriented spatial trivalent graph. *Y-oriented* R1–R6 *moves* are R1–R6 moves between two diagrams with Y-orientations which are identical except in the disk where the move applied. Two S^1-oriented handlebody-links are equivalent if and only if their diagrams are related by a finite sequence of Y-oriented R1–R6 moves [Ishii 2015a]. Note that in Figure 1 (R6), if all end points are oriented downward, then either choice of the two possible orientations of the middle edge makes the diagram Y-oriented locally. Thus reversing an orientation of this edge can be regarded as applying Y-oriented R6 moves twice. This is the case whenever both orientations of an edge give Y-orientations.

Let $X = \coprod_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. Let D be a diagram of an S^1-oriented handlebody-link H. We denote by $\mathscr{A}(D)$ the set of arcs of D, where an arc is a piece of a curve each of whose endpoints is an undercrossing or a vertex. We denote by $\mathscr{R}(D)$ the set of complementary regions of D. In this paper, an orientation of an arc is represented by the normal orientation obtained by rotating the usual orientation counterclockwise by $\pi/2$ on the diagram. An *X-coloring* C of a diagram D is an assignment of an element of X to each arc $\alpha \in \mathscr{A}(D)$ satisfying the conditions depicted in the left three diagrams in Figure 3 at each crossing and each vertex of D. An X_Y-coloring C of D is an extension of an X-coloring of D which assigns an element of Y to each region $R \in \mathscr{R}(D)$ satisfying the condition depicted in the rightmost diagram in Figure 3 at each arc. We denote by $\mathrm{Col}_X(D)$ (resp. $\mathrm{Col}_X(D)_Y$) the set of X-colorings (resp. X_Y-colorings) of D. Then we have the following proposition.

Proposition 7 [Ishii 2015a]—*Let $X = \coprod_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. Let D be a diagram of an S^1-oriented handlebody-link H. Let D' be a diagram obtained by applying one of the Y-oriented* R1–R6 *moves to the diagram D once. For an X-coloring (resp. X_Y-coloring) C of D, there is a unique X-coloring (resp. X_Y-coloring) C' of D' which coincides with C except near a point where the move applied.*

For an X_Y-coloring C of a diagram D of an S^1-oriented handlebody-link, we define the *local chains* $w(\xi;C) \in C_2(X)_Y$ at each crossing ξ and each vertex ξ of D as depicted in Figure 4. We define a chain $W(D;C) \in C_2(X)_Y$ by

$$W(D;C) = \sum_{\xi} w(\xi;C),$$

where ξ runs over all crossings and vertices of D. This is similar to the definitions found in [Carter et al. 2001] for links and surface-links, and in [Ishii and Iwakiri 2012] for handlebody-links.

Lemma 8— *The chain $W(D;C)$ is a 2-cycle of $C_*(X)_Y$. Further, for cohomologous 2-cocycles θ, θ' of $C^*(X;A)_Y$, we have*

$$\theta(W(D;C)) = \theta'(W(D;C)).$$

Proof: It is sufficient to show that $W(D;C)$ is a 2-cycle of $C_*(X)_Y$. We denote by $\mathcal{SA}(D)$ the set of semiarcs of D, where a semiarc is a piece of a curve each of whose endpoints is a crossing or a vertex. We denote by $\mathcal{SA}(D;\xi)$ the set of semiarcs incident to ξ, where ξ is a crossing or a vertex of D.

For a semiarc α, there is a unique region R_α facing α such that the normal orientation of α points from the region R_α to the opposite region with respect to α. For a semiarc α incident to a crossing or a vertex ξ, we define

$$\varepsilon(\alpha;\xi) := \begin{cases} 1 & \text{if the orientation of } \alpha \text{ points to } \xi, \\ -1 & \text{otherwise.} \end{cases}$$

Let $\chi_1, \ldots, \chi_4$ and $\omega_1, \omega_2, \omega_3$ be the semiarcs incident to a crossing χ and a vertex ω as depicted in Figure 5. From

$$\begin{aligned} \partial_2(w(\chi;C)) &= \sum_{\alpha \in \mathcal{SA}(D;\chi)} \varepsilon(\alpha;\chi)\langle C(R_\alpha)\rangle\langle C(\alpha)\rangle, \\ \partial_2(w(\omega;C)) &= \sum_{\alpha \in \mathcal{SA}(D;\omega)} \varepsilon(\alpha;\omega)\langle C(R_\alpha)\rangle\langle C(\alpha)\rangle, \end{aligned}$$

it follows that

$$\partial_2(W(D;C)) = \sum_{\chi} \partial_2(w(\chi;C)) + \sum_{\omega} \partial_2(w(\omega;C)) = 0,$$

where χ and ω, respectively, run over all crossings and vertices of D.

Lemma 9—*Let D be a diagram of an S^1-oriented handlebody-link H. Let D' be a diagram obtained by applying one of the Y-oriented* R1–R6 *moves to the diagram D once. Let C be an X_Y-coloring of D, let C' be the unique X_Y-coloring of D' such that C and C' coincide except near a point where the move applied. Then we have* $[W(D;C)] = [W(D';C')]$ *in* $H_2(X)_Y$.

Proof: We have the invariance under the Y-oriented R1 and R4 moves, since the difference between $[W(D;C)]$ and $[W(D';C')]$ is an element of $D_2(X)_Y$. The invariance under the Y-oriented R2 move follows from the signs of the crossings which appear in the move. We have the invariance under the Y-oriented R3, R5, and R6 moves, since the difference between $[W(D;C)]$ and $[W(D';C')]$ is an image of ∂_3. See Figure 6 for Y-oriented R6 moves, where all arcs are directed from top to bottom.

For a 2-cocycle θ of $C^*(X;A)_Y$, we define

$$\mathscr{H}(D) := \{[W(D;C)] \in H_2(X)_Y \mid C \in \mathrm{Col}_X(D)_Y\},$$
$$\Phi_\theta(D) := \{\theta(W(D,C)) \in A \mid C \in \mathrm{Col}_X(D)_Y\}$$

as multisets. By Lemmas 8 and 9, we have the following theorem.

Theorem 10—*Let D be a diagram of an S^1-oriented handlebody-link H. Then $\mathscr{H}(D)$ and $\Phi_\theta(D)$ are invariants of H.*

For an S^1-oriented handlebody-link H, let H^* be the mirror image of H, and $-H$ be the S^1-oriented handlebody-link obtained from H by reversing its S^1-orientation. Then we also have

$$\mathscr{H}(-H^*) = -\mathscr{H}(H), \quad \Phi_\theta(-H^*) = -\Phi_\theta(H),$$

where $-S = \{-a \mid a \in S\}$ for a multiset S. It is desirable to further study these invariants and applications to handlebody-links.

6. For unoriented handlebody-links

Let $X = \coprod_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. Let D be a diagram of an (unoriented) handlebody-link H. An $(X, \uparrow)$-color C_α of an arc $\alpha \in \mathscr{A}(D)$ is a map C_α from the set of orientations of the arc α to X such that $C_\alpha(-o) = C_\alpha(o)^{-1}$, where $-o$ is the inverse of an orientation o. An $(X, \uparrow)$-color C_α is represented by a pair of an orientation o of α and an element $C_\alpha(o) \in X$ on the diagram D. Two pairs (o, a) and $(-o, a^{-1})$ represent the same $(X, \uparrow)$-color (see Figure 7).

An $(X, \uparrow)$*-coloring* C of a diagram D is an assignment of an $(X, \uparrow)$-color C_α to each arc $\alpha \in \mathscr{A}(D)$ satisfying the conditions depicted in the left two diagrams in Figure 8 at each crossing and each vertex of D. An $(X, \uparrow)_Y$-coloring C of D is an extension of an $(X, \uparrow)$-coloring of D which assigns an element of Y to each region $R \in \mathscr{R}(D)$ satisfying the condition depicted in the rightmost diagram in Figure 8 at each arc. We denote by $\mathrm{Col}_{(X,\uparrow)}(D)$ (resp. $\mathrm{Col}_{(X,\uparrow)}(D)_Y$

the set of $(X, \uparrow)$-colorings (resp. $(X, \uparrow)_Y$-colorings) of D. The well-definedness of an $(X, \uparrow)$-coloring (resp. $(X, \uparrow)_Y$-coloring) follows from

$$(a^{-1})^{-1} = a, \quad a^{-1} * b = (a * b)^{-1}, \ (a * b) * b^{-1} = a,$$
$$b(ab)^{-1} = a^{-1}, \ (ab)^{-1}a = b^{-1}.$$

The first three equalities are the defining conditions of a *good involution* considered in [Kamada 2007; Kamada and Oshiro 2010]. They used the notion of a good involution precisely to allow for appropriate changes of orientations. Following their arguments, we can show the following proposition in the same way as Proposition 7.

