Applied Mathematical Modelling 102 (2022) 35–61

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

A two-stage stochastic variational inequality model for storage and dynamic distribution of medical supplies in epidemic management

Min Li, Chao Zhang*, Mingxv Ding, Ruipu Lv

Department of Applied Mathematics Beijing Jiaotong University, Beijing 100044, China

ARTICLE INFO

Article history:
Received 28 December 2020
Revised 14 September 2021
Accepted 23 September 2021
Available online 1 October 2021

Keywords:
Two-stage stochastic equilibrium
Two-stage stochastic variational inequality
Progressive hedging method
Storage and dynamic distribution of medical supplies
COVID-19 Pandemic

ABSTRACT

The storage and distribution of medical supplies are important parts of epidemic prevention and control. This paper first proposes a new nonsmooth two-stage stochastic equilibrium model of medical supplies in epidemic management. The first stage addresses the storage in the pre-disaster phase, and the second stage focuses on the dynamic distribution by enrolling competitions among multiple hospitals over a period of time in the post-disaster phase. The uncertainties are the numbers of infected people treated in multiple hospitals during the period of time, which are time-varying around a nominal distribution predicted by historical experience. The two-stage stochastic equilibrium model is further approximated and transformed to a monotone two-stage stochastic variational inequality (SVI) model that is computationally tractable, with the aid of a smooth approximation technique. We employ the progressive hedging method (PHM) to solve a case study in the city of Wuhan in China suffered from the COVID-19 pandemic. Numerical results are presented to demonstrate the effectiveness of the proposed model in planning the storage and dynamic distribution of medical supplies in epidemic management.

1. Introduction

In 2020, there are already more than 80 million people infected by the COVID-19pandemic and 1.75 million deaths, and the number of infected people is still rising dramatically. Enormous losses in both lives and economics have been caused by COVID-19, and insufficient medical supplies worsen the treatment of patients. Currently, growing attention has been paid to the management of medical supplies in an epidemic.

In this paper, we focus on the storage of medical supplies before an epidemic occurs, and the dynamic distribution of medical supplies after an epidemic really outbreaks. The government is the decision maker for the storage plan before an epidemic occurs, by taking consideration of the competitions of the hospitals under all possible scenarios of an epidemic. The government is considered to be the leader to determine the storage plan, and the hospitals are the followers to compete each other for providing possible dynamic distribution plans after the storage plan has been given and the scenario of the epidemic has been grasped. The government eventually decides an optimal dynamic distribution plan for the whole society among all the possible dynamic distribution plans that can be obtained from the competitions of the hospitals.

* Corresponding author.
E-mail addresses: dtlimin2009@163.com (M. Li), zc.njtu@163.com (C. Zhang), sdd97825@126.com (M. Ding), 19121575@bjtu.edu.cn (R. Lv).

https://doi.org/10.1016/j.apm.2021.09.033
0307-904X/

In an epidemic, the number of infected people that are treated in every hospital every day is difficult to predict accurately such that decisions have to be taken under uncertainty. Moreover, it is better to consider the dynamic distribution over a period of time, rather than the static distribution only once. Because the effective treatment of infected people, the prevention and control of an epidemic, and the increasing of production capacity for medical supplies all need a key period which lasts two weeks or even longer, and the demand of medical supplies of a day depends also on distributions of the previous days. Few studies have explored the dynamic decision problems over a period [1–6].

When an epidemic occurs, without storing medical supplies pre-disaster, insufficient or delayed medical supplies cannot meet the demand and will lead to heavy loss of life. On the other hand, storing too many medical supplies pre-disaster will lead to huge economic losses. Therefore, how to store pre-disaster and distribute medical supplies post-disaster of an epidemic appropriately is a challenging task. Under such circumstances, two-stage stochastic programming (SP) is one of the most popular modeling approaches to support the decision making process, see [7–11], as it allows the modeler to represent the pre- and post-disaster phase together via the first- and second-stage decision variables.