Proposition 11—*Let $X = \coprod_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. Let D be a diagram of a handlebody-link H. Let D′ be a diagram obtained by applying one of the* R1–R6 *moves to the diagram D once. For an $(X, \uparrow)$-coloring (resp. $(X, \uparrow)_Y$-coloring) C of D, there is a unique $(X, \uparrow)$-coloring (resp. $(X, \uparrow)_Y$-coloring) C′ of D′ which coincides with C except near a point where the move applied.*

Let $D_n^{\uparrow}(X)_Y$ be the subgroup of $P_n(X)_Y$ generated by the elements of the form

$$\langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \langle \boldsymbol{a} \rangle \cdots \langle \boldsymbol{a}_k \rangle + \langle y \rangle \langle \boldsymbol{a}_1 \rangle \cdots \langle \boldsymbol{a} \rangle_i^{-1} \cdots \langle \boldsymbol{a}_k \rangle,$$

where $\langle a_1, \ldots, a_m \rangle_i^{-1}$ denotes

$$\begin{cases} * \, a_1 \langle a_1^{-1}, a_1 a_2, a_3, \ldots, a_m \rangle & \text{if } i = 1, \\ \langle a_1, \ldots, a_{i-2}, a_{i-1} a_i, a_i^{-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_m \rangle & \text{if } i \neq 1, m, \\ \langle a_1, \ldots, a_{m-2}, a_{m-1} a_m, a_m^{-1} \rangle & \text{if } i = m. \end{cases}$$

The chain group $D_n^{\uparrow}(X)_Y$ will be called the *group of orientation degeneracies*. For example, $D_1^{\uparrow}(X)_Y$ is generated by the elements of the form

$$\langle y \rangle \langle a \rangle + \langle y * a \rangle \langle a^{-1} \rangle,$$

and $D_2^{\uparrow}(X)_Y$ is generated by the elements of the form

$$\langle y \rangle \langle a \rangle \langle b \rangle + \langle y * a \rangle \langle a^{-1} \rangle \langle b \rangle, \ \langle y \rangle \langle a \rangle \langle b \rangle + \langle y * b \rangle \langle a * b \rangle \langle b^{-1} \rangle,$$
$$\langle y \rangle \langle a, b \rangle + \langle y * a \rangle \langle a^{-1}, ab \rangle, \quad \langle y \rangle \langle a, b \rangle + \langle y \rangle \langle ab, b^{-1} \rangle.$$

We remark that the elements of the form

$$\langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots\langle a_1,\ldots,a_m\rangle\cdots\langle \boldsymbol{a}_k\rangle-(-1)^{m(m+1)/2}\langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots*(a_1\cdots a_m)\langle a_m^{-1},\ldots,a_1^{-1}\rangle\cdots\langle \boldsymbol{a}_k\rangle$$

belong to $D_n^{\uparrow}(X)_Y$. Furthermore, we can prove that the elements of the form

$$\langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots\langle a_1,\ldots,a_m\rangle\cdots\langle \boldsymbol{a}_k\rangle-(-1)^{i(i+1)/2}\langle y\rangle\langle \boldsymbol{a}_1\rangle\cdots*(a_1\cdots a_i)\langle a_i^{-1},\ldots,a_1^{-1},a_1\cdots a_{i+1},a_{i+2},\ldots,a_m\rangle\cdots\langle \boldsymbol{a}_k\rangle$$

belong to $D_n^{\uparrow}(X)_Y$ by induction.

Lemma 12— $D_*^{\uparrow}(X)_Y=(D_n^{\uparrow}(X)_Y,\partial_n)$ *is a subcomplex of* $P_*(X)_Y$.

Proof: We have $\partial_n(D_n^{\uparrow}(X)_Y)\subset D_{n-1}^{\uparrow}(X)_Y$, since

$$\begin{aligned}&\partial(\langle a_1,\ldots,a_m\rangle+*a_1\langle a_1^{-1},a_1a_2,a_3,\ldots,a_m\rangle)\\&=\langle a_1,a_2a_3,a_4,\ldots,a_m\rangle+*a_1\langle a_1^{-1},a_1a_2a_3,a_4,\ldots,a_m\rangle\\&+\sum_{i=3}^{m-1}(-1)^i\Big(\langle a_1,\ldots,a_ia_{i+1},a_{i+2},\ldots,a_m\rangle+*a_1\langle a_1^{-1},a_1a_2,a_2,\ldots,a_ia_{i+1},a_{i+2},\ldots,a_m\rangle\Big)\\&+(-1)^m(\langle a_1,\ldots,a_{m-1}\rangle+*a_1\langle a_1^{-1},a_1a_2,a_3,\ldots,a_{m-1}\rangle)\end{aligned}$$

and

$$\begin{aligned}&\partial(\langle a_1,\ldots,a_m\rangle+\langle a_1,\ldots,a_{i-1}a_i,a_i^{-1},a_ia_{i+1},a_{i+2},\ldots,a_m\rangle)\\&=*a_1\langle a_2,\ldots,a_m\rangle+*a_1\langle a_2,\ldots,a_{i-1}a_i,a_i^{-1},a_ia_{i+1},a_{i+2},\ldots,a_m\rangle\\&+\sum_{j=1}^{i-2}(-1)^j\Big(\langle a_1,\ldots,a_ja_{j+1},a_{j+2},\ldots,a_m\rangle+\langle a_1,\ldots,a_ja_{j+1},a_{j+2},\ldots,a_{i-1}a_i,a_i^{-1},a_ia_{i+1},a_{i+2},\ldots,a_m\rangle\Big)\\&+\sum_{j=i+1}^{m-1}(-1)^j\Big(\langle a_1,\ldots,a_ja_{j+1},a_{j+2},\ldots,a_m\rangle+\langle a_1,\ldots,a_{i-1}a_i,a_i^{-1},a_ia_{i+1},a_{i+2},\ldots,a_ja_{j+1},\ldots,a_m\rangle\Big)\\&+(-1)^m(\langle a_1,\ldots,a_{m-1}\rangle+\langle a_1,\ldots,a_{i-1}a_i,a_i^{-1},a_ia_{i+1},a_{i+2},\ldots,a_{m-1}\rangle).\end{aligned}$$

Thus $D_*^{\uparrow}(X)_Y$ is a subcomplex of $P_*(X)_Y$.

Definition 13—We set $C_n^{\uparrow}(X)_Y=P_n(X)_Y/(D_n(X)_Y+D_n^{\uparrow}(X)_Y)$. The quotient complex $(C_n^{\uparrow}(X)_Y,\partial_n)$ is denoted by $C_*^{\uparrow}(X)_Y$. For an abelian group A, we define the cochain complex $C^*_{\uparrow}(X;A)_Y=\mathrm{Hom}(C_*^{\uparrow}(X)_Y,A)$. We denote by $H_n^{\uparrow}(X)_Y$ the n-th homology group of $C_*^{\uparrow}(X)_Y$.

For an $(X, \uparrow)_Y$-coloring C of a diagram D for a handlebody-link, we define the *local chains* $w(\xi;C)$ at each crossing ξ and each vertex ξ of D as depicted in Figure 4. The local chain is well-defined, since

$$-\langle y * a\rangle\langle a^{-1}\rangle\langle b\rangle = \langle y\rangle\langle a\rangle\langle b\rangle = -\langle y * b\rangle\langle a * b\rangle\langle b^{-1}\rangle,$$
$$-\langle y * a\rangle\langle a^{-1}, ab\rangle = \langle y\rangle\langle a, b\rangle = -\langle y\rangle\langle ab, b^{-1}\rangle$$

in $C_2^{\uparrow}(X)_Y$ (see Figure 9). Then we can define the chain $W(D;C) \in C_2^{\uparrow}(X)_Y$ in the same way as $W(D;C) \in C_2(X)_Y$, and obtain invariants $\mathcal{H}(H)$, $\Phi_\theta(H)$ for an (unoriented) handlebody-link H.

7. Simplicial decomposition

The goal of this section is to prove Lemma 15 stating that $D_*(X)_Y$ is a subcomplex. The formula of $D_2(X)_Y$, when $\langle y\rangle$ is omitted, is written as

$$\langle a\rangle\langle b\rangle - \langle a, b\rangle + \langle b, a * b\rangle,$$

and its geometric interpretation is depicted in Figure 10. In (A), a colored triangle representing $\langle a, b\rangle$ is depicted, as well as its dual graph with a trivalent vertex. The colorings of such a graph were discussed in Section 5. A colored square representing $\langle a\rangle\langle b\rangle$ is depicted in (B), with the dual graph that corresponds to a crossing. In (C), a triangulation of the square is depicted, and after triangulation it represents $\langle a, b\rangle - \langle b, a * b\rangle$. Thus the triangulation corresponds to the above formula. This decomposition is found in [Carter et al. 2003].

At the same time, this equation corresponds to Y-oriented R4 moves in Figure 1 as follows. In Figure 11, colored diagrams of Y-oriented R4 moves are depicted. In the left diagram, the left-hand side represents the chain $\langle a\rangle\langle b\rangle + \langle b, a * b\rangle$ and the right-hand side represents $\langle a, b\rangle$. In the right diagram, the left-hand side represents the chain $-\langle a\rangle\langle b\rangle - \langle b, a * b\rangle$ and the right-hand side represents $-\langle a, b\rangle$. Thus the above equality is needed for colored diagrams to define equivalent chains in the quotient complex. A geometric interpretation of the last expression of $D_3(X)_Y$ omitting $\langle y\rangle$,

$$\langle a\rangle\langle b, c\rangle - \langle a, b, c\rangle + \langle b, a * b, c\rangle - \langle b, c, a * (bc)\rangle$$

is found in Figure 12. The symbol $\langle a\rangle$ is represented by the horizontal 1-simplex, $\langle b, c\rangle$ is represented by the right triangular face, and $\langle a\rangle\langle b, c\rangle$ is represented by a prism. The term $\langle a, b, c\rangle$ corresponds to the right top tetrahedron in the prism. The expressions of the form $\langle\langle a\rangle\langle b, c\rangle\rangle_i$ provides a triangulation of a product of simplices. Each term corresponds to

$$\begin{aligned}\langle a,b,c\rangle &= \langle\langle a\rangle\,\langle b,c\rangle\rangle_1,\\ \langle b,a*b,c\rangle &= -\langle\langle a\rangle\,\langle b,c\rangle\rangle_2,\\ \langle b,c,a*(bc)\rangle &= \langle\langle a\rangle\,\langle b,c\rangle\rangle_3\,.\end{aligned}$$

Below we use the notation

$$\begin{aligned}\partial_{(0)}\langle x_1,\dots,x_m\rangle &= *\,x_1\langle x_2,\dots,x_m\rangle,\\ \partial_{(i)}\langle x_1,\dots,x_m\rangle &= (-1)^i\langle x_1,\dots,x_ix_{i+1},\dots,x_m\rangle,\\ \partial_{(m)}\langle x_1,\dots,x_m\rangle &= (-1)^m\langle x_1,\dots,x_{m-1}\rangle\,.\end{aligned}$$

Then the boundaries of $\langle\langle a\rangle\,\langle b,c\rangle\rangle_i$ are computed as

$$\langle\langle a\rangle,\langle b,c\rangle\rangle_i \overset{\partial}{\mapsto} \partial_{(0)}\langle\langle a\rangle\,\langle b,c\rangle\rangle_i + \partial_{(1)}\langle\langle a\rangle\,\langle b,c\rangle\rangle_i + \partial_{(2)}\langle\langle a\rangle\,\langle b,c\rangle\rangle_i + \partial_{(3)}\langle\langle a\rangle\,\langle b,c\rangle\rangle_i$$

and the right-hand sides for $i = 1, 2, 3$ are computed as follows:

$$\begin{aligned}&\langle\langle a\rangle\,\langle b,c\rangle\rangle_1 \overset{\partial}{\mapsto} *\,a\langle b,c\rangle - \langle ab,c\rangle + \langle a,bc\rangle - \langle a,b\rangle\\ &= \langle(\partial_{(0)}\langle a\rangle)\,\langle b,c\rangle\rangle_1 + \partial_{(1)}\langle\langle a\rangle\,\langle b,c\rangle\rangle_1 - \langle\langle a\rangle\,\partial_{(1)}\langle b,c\rangle\rangle_1 - \langle\langle a\rangle\,\partial_{(2)}\langle b,c\rangle\rangle_1,\\ &\langle\langle a\rangle\,\langle b,c\rangle\rangle_2 \overset{\partial}{\mapsto} -\,*\,b\langle a*b,c\rangle + \langle b(a*b),c\rangle - \langle b,(a*b)c\rangle + \langle b,a*b\rangle\\ &= -\langle\langle a\rangle\,\partial_{(0)}\langle b,c\rangle\rangle_1 - \partial_{(1)}\langle\langle a\rangle\,\langle b,c\rangle\rangle_1 - \partial_{(2)}\langle\langle a\rangle\,\langle b,c\rangle\rangle_3 - \langle\langle a\rangle\,\partial_{(2)}\langle b,c\rangle\rangle_2,\\ &\langle\langle a\rangle\,\langle b,c\rangle\rangle_3 \overset{\partial}{\mapsto} *\,b\langle c,a*(bc)\rangle - \langle bc,a*(bc)\rangle + \langle b,c(a*(bc))\rangle - \langle b,c\rangle\\ &= -\langle\langle a\rangle\,\partial_{(0)}\langle b,c\rangle\rangle_2 - \langle\langle a\rangle\,\partial_{(1)}\langle b,c\rangle\rangle_2 + \partial_{(2)}\langle\langle a\rangle\,\langle b,c\rangle\rangle_3 + \langle(\partial_{(1)}\langle a\rangle)\,\langle b,c\rangle\rangle_1,\end{aligned}$$

where $\langle(\partial_{(i)}\langle a\rangle)\,\langle b,c\rangle\rangle_1$ is regarded as $(\partial_{(i)}\,\langle a\rangle)\langle b,c\rangle$. The canceling terms of the form $\partial_{(i)}\,\langle\langle a\rangle\,\langle b,c\rangle\rangle_j$ in the above boundaries correspond to internal triangles in Figure 12 that are shared by a pair of tetrahedra. Other terms are of the form $\langle\partial_{(i)}\,\langle a\rangle\,\langle b,c\rangle\rangle_j$ or $\langle\langle a\rangle\partial_{(i)}\,\langle b,c\rangle\rangle_j$, and they are outer triangles that constitute the boundary of the prism. The expression $\langle\partial_{(i)}\,\langle a\rangle\,\langle b,c\rangle\rangle_j$ represents the two triangles on the right and the left in Figure 12, since this represents

$$(\text{boundary of the interval represented by } \langle a\rangle) \times (\text{the triangle represented by } \langle b,c\rangle)\,.$$

Thus the outer boundary follows the pattern of Leibniz rule.

In terms of the coloring invariant of graphs, as in the case of the preceding relation for the Y-oriented R4 move, this relation corresponds to an equivalence of colored 2-complexes called foams, which are higher-dimensional analogues of the move depicted in Figure 11. See [Carter and Ishii 2012] for more on colored foams.

Lemma 14— *For* $\langle a\rangle = \langle a_1,\dots,a_s\rangle$ *and* $\langle b\rangle = \langle b_1,\dots,b_t\rangle$ *where* $a_i, b_j \in G_\lambda$, *we have*

$$\partial\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle = \langle(\partial\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle + (-1)^{|\boldsymbol{a}|}\langle\langle \boldsymbol{a}\rangle\langle\partial(\boldsymbol{b})\rangle\rangle,$$

where $\langle\langle\cdot\rangle\langle\cdot\rangle\rangle$ *is linearly extended.*

Proof: By definition, we have

$$\partial\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle = \sum_{i=0}^{s+t}\partial_{(i)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle = \sum_{i=0}^{s+t}\sum_{1\le i_1<\cdots<i_s\le s+t}\partial_{(i)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_s}.$$

Direct computations show that

$$\partial_{(0)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_s} = \begin{cases}\langle(\partial_{(0)}\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle_{i_2-1,\ldots,i_s-1} & \text{if } (i_1=1),\\ (-1)^s\langle\langle \boldsymbol{a}\rangle(\partial_{(0)}\langle \boldsymbol{b}\rangle)\rangle_{i_1-1,\ldots,i_s-1} & \text{if } (i_1>1),\end{cases}$$

$$\partial_{(i)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_s} = \begin{cases}\langle(\partial_{(k)}\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_k,i_{k+2}-1,\ldots,i_s-1} & \text{if } (i_k=i<i+1=i_{k+1}),\\ -\partial_{(i)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_{k-1},i_k+1,i_{k+1},\ldots i_s} & \text{if } (i_k=i<i+1<i_{k+1}),\\ -\partial_{(i)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_k,i_{k+1}-1,i_{k+2},\ldots,i_s} & \text{if } (i_k<i<i+1=i_{k+1}),\\ (-1)^s\langle\langle \boldsymbol{a}\rangle(\partial_{(i-k)}\langle \boldsymbol{b}\rangle)\rangle_{i_1,\ldots,i_k,i_{k+1}-1,\ldots,i_s-1} & \text{if } (i_k<i<i+1<i_{k+1}),\end{cases}$$

$$\partial_{(s+t)}\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_s} = \begin{cases}\langle(\partial_{(s)}\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_{s-1}} & \text{if } (i_s=s+t),\\ (-1)^s\langle\langle \boldsymbol{a}\rangle(\partial_{(t)}\langle \boldsymbol{b}\rangle)\rangle_{i_1,\ldots,i_s} & \text{if } (i_s<s+t).\end{cases}$$

The terms of the form $-\partial_{(i)}\langle\langle a\rangle\langle b\rangle\rangle$ i₁,…,i_{k–1},i_k+1,i_{k+1},…,i_s $(i_k = i < i+1 < i_{k+1})$ and $-\partial_{(i)}\langle\langle a\rangle\langle b\rangle\rangle$ i₁,…,i_k,i_{k+1}–1,i_{k+2},…,i_s $(i_k < i < i+1 = i_{k+1})$ cancel in pairs. The other terms are organized as

$$\begin{aligned}&\sum_{\substack{1\le i_1<\cdots\\<i_{s-1}\le s+t-1}}\sum_{i=0}^{s}\langle(\partial_{(i)}\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_{s-1}} + \sum_{\substack{1\le i_1<\cdots\\<i_s\le s+t-1}}\sum_{i=0}^{t}(-1)^s\langle\langle \boldsymbol{a}\rangle(\partial_{(i)}\langle \boldsymbol{b}\rangle)\rangle_{i_1,\ldots,i_s}\\ &= \sum_{\substack{1\le i_1<\cdots\\<i_{s-1}\le s+t-1}}\langle(\partial\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle_{i_1,\ldots,i_{s-1}} + \sum_{\substack{1\le i_1<\cdots\\<i_s\le s+t-1}}(-1)^s\langle\langle \boldsymbol{a}\rangle(\partial\langle \boldsymbol{b}\rangle)\rangle_{i_1,\ldots,i_s}\\ &= \langle(\partial\langle \boldsymbol{a}\rangle)\langle \boldsymbol{b}\rangle\rangle + (-1)^s\langle\langle \boldsymbol{a}\rangle(a\langle \boldsymbol{b}\rangle)\rangle,\end{aligned}$$

where $\langle\cdot\rangle$ i₁,…,i_s is linearly extended.

Since the Leibniz rule holds (by the preceding Lemma 14), we have the following.

Lemma 15—*$D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $P_*(X)_Y$.*

8. Chain map for simplicial decomposition

In this section we examine relations between group and MCQ homology theories.

8.1. Simplicial decomposition (general case)

We observe an associativity of the notation $\langle\langle a\rangle\langle b\rangle\rangle$ defined in Section 3, and extend the notation to multi-tuples. For an expression of the form $\langle a\rangle\langle b\rangle\langle c\rangle$ in a chain in $P_*(X)_Y$, where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \cup_{m \in \mathbb{N}} G_\lambda^m$, it is easy to see that we have the following.

Lemma 16

$$\langle\langle\langle \boldsymbol{a}\rangle\langle \boldsymbol{b}\rangle\rangle\langle \boldsymbol{c}\rangle\rangle = \langle\langle \boldsymbol{a}\rangle\langle\langle \boldsymbol{b}\rangle\langle \boldsymbol{c}\rangle\rangle\rangle.$$

By Lemma 16, we can define $\langle\langle a\rangle\langle b\rangle\langle c\rangle\rangle$ by $\langle\langle\langle a\rangle\langle b\rangle\rangle\langle c\rangle\rangle = \langle\langle a\rangle\langle\langle b\rangle\langle c\rangle\rangle\rangle$. Moreover, for an expression of the form $\langle a_1\rangle \cdots \langle a_k\rangle$ in a chain in $P_*(X)_Y$, where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k \in \cup_{m \in \mathbb{N}} G_\lambda^m$, we can define $\langle\langle a_1\rangle \cdots \langle a_k\rangle\rangle$ inductively. By Lemma 14, this notation is compatible with the boundary homomorphism ∂ in the following sense.