Another important issue is that an unfair distribution of supplies will make humans' lives different and will also cause social confliction. It is a natural requirement that the distribution of medical supplies should be fair, which means that there should be no privilege for certain groups of individuals. Fair distribution is usually captured by the concepts of equity [12], including *equitability* and *balance*. *Equitability* means servicing a set of indistinguishable entities. *Balance* means servicing a set of entities while they are different from each other regarding their demands, claims, and preferences. In balance, an ideal solution may give each entity a different proportion of the total assignment. The proportion, however, is given by disaster managers, rather than decided by agents.

In this paper, we concentrate on a certain period of time that the number of infected people is high, and the hospitals are extremely short of medical supplies. The nominal values of the numbers of infected people during the period are assumed to be known ahead based on historical information. There are finite possibilities of the unknown numbers of infected people that are within a band around the nominal values.

We first develop a two-stage stochastic equilibrium model. The first stage determines the storage plan that has to be made before an epidemic occurs. It tries to minimize the storage costs as well as the costs occurred from all possible scenarios of distributions to hospitals in the second stage. The second stage is a non-cooperative multi-agent game model that determines the dynamic distribution of medical supplies based on time-variant demand. In the second stage, each hospital competes to minimize its own costs composed of transportation costs, purchase costs, as well as penalty costs if the demand is unmet during the period of time. The term of penalty costs indeed plays an important role in fairness.

There are two main difficulties of solving the two-stage stochastic equilibrium model. Firstly, the minimization problem of one hospital contains the other hospitals' strategies, not yet known at the decision horizon. Thus the minimization problem cannot be solved directly. One hospital's objective is in conflict with the others'. Thus one hospital has to make its decision by taking other hospitals' decisions into account. Hospitals compete to reach a steady state. That is, no hospital can further cut down its costs by unilaterally changing its own strategy for distribution. Secondly, the term of penalty costs in the objective function of the minimization problem for each hospital involves the max operator that is nonsmooth.

In order to overcome the above two difficulties, we use a smoothing function to approximate the term of penalty costs. We then transform the optimistic version of the two-stage stochastic equilibrium model into an equivalent two-stage SP, and then transform the two-stage SP equivalently into a two-stage SVI. It is worth mentioning that the multi-stage SVI was introduced by Rockafellar and Wets [13] in 2017, which is powerful to model general multi-stage SP problems and equilibrium problems. The two-stage SVI model in this paper is shown to be monotone and is solved by the PHM [14] proposed by Rockafellar and Sun in 2019, and the efficient semismooth Newton method is adopted for solving the problem corresponding to each scenario in the PHM. Numerical results demonstrate the effectiveness of the two-stage SVI model in describing the storage and dynamic distribution of medical supplies in epidemic prevention and control.

The main contributions of the paper are summarized as follows.

- We develop a two-stage stochastic equilibrium model for the storage and dynamic distribution of medical supplies in a period of time.
- We use a smoothing function to approximate the term of penalty costs in the nonsmooth two-stage stochastic equilibrium model, and transform its optimistic version into an equivalent smooth monotone two-stage SVI model that is computationally tractable. The PHM is employed to solve the two-stage SVI model, which has global convergence property. The subproblem for each scenario is solved by the semismooth Newton method.
- A realistic application to the city of Wuhan in China that was suffered from COVID-19 is considered in numerical experiments, which demonstrates that our proposed two-stage SVI model solved by the PHM provides good decisions of the storage and dynamic distribution of medical supplies.

The rest of the paper is organized as follows. In Section 2, the relevant literature is reviewed. Section 3 develops the proposed two-stage stochastic equilibrium model. In Section 4, we present its smooth approximation and transform its optimistic version equivalently to a smooth two-stage SP. We then transform the smooth two-stage SP equivalently to a monotone two-stage SVI. Section 5 presents the PHM with the updating rule for the smoothing parameter to solve the two-stage SVI model. Section 6 implements the model on real data in Wuhan suffered from the COVID-19 pandemic and analyzes the important role of the penalty in fairness. The conclusions and directions for future research are provided in the final section.

2. Literature review

This section presents a review of the literature related to our problem including two-stage SP models for disaster management, dynamic distribution, fairness, and stochastic variational inequalities (SVIs).

Disaster management has a two-stage nature: choosing the level of preparedness such as the location and inventory level of medical supplies before the disaster occurs, and then reacting once the uncertainty has been revealed. Barbarosoğlu and Arda [8] introduced for the first time a two-stage SP model for both the pre-disaster and post-disaster stage where the supply capacities and demands are considered as random variables captured by a set of scenarios. Liu et al. [15] modeled the network retrofit problem as a two-stage SP to optimize a mean-risk objective of the system losses. Fan and Liu [16] formulated a two-stage SP with equilibrium constraints on pre-disaster transportation network protection problems against uncertain future disasters. They demonstrated the applicability of the progressive hedging-based method for solving the two-stage SP model. Mete and Zabinsky [17] developed a two-stage SP for the storage and distribution of medical supplies in disaster management. Noyan [10] considered a risk-averse two-stage SP model for disaster management and discussed the importance of incorporating a risk measure to derive optimal decisions. Two decomposition algorithms based on the generic Benders decomposition approach are constructed to solve such problems. We refer to the excellent survey papers [9], [11] and the references therein for more literature of two-stage SP models for disaster management.

Dynamic distribution of emergency supplies is important. It is necessary to adjust the plans incorporating new information along time horizons. Rawls and Turnquist [1] considered the dynamic allocation to satisfy short-term demands for emergency supplies. Bozorgi-Amiri and Khorsi [6] considered a multi-objective dynamic SP model by integrating pre-disaster plans and post-disaster decisions. The above papers mainly dedicated to the short-term dynamic distribution of emergency supplies. Little attention has been paid to the long-term preparedness of emergency supplies. Yang et al. [2] proposed a distributionally robust optimization model for the dynamic distribution of emergency supplies under demand uncertainty within a relatively long period of time.

Fairness is important for medical supplies. The balance concept related to fairness is often considered in various real life allocation problems [18]. In a disaster, different locations have different priorities because of different amounts of population and degrees of severity suffered from the disaster [19], [20]. Hence, balance should be satisfied among demand points in the distribution of relief commodities for preventing a possible social disaster [21]. Bertsimas et al. [22] discussed different fairness concepts that are used to ensure fair allocation of resources. They also focused on balancing efficiency and equity in resource allocation settings [23]. Bertsimas et al. [24] proposed a modeling framework for general dynamic resource allocation problems where there is a concern of equitably distributing the delay among the resource requests. Cayirli and Veral [25] stated that fairness across patients need to be considered while designing appointment systems. Turkcan et al. [26] introduced a constrained model for sequential clinical scheduling. The proposed unfairness measures are based on the expected waiting times at each slot and the number of patients in the system at the beginning of each slot.

Two-stage SVI has wide applications in economics, traffic network, electricity markets, supply chain problems, finance, and risk management under uncertain environment. Rockafellar and Wets [13] first introduced the formal definition of multi-stage SVIs in 2017. Two-stage SVI involves making a “here-and-now” decision at present to meet the uncertainty that is revealed later. This is one of the motivations of both two-stage SP and two-stage SVI. Chen, Pong and Wets [27] investigated a two-stage SVI model, and an expected residual minimization procedure is used to formulate the two-stage SVI into a two-stage SP with recourse. Chen et al. [28] provided a discrete scheme of two-stage stochastic linear complementarity problem, a special case of two-stage SVI, where the underlying random data are continuously distributed. Chen et al. [29] investigated the sample average approximation of two-stage stochastic generalized equation, which includes two-stage SVI as a special case. Rockafellar and Wets [30] developed the PHM for solving multi-stage convex SP for the first time in 1991. In 2019, Rockafellar and Sun [14] proposed the PHM for solving monotone multi-stage SVIs with a finite collection of scenarios for the random vector and showed the convergence of the PHM. In 2020, Rockefallar and Sun [31] considered the multistage Lagrangian SVI problem. In 2020, Li and Zhang [32] considered a two-stage SVI arising from a general convex two-stage SP, where the random variables have continuous distributions. We outline below some references that relate to the two-stage SVIs arising from non-cooperative multi-agent games. In 2017, Pang et al. [33] formally introduced and studied a non-cooperative multi-agent game under uncertainty and focused mainly on a two-stage setting of the game where each agent is risk-averse. Jiang et al. [34] in 2019 developed a two-stage SVI model for a production and supply competition of a homogenous product under uncertainty in an oligopolistic market. Zhang et al. [35] in 2019 studied a model of two-stage N-player non-cooperative game under uncertainty and showed that it is equivalent to a SVI model. The PHM is employed to solve the monotone SVI, as well as the nonmonotone case where an elicitability condition holds with convergence guarantee. We refer to the excellent survey paper by Sun and Chen [36] in 2021 for more literature of the theory, algorithms and applications for two-stage SVIs.