Lemma 17

$$\partial\langle\langle \boldsymbol{a}_1\rangle\cdots\langle \boldsymbol{a}_k\rangle\rangle = \langle\partial(\langle \boldsymbol{a}_1\rangle\cdots\langle \boldsymbol{a}_k\rangle)\rangle.$$

We give a direct formula (instead of induction) for the notation $\langle\langle a_1\rangle \cdots \langle a_k\rangle\rangle$ later in Section 8.3.

8.2. Chain map (from MCQ to group)

Let $X = \coprod_{\lambda\in\Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. Let $P_n^G(X)_Y$ be the subgroup of $P_n(X)_Y$ generated by the elements of the form $\langle y\rangle\langle a\rangle$. Let $D_n^G(X)_Y$ and $D_n^{G,\uparrow}(X)_Y$ be respectively $P_n^G(X)_Y \cap D_n(X)_Y$ and $P_n^G(X)_Y \cap D_n^{\uparrow}(X)_Y$, which are the subgroups of $P_n^G(X)_Y$. Note that $D_n^G(X)_Y = P_n^G(X)_Y \cap D_n(X)_Y$ is the trivial group. We put

$$C_n^G(X)_Y := P_n^G(X)_Y / D_n^G(X)_Y = P_n^G(X)_Y,$$
$$C_n^{G,\uparrow}(X)_Y := P_n^G(X)_Y / (D_n^G(X)_Y + D_n^{G,\uparrow}(X)_Y) = P_n^G(X)_Y / D_n^{G,\uparrow}(X)_Y.$$

Then $C_*^G(X)_Y = (C_n^G(X)_Y, \partial_n)$ and $C_*^{G,\uparrow}(X)_Y = (C_n^{G,\uparrow}(X)_Y, \partial_n)$ are chain complexes. If X is a group (regarded as $X = \coprod_{\lambda\in\Lambda} G_\lambda$ with Λ a singleton) and Y is a singleton, $C_*^G(X)_Y$ is essentially the same as the chain complex of the usual group homology. For an abelian group A, we define the cochain complexes

$$C_G^*(X;A)_Y = \mathrm{Hom}(C_*^G(X)_Y, A) \quad \text{and} \quad C_{G,\uparrow}^*(X;A)_Y = \mathrm{Hom}(C_*^{G,\uparrow}(X)_Y, A).$$

When X is a multiple conjugation quandle consisting of a single group, define homomorphisms $\Delta: P_*(X)_Y \to P_*^G(X)_Y$ by

$$\Delta(\langle a_1\rangle \cdots \langle a_m\rangle) := \langle\langle a_1\rangle \cdots \langle a_m\rangle\rangle.$$

Then by Lemma 17 and from these definitions, we have the following.

Proposition 18—*The homomorphisms $\Delta: P_*(X)_Y \to P_*^G(X)_Y$ give rise to a chain homomorphism. Furthermore, Δ induces the chain homomorphisms $\Delta: C_*(X)_Y \to C_*^G(X)_Y$ and $\Delta: C_*^{\uparrow}(X)_Y \to C_*^{G,\uparrow}(X)_Y$.*

When $n = 0, 1$, the induced homomorphisms $\Delta: C_n(X)_Y \to C_n^G(X)_Y$ and $\Delta: C_n^{\uparrow}(X)_Y \to C_n^{G,\uparrow}(X)_Y$ are identities. Furthermore $H_n(X)_Y \cong H_n^G(X)_Y$ and $H_n^{\uparrow}(X)_Y \cong H_n^{G,\uparrow}(X)_Y$ for $n = 0, 1$. We note that the chain homomorphisms Δ are defined only for an MCQ consisting of a single group. In this case, we also have the cochain homomorphisms $\Delta: C_G^*(X;A)_Y \to C^*(X;A)_Y$ and $\Delta: C_{G,\uparrow}^*(X;A)_Y \to C_{\uparrow}^*(X;A)_Y$ for an abelian group A. Hence, for a given cocycle of group homology theory, we can obtain that of our theory through Δ. This approach will be discussed in Section 10.

Remark 19—We point out here that for a group $X = \mathbb{Z}_3$ and a trivial X-set Y, there is a group 2-cocycle η that satisfies the conditions in $C_{G,\uparrow}^2(X)_Y$ (coming from $D_n^{G,\uparrow}(X)_Y$),

$$\eta\langle a,b\rangle + \eta\langle a^{-1}, ab\rangle = 0 \quad \text{and} \quad \eta\langle a,b\rangle + \eta\langle ab, b^{-1}\rangle = 0.$$

Specifically, let $\eta: \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{Z}_3$ denote the function that has values $\eta(1,1) = 1$, $\eta(2,2) = 2$ and $\eta(g,h) = 0$ otherwise. It is a direct calculation that the condition above is satisfied. Furthermore, to see that η is a cocycle, consider the generating cocycle over $G = \mathbb{Z}_p$ where p is a prime that is defined by

$$\eta_0(x,y) = (1/p)(\bar{x} + \bar{y} - \overline{x+y}) \pmod{p},$$

where $\bar{x}$ is an integer $0 \le \bar{x} < p$ such that $\bar{x} = x \pmod{p}$. It is known that η_0 is a generating 2-cocycle for $H_G^2(\mathbb{Z}_p; \mathbb{Z}_p)$ for prime p. For $p = 3$, let ζ be a 1-chain defined by $\zeta(0) = 0$ and $\zeta(1)+\zeta(2) = 2$. Then one can easily compute that $\eta = \eta_0 + \delta\zeta$. Hence there is a 2-cocycle $\eta \in C_{G,\uparrow}^2(X)_Y$ of our theory that is cohomologous to the standard group 2-cocycle η_0.

8.3. Simplicial decomposition (direct formula)

We give a direct formula (instead of induction) for the notation $\langle\langle a_1\rangle \cdots \langle a_k\rangle\rangle$. To the term $\langle\langle a\rangle\langle b\rangle\rangle$ i₁,…,i_s, we associate a vector $\boldsymbol{v} = (v_1, \dots, v_n) \in \{1, 2\}^n$ by defining $v_i = 1$ if $i = i_j$ for some j, and otherwise $v_i = 2$, where $n = s + t$. In the term

$$c_i = \begin{cases} a_k * (b_1 \cdots b_{i-k}) & \text{if } i = i_k, \\ b_{i-k} & \text{if } i_k < i < i_{k+1}, \end{cases}$$

the first entry with a_k in it corresponds to $v_i = 1$ and the second with b_{i-k} to $v_i = 2$. We note that the term a_k came from the first part $\langle a\rangle$ in $\langle\langle a\rangle\langle b\rangle\rangle$ i₁,…,i_s so that $v_i = 1$ is assigned, and the term b_{i-k} belongs to the second part $\langle b\rangle$ receiving $v_i = 2$.

Example 20—For the term $\langle a\rangle\langle b, c\rangle$ discussed for Figure 12, the terms $\langle a, b, c\rangle$, $-\langle b, a * b, c\rangle$, and $\langle b, c, a * (bc)\rangle$ correspond to the vectors (1, 2, 2), (2, 1, 2), and (2, 2, 1), respectively. Note that (2, 1, 2) is obtained from (1, 2, 2) by a transposition of the first two entries, and this is reflected in Figure 12 by the fact that the tetrahedra represented by these vectors share a triangular internal face. We indicate by an edge between two vectors when one is obtained from the other by a transposition of consecutive entries. In this case we draw the graph:

$$(1, 2, 2) \text{ --- } (2, 1, 2) \text{ --- } (2, 2, 1)\,.$$

For $\langle a, b\rangle\langle c, d\rangle$, the terms $\langle\langle a\rangle\langle b\rangle\rangle$ i₁,…,i_s are listed as $\langle a, b, c, d\rangle$, $-\langle a, c, b*c, d\rangle$, $\langle c, a*c, b*c, d\rangle$, $\langle a, c, d, b*(cd)\rangle$, $-\langle c, a*c, d, b*(cd)\rangle$, $\langle c, d, a*(cd), b*(cd)\rangle$, and these correspond to vectors

$$(1, 1, 2, 2),\ (1, 2, 1, 2),\ (2, 1, 1, 2),\ (1, 2, 2, 1),\ (2, 1, 2, 1),\ (2, 2, 1, 1),$$

respectively. They are connected by edges as

$$\begin{array}{ccccc} & & (2, 1, 1, 2) & & \\ & / & & \backslash & \\ (1, 1, 2, 2) - (1, 2, 1, 2) & & & & (2, 1, 2, 1) - (2, 2, 1, 1) \\ & \backslash & & / & \\ & & (1, 2, 2, 1) & & \end{array}$$

indicating which simplices share internal faces. Note that from a vector $\boldsymbol{v} = (v_1, \dots, v_n) \in \{1, 2\}^n$ the subscripts $i_1, \dots, i_s$ in $\langle\langle a\rangle\langle b\rangle\rangle$ i₁,…,i_s are recovered by the condition v_i _j = 1.

For an expression of the form $\langle a_1\rangle \cdots \langle a_k\rangle$ in a chain in $P_*(X)_Y$, where

$$\boldsymbol{a}_1, \dots, \boldsymbol{a}_k \in \bigcup_{m \in \mathbb{N}} G_\lambda^m,$$

we put $n = |a_1| + \cdots + |a_k|$ and consider vectors $\boldsymbol{v} = (v_1, \ldots, v_n) \in \{1, \ldots, k\}^n$, and denote by $\#_j^i \boldsymbol{v}$ the number of j's in $v_1, \ldots, v_i$. Then for a given $\boldsymbol{v}$ define $i(j, 1) < \cdots < i(j, n_j)$ by the condition that $v_{i(j,1)} = \cdots = v_{i(j,n_j)} = j$.