3. A new two-stage stochastic equilibrium model

We first propose a new two-stage stochastic equilibrium model for the storage and dynamic distribution of medical supplies in an epidemic within a period of time. In the first stage, the government needs to decide the amounts of storage of the different types of medical supplies in the hospitals before an epidemic occurs. The manager tries to minimize the total cost of hospitals, including the sum of the storage costs, and the costs of all hospitals in the second stage for the dynamic distribution during the period of time, subject to several constraints such as the budget limits for epidemic prevention, and the capacities of storage of the hospitals, etc. Storing a large amount of medical supplies ahead is beneficial for timely and effective aid to a sudden epidemic. However, this strategy may become a burden of social economy. It is challenging to find a good trade-off between maximizing the demand coverage and minimizing the total cost in the first stage.

The demands of a hospital for all kinds of medical supplies on one day are considered as stochastic variables, which are not known before this day. The decisions of the first stage have to be made without grasping the realizations of the stochastic variables. However, it is reasonable to assume that the nominal values of the stochastic variables during the period of time we considered are known with the aid of historical experience.

In the second stage, the realizations of the stochastic demand vector are grasped. The government has to decide further the distribution of medical supplies for each hospital according to the storage capacities, the actual demands, as well as the other hospitals' decisions. The medical supplies are classified into two categories – government distribution and social non-targeted donation. The social non-targeted donation refers to the medical supplies donated from organizations, companies and individuals for the epidemic, without specifying the medical supplies to which hospitals. The hospitals only need to pay the transportation costs for these medical supplies. The goal of the second stage of a hospital is to minimize the total cost of its own. The total cost of a hospital include the transportation costs, the purchase costs, and the penalty costs for unmet demands. The overall effects of the second-stage decisions under all possible scenarios are enrolled in the costs of all hospitals in the first-stage objective function.

To build the two-stage stochastic equilibrium model for the storage and dynamic distribution of medical supplies in this section and also transform the model in the next section, we introduce some notations as follows.

Sets:

$\mathcal{I} = \{1, \ldots, m\}$: set of hospitals;

$\mathcal{J} = \{1, \ldots, n\}$: set of medical supply types;

$\mathcal{T} = \{1, \ldots, l\}$: set of time horizons.

Random Variables and Random Vectors:

ξ_{it}: number of infected people in hospital i on day t;

$\xi = (\xi_{it},\ \forall i, t)^T = (\xi_{11}, \ldots, \xi_{1l}, \ldots, \xi_{m1}, \ldots, \xi_{ml})^T$: a random vector $\xi : \Omega \to \Xi \subset R^{ml}$ defined in the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a support set Ξ.

Parameters and Related Vectors:

$d_{ijt}(\xi_{it})$: demand of hospital i for medical supply j on day t;

a_{ij}: unit reserve price of hospital i for medical supply j;

$a = (a_{ij},\ \forall i, j)^T$: vector of unit reserve prices;

b_{ijt}: unit transport price of hospital i for medical supply j on day t;

$b = (b_{ijt},\ \forall i, j, t)^T$: vector of unit transport prices;

c_{ij}: penalty coefficient (penalty cost per unit) of hospital i for medical supply j each day;

$p_{jt}(\xi)$: unit price of medical supply j on day t;

$p^i(\xi) = (p_{jt}(\xi),\ \forall j, t)^T$: vector of unit prices of medical supplies for hospital i;

$x_{ij\max}$: storage capacity of hospital i for medical supply j;