With these notations in hand, we temporarily define $\langle\langle a_1\rangle \cdots \langle a_k\rangle\rangle'$ by

$$\sum_{\substack{\boldsymbol{v} \in \{1, \ldots, k\}^n \\ \#_j^n \boldsymbol{v} = n_j\,(j = 1, \ldots, k)}} (-1)^{\sum_{j=1}^{k-1} \sum_{t=1}^{n_j} (i(j,t) - t - \sum_{s=1}^{j-1} n_s)} \langle c_1, \ldots, c_n\rangle$$

for $\langle a_1\rangle \cdots \langle a_k\rangle = \langle a_{1,1}, \ldots, a_{1,n_1}\rangle \cdots \langle a_{k,1}, \ldots, a_{k,n_k}\rangle$, where

$$c_i = a_{v_i, \#_{v_i}^i \boldsymbol{v}} * \prod_{s = v_i + 1}^{k} \prod_{t=1}^{\#_s^i \boldsymbol{v}} a_{s,t}\,.$$

Then we have $\langle\langle a_1\rangle \cdots \langle a_k\rangle\rangle' = \langle\langle a_1\rangle \cdots \langle a_k\rangle\rangle$, from the fact that simplices of both sides are in one-to-one correspondence with vectors $\boldsymbol{v} = (v_1, \ldots, v_n) \in \{1, \ldots, k\}^n$, and the signs correspond to the number of transpositions, modulo 2, of a given vector $\boldsymbol{v}$ from the vector $(1, \ldots, 1, 2, \ldots, 2, \ldots, k, \ldots, k)$.

Example 21—The terms of $\langle\langle a\rangle \langle b, c\rangle \langle d\rangle\rangle$ consist of

$$\begin{array}{lll} \langle a, b, c, d\rangle, & \langle b, a * b, c, d\rangle, & \langle a, b, d, c * d\rangle, \\ \langle b, c, a * (bc), d\rangle, & \langle b, a * b, d, c * d\rangle, & \langle a, d, b * d, c * d\rangle, \\ \langle b, c, d, a * (bcd)\rangle, & \langle b, d, a * (bd), c * d\rangle, & \langle d, a * d, b * d, c * d\rangle, \\ \langle b, d, c * d, a * (bcd)\rangle, & \langle d, b * d, a * (bd), c * d\rangle, & \langle d, b * d, c * d, a * (bcd)\rangle, \end{array}$$

which, respectively, correspond to the vectors

$$\begin{array}{lll} (1,2,2,3), & (2,1,2,3), & (1,2,3,2), \\ (2,2,1,3), & (2,1,3,2), & (1,3,2,2), \\ (2,2,3,1), & (2,3,1,2), & (3,1,2,2), \\ (2,3,2,1), & (3,2,1,2), & (3,2,2,1)\,. \end{array}$$

The graph representing shared faces is depicted in Figure 13.

9. Relationship between MCQ and IIJO

Let $X = \coprod_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation quandle, and let Y be an X-set. Let $P_n^{\mathrm{IIJO}}(X)_Y$ be the subgroups of $P_n(X)_Y$ generated by the elements of the form $\langle y\rangle \langle a_1\rangle \cdots \langle a_n\rangle$. Then

$P_*^{\mathrm{IIJO}}(X)_Y = (P_n^{\mathrm{IIJO}}(X)_Y, \partial_n)$ is a subcomplex of $P_*(X)_Y$. Let $D_n^{\mathrm{IIJO}}(X)_Y$ be the subgroup of $P_n^{\mathrm{IIJO}}(X)_Y$ generated by the elements of the forms

$$\langle y \rangle \langle a_1 \rangle \cdots \langle b_1 \rangle \langle b_2 \rangle \cdots \langle a_n \rangle, \quad \langle y \rangle \langle a_1 \rangle \cdots \partial \langle b_1, b_2 \rangle \cdots \langle a_n \rangle$$

for $a_1, \ldots, a_n \in X$ and $b_1, b_2 \in G_\lambda$. We note that the former elements relate to the invariance under the R1 and R4 move, and that the latter elements relate to the invariance under the R5 move and reversing orientation.

Lemma 22— $D_*^{IIJO}(X)_Y = (D_n^{IIJO}(X)_Y, \partial_n)$ *is a subcomplex of* $P_*^{IIJO}(X)_Y$.

Proof: This follows from

$$\partial(\langle b_1 \rangle \langle b_2 \rangle) = \partial \langle b_1, b_2 \rangle - \partial \langle b_2, b_1 * b_2 \rangle, \quad \partial(\partial \langle b_1, b_2 \rangle) = 0$$

for $b_1, b_2 \in G_\lambda$.

We put

$$C_n^{\mathrm{IIJO}}(X)_Y = P_n^{\mathrm{IIJO}}(X)_Y / D_n^{\mathrm{IIJO}}(X)_Y .$$

Then $C_*^{\mathrm{IIJO}}(X)_Y = (C_n^{\mathrm{IIJO}}(X)_Y, \partial_n)$ is a chain complex. If X is obtained from a G-family of quandles as in Example 2, $C_*^{\mathrm{IIJO}}(X)_Y$ is the chain complex defined in [Ishii et al. 2013]. For an abelian group A, we define the cochain complexes

$$C_{\mathrm{IIJO}}^*(X; A)_Y = \mathrm{Hom}(C_*^{\mathrm{IIJO}}(X)_Y, A) .$$

We note that a natural projection $\mathrm{pr}_*: P_*(X)_Y \to P_*^{\mathrm{IIJO}}(X)_Y$ does not induce a chain homomorphism $\mathrm{pr}_*: C_*(X)_Y \to C_*^{\mathrm{IIJO}}(X)_Y$, since IIJO homology theory is invariant under the invariance for reversing orientations. (See Table 1.) It is seen, however, that this map induces the chain homomorphism $\mathrm{pr}_*: C_*^{\dagger}(X)_Y \to C_*^{\mathrm{IIJO}}(X)_Y$ and the cochain homomorphism $\mathrm{pr}^*: C_{\mathrm{IIJO}}^*(X; A)_Y \to C_{\dagger}^*(X; A)_Y$ for an abelian group A. Hence, for a given cocycle of IIJO homology theory (with some modification for a multiple conjugation quandle as above), we can obtain that of our theory through pr^*. This implies that our invariant is a generalization of the IIJO quandle cocycle invariant.

10. Towards finding 2-cocycles

We discuss approaches to finding 2-cocycles that are not induced from the IIJO (co)homology theory. Let G be a group, M a right G-module, and A an abelian group. The module M and the set $X = M \times G$ $(= \coprod_{x \in M} \{x\} \times G)$ can be considered as a G-family of quandles and a multiple conjugation quandle as in Example 2, respectively.

We take an X-set Y as a singleton $\{y_0\}$ and suppress the notation $\langle y_0 \rangle$. For a 2-cocycle $\psi \in P^2(X; A)_Y$, we denote $\psi(\langle (x, g) \rangle \langle (y, h) \rangle)$ by $\phi((x, g), (y, h))$, and $\psi(\langle (x, g), (x, h) \rangle)$ by $\eta_x(g, h)$. Then the 2-cocycle conditions are written as

1. $\eta_x(g, h) + \eta_x(gh, k) = \eta_x(h, k) + \eta_x(g, hk)$,
2. $\phi((x, g), (y, k)) + \phi((x, h), (y, k)) - \phi((x, gh), (y, k)) = \eta_x(g, h) - \eta_{x *^{k} y}(g * k, h * k)$,
3. $\phi((x, g), (y, h)) + \phi((x *^h y, g * h), (y, k)) = \phi((x, g), (y, hk))$,
4. $\phi((x, g), (y, h)) + \phi((x *^h y, g * h), (z, k)) = \phi((x, g), (z, k)) + \phi((x *^k z, g * k), (y *^k z, h * k))$,

   where $x, y, z \in M$ and $g, h, k \in G$. Furthermore, for a 2-cochain $\psi \in P^2(X; A)_Y$, the condition that ψ is a 2-cochain in $C^2(X; A)_Y$ is written as

5. $\phi((x, g), (x, h)) = \eta_x(g, h) - \eta_x(h, g * h)$,

   where $x \in M$ and $g, h \in G$.

Towards constructing MCQ 2-cocycles that are not from the IIJO homology, first we note that if ϕ above is an IIJO 2-cocycle, then ϕ satisfies the conditions (3),(4), and the condition that the LHS of (2) vanishes. By considering $\psi' = \psi - \phi$, we obtain an MCQ 2-cocycle ψ' that consists only of terms of η_x for $x \in M$. Thus we first consider such a case in Example 23 below. In this case, we can take an approach described in Section 8 for finding MCQ cocycles from group cocycles.

Example 23—For a 2-cochain $\psi \in P^2(X; A)_Y$ with the assumption

(0) $\psi(\langle (x, g) \rangle \langle (y, h) \rangle)$ $(= \phi((x, g), (y, h))) = 0$,

we discuss what conditions are needed for the 2-cochain ψ being a 2-cocycle in $P^2(X; A)_Y$. When we use the notation $\eta_x(g, h)$ for $\psi(\langle (x, g), (x, h) \rangle)$, the 2-cocycle conditions are written as

(1) $\eta_x(g, h) + \eta_x(gh, k) = \eta_x(h, k) + \eta_x(g, hk)$,

(2′) $\eta_x(g, h) - \eta_{x *^{k} y}(g * k, h * k) = 0$,

where $x, y \in M$ and $g, h, k \in G$. We note that the condition (0) implies (3) and (4). Furthermore, for a 2-cochain $\psi \in P^2(X; A)_Y$ with the assumption (0), the condition that ψ is a 2-cochain in $C_2(X; A)_Y$ are written as

(5′) $\eta_x(g, h) - \eta_x(h, g * h) = 0$, where $x \in M$ and $g, h \in G$. Hence if ψ satisfies (0),(1), (2′) and (5′), then ψ is a 2-cocycle in $C^2(X; A)_Y$ and defines an invariant for handlebody-knots.

If $y = x$, then (2′) implies $\eta_x(g * k, h * k) = \eta_x(g, h)$, called the *right invariance* of η_x. If $x = 0$, then (2′) with right invariance implies $\eta_{y \cdot (1-k)} \equiv \eta_0$, which is another necessary condition for the condition (2′). Hence if any element in M can be represented by the form $y \cdot (1 - k)$ for some $y \in M$ and $k \in G$, then we have $\eta_x \equiv \eta_0$ for any $x \in M$. In this case, we can check that the 2-cocycle ψ in $C^2(X; A)_Y$ comes from the dual of the composition of the chain homomorphisms

$$C_*(X)_Y \xrightarrow{\text{pr}_2} C_*(G)_Y \xrightarrow{\Delta} C_*^G(G)_Y,$$

where a chain homomorphism pr_2 is induced from a natural projection into the second factor and the chain homomorphism Δ was defined in Section 8.2. In this case, ψ assigned at a crossing is decomposed into a pair of weights η corresponding to trivalent vertices as depicted in Figure 10 (B) and (C). Hence the resulting invariant is equivalent to the invariant of the trivalent graph obtained by replacing all crossings with vertices, that is, embedded in the 2-sphere without crossing. Such an embedded graph is equivalent to a circle with small bubbles, and has trivial invariant value ($W(D; C) = 0$ for any coloring C). Thus, in this case, ψ defines a trivial invariant for handlebody-knots by the group 2-cocycle η_0, whose cohomology class may not be zero in $H_G^2(G; A)_Y$.

If the condition that any element in M can be represented by the form $y \cdot (1-k)$ for some $y \in M$ and $k \in G$ is not satisfied, then ψ satisfying (0), (1), (2′) and (5′) may give rise to a nontrivial invariant for handlebody-links.

Example 24—In contrast to Example 23, next we consider the case when ϕ is not an IIJO 2-cocycle, so that the LHS of (2) does not vanish for ϕ.

For any G-invariant A-bilinear map $f: M^2 \to A$, Theorem 5.2 of [Nosaka 2013] claimed that the map $\phi_f: X^2 \to A$ defined by

$$\phi_f((x, g), (y, h)) := f(x - y, y \cdot (1 - h^{-1}))$$

satisfies the conditions (3) and (4) above. For the G-invariant A-bilinear map f, if we can find maps η_x such that the conditions (1) and (2) are also satisfied, then we obtain a 2-cocycle, which may be new. We remark here that ϕ_f itself can be modified as in [Nosaka 2013, Corollary 4.7] (by using an additive homomorphism form G to some commutative ring) so that the conditions (1) and (2) are also satisfied under the assumption $\eta_x \equiv 0$ for any $x \in M$.

The condition (1) merely says that η_x is a usual group 2-cocycle for any $x \in M$. The condition (2) is equivalent to

(2″) $f(x-y, y\cdot(1-k^{-1})) = \eta_x(g, h) - \eta_{x *^{k} y}(g * k, h * k)$

from the definition of f. If $y = x$, then (2″) implies that η_x is right invariant in the sense that $\eta_x(g * k, h * k) = \eta_x(g, h)$ as above. If $y = 0$, then (2″) with the right invariance implies $\eta_{x \cdot k} \equiv \eta_x$, called the *orbit dependence* of η_x. Thus we obtain these two necessary conditions for the condition (2″).

We examine the following specific examples. For a prime number p, let G denote $SL(2, \mathbb{Z}_p)$ that acts on $M = (\mathbb{Z}_p)^2$ from the right. For $A = \mathbb{Z}_p$, the map $f: M^2 \to A$ defined by $f(x, y) := \det\begin{pmatrix} x \\ y \end{pmatrix}$ is a G-invariant A-bilinear map, where $x, y \in M$ are row vectors on which G acts on the right, and det denotes the determinant. This setting is motivated from [Nosaka 2013, Proposition 4.5].

First, we consider the case where $p = 2$. Define $m: M \to A$ by

$$m(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0. \end{cases}$$

Then we can check that

$$\phi_f((x, g), (y, h)) = -m(x) + m(x *^h y)$$

for any $x, y \in M$ and $g, h \in G$. Take $\eta_x(g, h)$ to be $m(x)$ for any $x \in M$ and $g, h \in G$. Then we can show that the 2-cochain ψ, defined by ϕ_f and η_x, is a 2-coboundary as follows. Define a 1-cochain $\tilde{m} \in P^1(X; A)$ by $\tilde{m}(\langle\langle(x, g)\rangle\rangle) := m(x)$. Then the 2-coboundary $\delta\tilde{m} \in P^2(X; A)$ is written as

$$\begin{aligned} (\delta\tilde{m})(\langle\langle(x, g)\rangle\, \langle(y, h)\rangle\rangle) &= -m(x) + m(x *^h y), \\ (\delta\tilde{m})(\langle\langle(x, g), (x, h)\rangle\rangle) &= m(x), \end{aligned}$$

where $x, y \in M$ and $g, h \in G$. This implies that $\psi = \delta\tilde{m}$.

Second, we consider the case where $p > 2$. If $x = (0, 0)$ and $k = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, the condition (2″) implies $\eta_{2y}(g, h) = \eta_0(g, h)$ for any $y \in M$ and $g, h \in G$. Since p is odd, we have that $\eta_x \equiv \eta_0$ for any $x \in M$. If we substitute $y = (1, 0)$ and $k = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ for (2″), then LHS is 1 and RHS is 0, which turns out to be a contradiction. Hence there is no choice of η_x such that the conditions (1) and (2″) are satisfied.

Although our attempts have not resulted in new nontrivial 2-cocycles, it appears useful to record our approaches and facts we have found, for future endeavors towards constructing new cocycles using these approaches. Further studies are desirable on this homology theory, as it unifies group and quandle homology theories for a structure of multiple conjugation quandles, which have ample interesting examples and applications to handlebody-links.

Acknowledgments

Carter was partially supported by the Simons Foundation. Ishii was partially supported by JSPS KAKENHI Grant Number 24740037. Saito was partially supported by (U.S.) NIH R01GM109459. Tanaka was partially supported by JSPS KAKENHI Grant Number 26400082. The authors are grateful to Daniel Moskovich for valuable comments on exposition, and Yusuke Iijima for pointing out an error in an earlier version.

References

[Brieskorn 1988]. Brieskorn, E. Braids. Santa Cruz, CA: 1986. Automorphic sets and braids and singularities; 45–115. Contemp. Math. 78, Amer. Math. Soc., Providence, RI, 1988. MR Zbl

[Carter and Ishii 2012]. Carter JS, Ishii A. A knotted 2-dimensional foam with non-trivial cocycle invariant. 2012

[Carter et al 2001]. Carter JS, Kamada S, Saito M. Geometric interpretations of quandle homology. J Knot Theory Ramifications. 10(3)2001; :345–386.

[Carter et al 2003]. Carter JS, Jelsovsky D, Kamada S, Langford L, Saito M. Quandle cohomology and state-sum invariants of knotted curves and surfaces. Trans Amer Math Soc. 355(10)2003; :3947–3989.

[Fenn and Rourke 1992]. Fenn R, Rourke C. Racks and links in codimension two. J Knot Theory Ramifications. 1(4)1992; :343–406.

[Fenn et al 1995]. Fenn R, Rourke C, Sanderson B. Trunks and classifying spaces. Appl Categ Structures. 3(4)1995; :321–356.

[Ishii 2008]. Ishii A. Moves and invariants for knotted handlebodies. Algebr Geom Topol. 8(3)2008; :1403–1418.

[Ishii 2015a]. Ishii A. The Markov theorems for spatial graphs and handlebody-knots with Y-orientations. Internat J Math. 26(14)2015; :1550116, 23.

[Ishii 2015b]. Ishii A. A multiple conjugation quandle and handlebody-knots. Topology Appl. 196(part B)2015; :492–500.

[Ishii and Iwakiri 2012]. Ishii A, Iwakiri M. Quandle cocycle invariants for spatial graphs and knotted handlebodies. Canad J Math. 64(1)2012; :102–122.

[Ishii et al 2013]. Ishii A, Iwakiri M, Jang Y, Oshiro K. A *G*-family of quandles and handlebody-knots. Illinois J Math. 57(3)2013; :817–838.

[Joyce 1982]. Joyce D. A classifying invariant of knots, the knot quandle. J Pure Appl Algebra. 23(1)1982; :37–65.

[Kamada 2007]. Kamada, S. Intelligence of low dimensional topology 2006. World Scientific; Hackensack, NJ: 2007. Quandles with good involutions, their homologies and knot invariants; 101–108. Ser. Knots Everything 40MR Zbl

[Kamada and Oshiro 2010]. Kamada S, Oshiro K. Homology groups of symmetric quandles and cocycle invariants of links and surface-links. Trans Amer Math Soc. 362(10)2010; :5501–5527.

[Matveev 1982]. Matveev SV. Distributive groupoids in knot theory. Mat Sb (NS). 119(1)1982; :78–88.

[Nosaka 2013]. Nosaka T. Quandle cocycles from invariant theory. Adv Math. 2452013; :423–438.

[Przytycki 2011]. Przytycki JH. Distributivity versus associativity in the homology theory of algebraic structures. Demonstratio Math. 44(4)2011; :823–869.

[Takasaki 1943]. Takasaki M. Abstraction of symmetric transformations. Tôhoku Math J. 491943; :145–207.

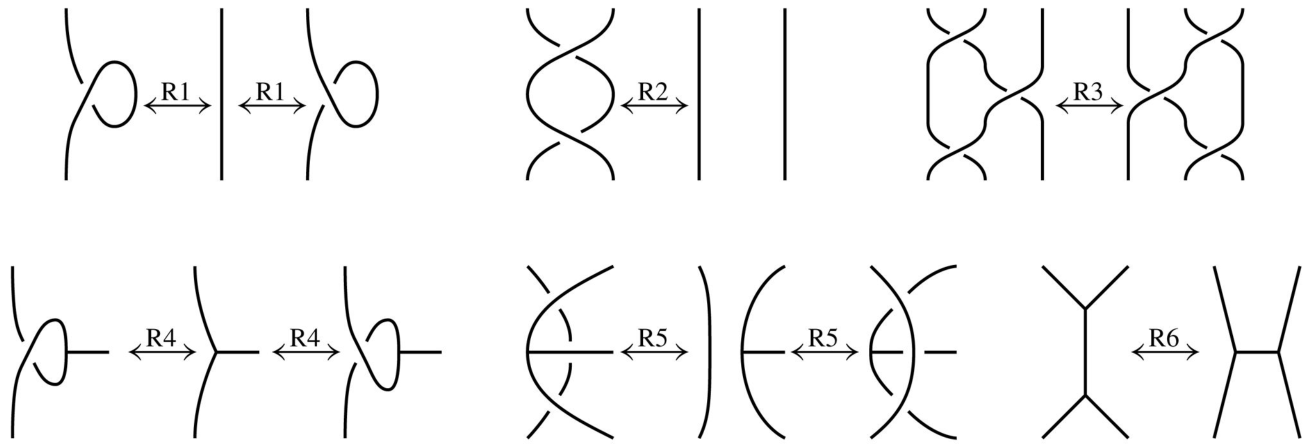


Figure 1.
Reidemeister moves for handlebody-links.

Figure 2.
Y-orientation.

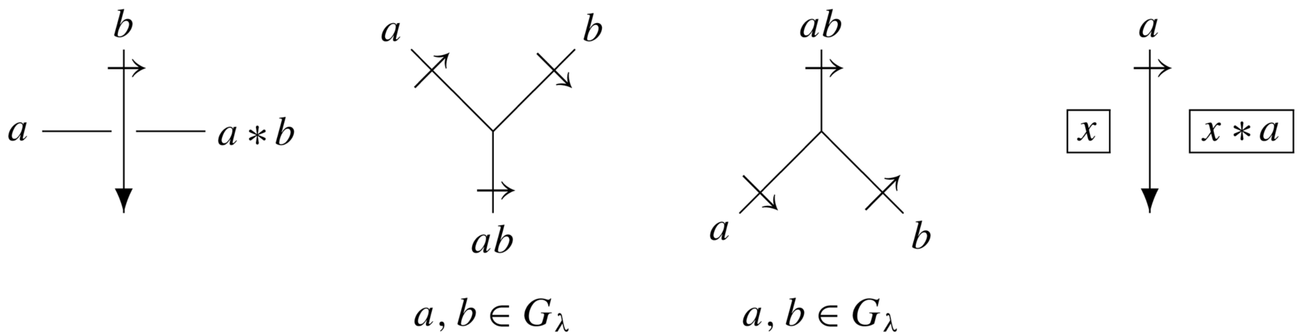


Figure 3.
Rules of a coloring.

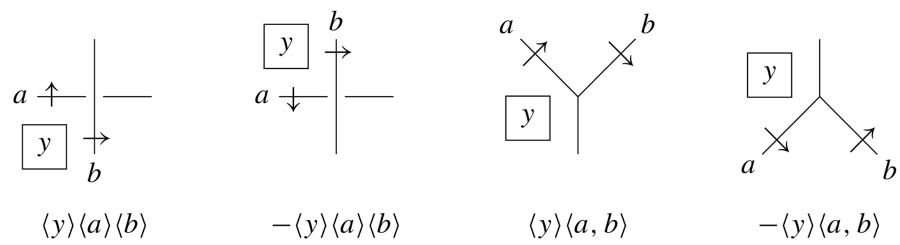


Figure 4.
Local chains represented by crossings and vertices.

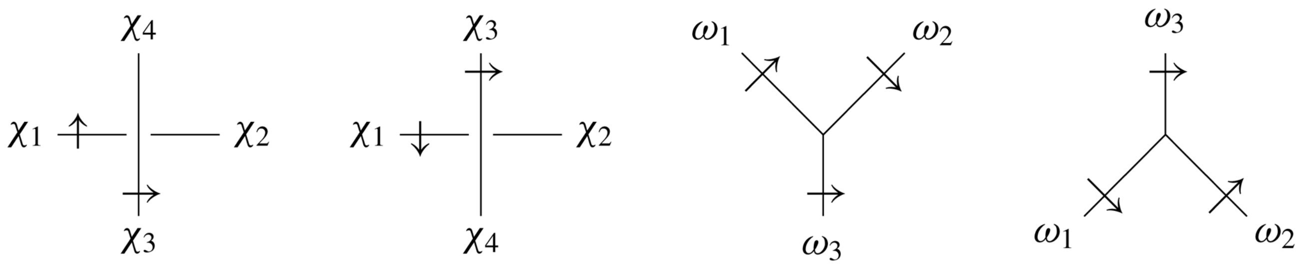
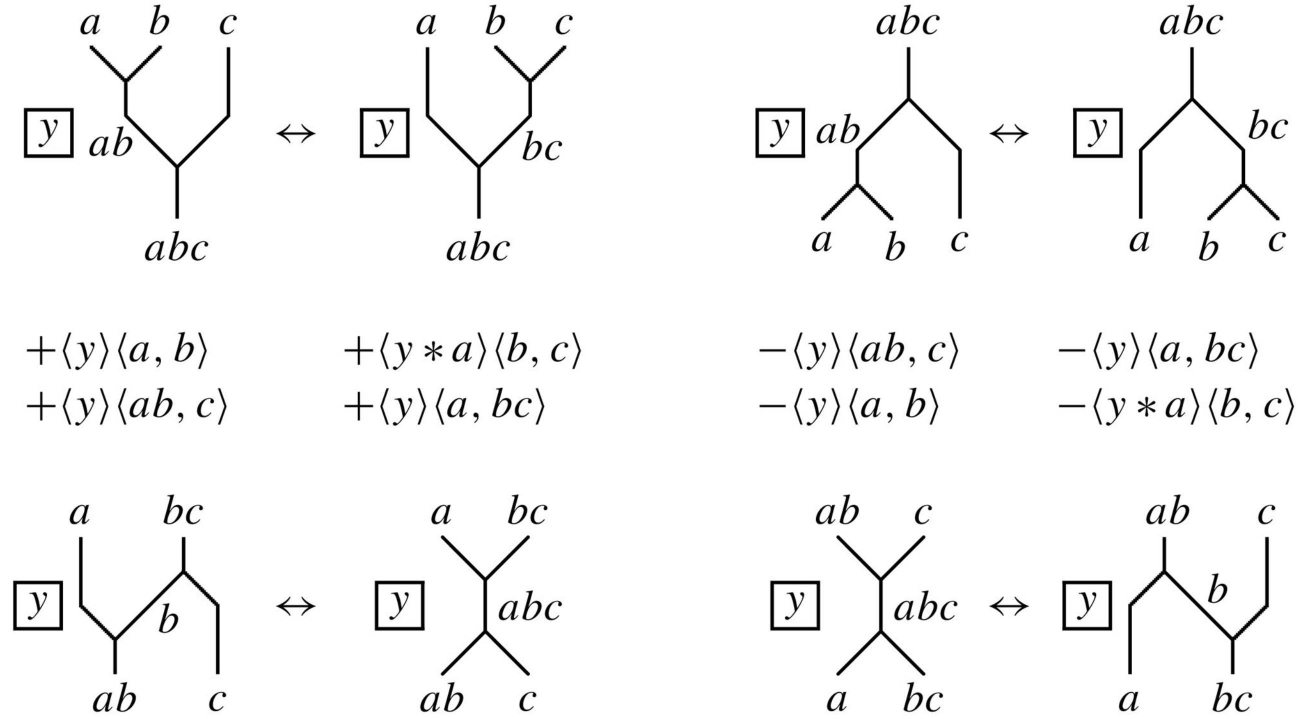


Figure 5.
Semiarcs near crossings and vertices.



$$+\langle y\rangle\langle a,b\rangle \qquad +\langle y*a\rangle\langle b,c\rangle \qquad -\langle y\rangle\langle ab,c\rangle \qquad -\langle y\rangle\langle a,bc\rangle$$
$$+\langle y\rangle\langle ab,c\rangle \qquad +\langle y\rangle\langle a,bc\rangle \qquad -\langle y\rangle\langle a,b\rangle \qquad -\langle y*a\rangle\langle b,c\rangle$$

$$-\langle y*a\rangle\langle b,c\rangle \qquad +\langle y\rangle\langle a,bc\rangle \qquad +\langle y\rangle\langle ab,c\rangle \qquad -\langle y\rangle\langle a,b\rangle$$
$$+\langle y\rangle\langle a,b\rangle \qquad -\langle y\rangle\langle ab,c\rangle \qquad -\langle y\rangle\langle a,bc\rangle \qquad +\langle y*a\rangle\langle b,c\rangle$$

Figure 6.
Chains for Y-oriented R6 moves.

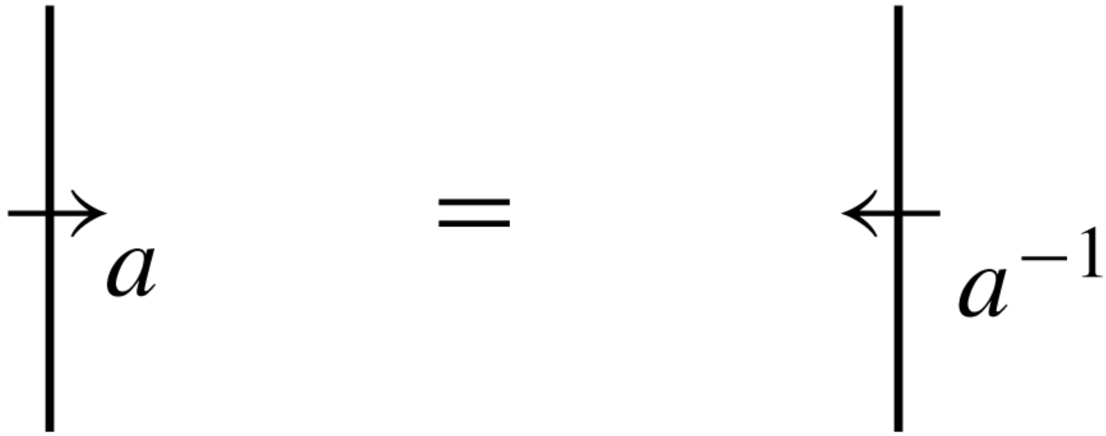
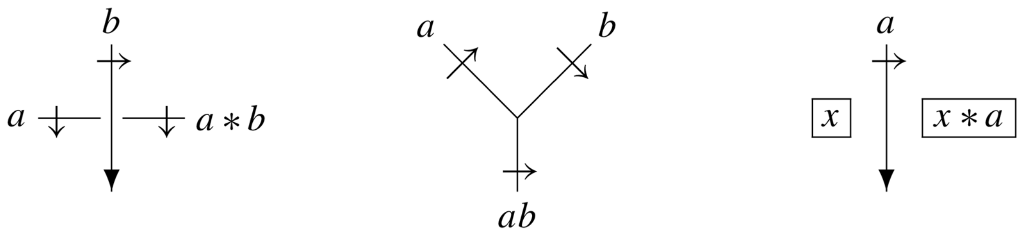


Figure 7.
$(X, \uparrow)$-color.



$a, b \in G_\lambda$

Figure 8.
Rules of an unoriented coloring.

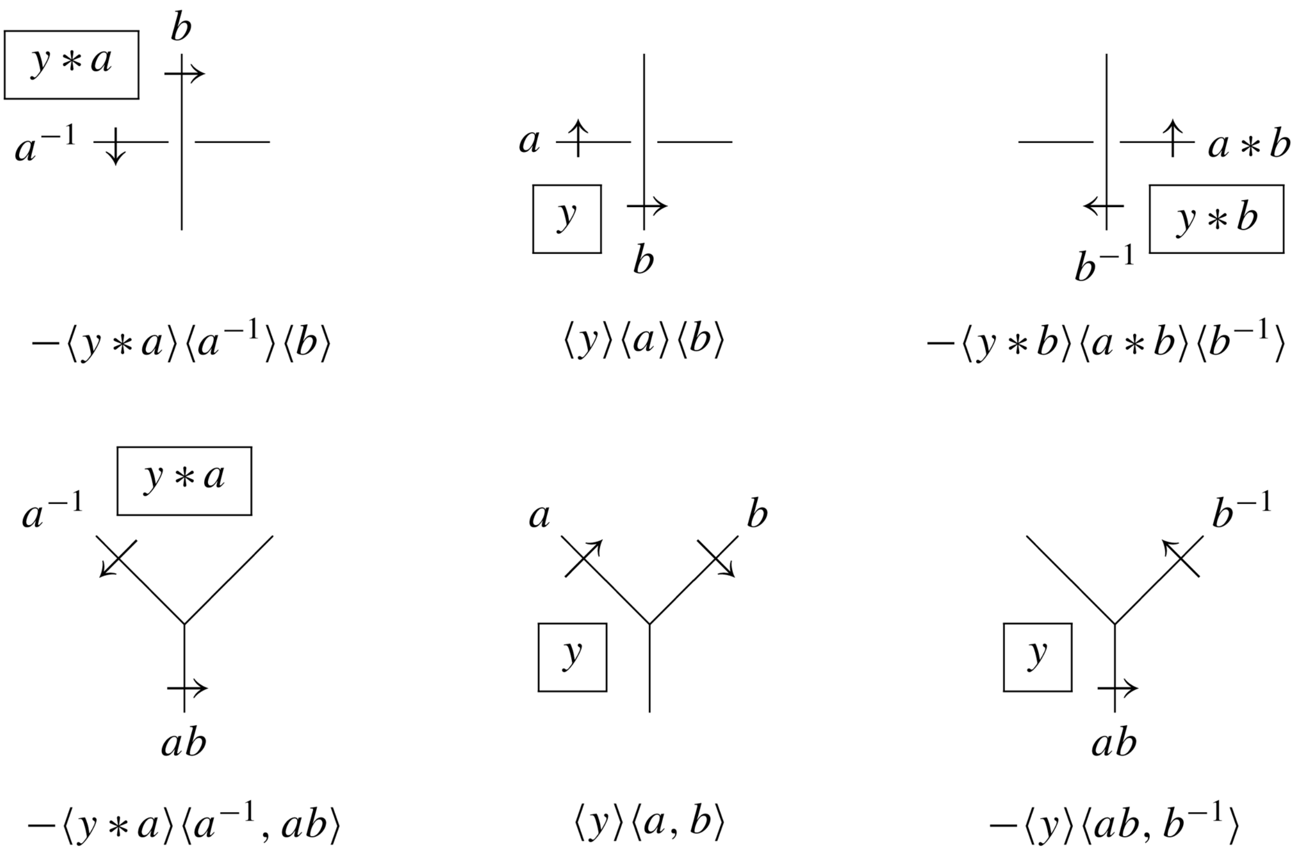


Figure 9.

Well-definedness of local chains for unoriented handle-body-links.

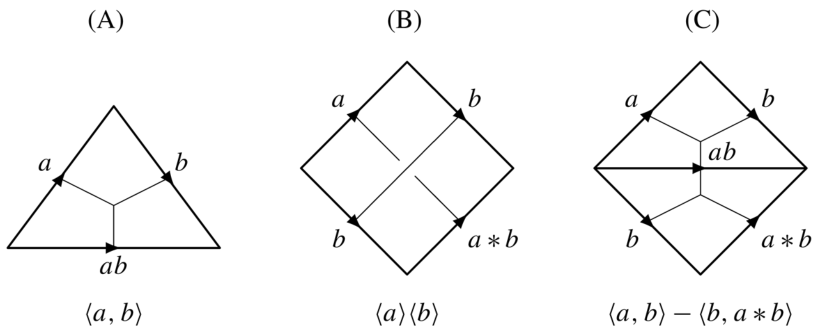


Figure 10.
Dividing a square into triangles.

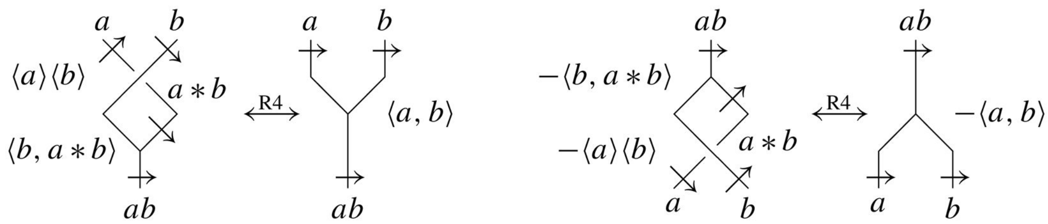


Figure 11.
Colors for Y-oriented R4 moves.

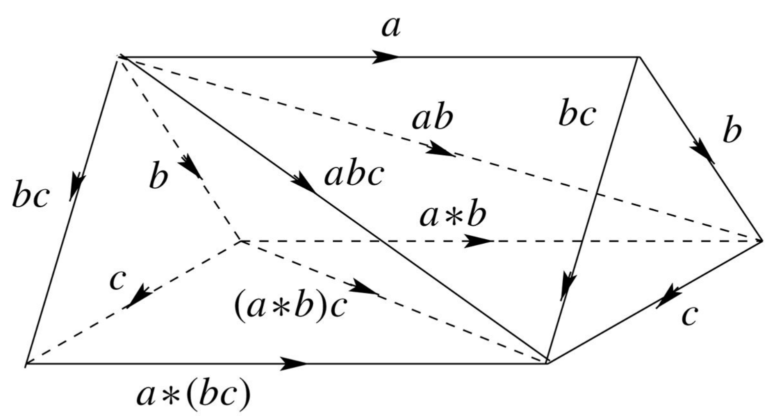


Figure 12.
Decomposition of a prism into tetrahedra.

Author Manuscript
Author Manuscript
Author Manuscript

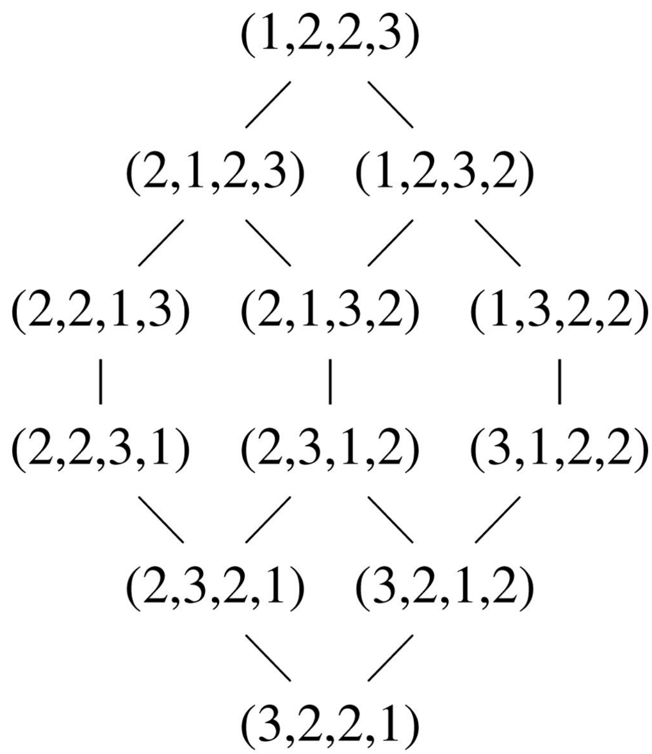


Figure 13.
Boundaries of $\langle a\rangle\langle b, c\rangle\langle d\rangle$.

Table 1

Comparison between IIJO theory and MCQ theory

IIJO	**2-boundary**	**degenerate** $D_2^{\mathrm{IIJO}}(X)_Y$	**cancelled by sign**	**zero by definition**
moves	R3	R4(⇝ R1), R5(⇝ ori.)	R2	R6

MCQ	**2-boundary**	**degenerate** $D_2(X)_Y$	**degenerate** $D_2^{\uparrow}(X)_Y$	**cancelled by sign**
moves	R3, R5, R6	R4(⇝ R1)	orientation	R2