$x^{\text{hosp}}_{\max} = (x_{ij\max},\ \forall i, j)^T$: upper bound vector for storage capacities;

$x_{j\max}$: maximal available amount of medical supply j;

$x^{\text{budg}}_{\max} = (x_{1\max}, \ldots, x_{n\max})^T$: upper bound vector for budgets;

$y_{jt\max}$: upper bound of medical supply j by government distribution on day t;

$y^{\text{dist}}_{\max} = (y_{jt\max},\ \forall j, t)^T$: upper bound vector from government distribution;

$z_{jt\max}$: upper bound of medical supply j by social non-targeted donation on day t;

$z^{\text{soci}}_{\max} = (z_{jt\max},\ \forall j, t)^T$: upper bound vector from social non-targeted donation.

Decision Variables:

x_{ij}: amount of medical supply j stored to hospital i;

$y_{ijt}(\xi)$: amount of medical supply j distributed to hospital i by government on day t under scenario ξ;

$z_{ijt}(\xi)$: amount of medical supply j distributed to hospital i by social non- targeted donation on day t under scenario ξ;

$s_{ijt}(\xi)$: amount of medical supply j used by hospital i from the storage on day t under scenario ξ.

Note: Here $x_{ij\max}, y_{jt\max}, z_{jt\max} > 0$ for all $i \in \mathcal{I},\ j \in \mathcal{J},\ t \in \mathcal{T}$. There may exist $x_{j\max} = 0$ for some j, since some medical supply is only available after the epidemic occurs, e.g., the inspection and testing articles for the COVID-19 pandemic.

Let us also define the decision vectors for the two stages as follows.

$$
\begin{aligned}
& x^i = (x_{i1}, x_{i2}, \ldots, x_{in})^T, \quad x = (x_{ij},\ \forall i, j)^T, \\
& y^i(\xi) = (y_{ijt}(\xi),\ \forall j, t)^T, \quad y(\xi) = (y_{ijt}(\xi),\ \forall i, j, t)^T,\ \forall \xi \in \Xi, \\
& z^i(\xi) = (z_{ijt}(\xi),\ \forall j, t)^T, \quad z(\xi) = (z_{ijt}(\xi),\ \forall i, j, t)^T,\ \forall \xi \in \Xi, \\
& s^i(\xi) = (s_{ijt}(\xi),\ \forall j, t)^T, \quad s(\xi) = (s_{ijt}(\xi),\ \forall i, j, t)^T,\ \forall \xi \in \Xi.
\end{aligned}
$$

Because the decision vector of hospital i is closely related to the decision vectors of the other hospitals, for $i = 1, 2, \ldots, m$, we define the vectors

$$\begin{aligned} x^{-i} &= ({x^{1}}^{T}, \ldots, {x^{i-1}}^{T}, {x^{i+1}}^{T}, \ldots, {x^{m}}^{T})^{T}, \\ y^{-i}(\xi) &= (y^{1}(\xi)^{T}, \ldots, y^{i-1}(\xi)^{T}, y^{i+1}(\xi)^{T}, \ldots, y^{m}(\xi)^{T})^{T}, \ \forall \xi \in \Xi, \\ z^{-i}(\xi) &= (z^{1}(\xi)^{T}, \ldots, z^{i-1}(\xi)^{T}, z^{i+1}(\xi)^{T}, \ldots, z^{m}(\xi)^{T})^{T}, \ \forall \xi \in \Xi, \\ s^{-i}(\xi) &= (s^{1}(\xi)^{T}, \ldots, s^{i-1}(\xi)^{T}, s^{i+1}(\xi)^{T}, \ldots, s^{m}(\xi)^{T})^{T}, \ \forall \xi \in \Xi, \end{aligned}$$

and we add superscript $*$ to the above vectors for the decision vectors of the other hospitals in an optimal solution. For instance,

$$x^{-i^{*}} = ({x^{1*}}^{T}, \ldots, {x^{i-1*}}^{T}, {x^{i+1*}}^{T}, \ldots, {x^{m*}}^{T})^{T}$$

is the decision vector for hospitals except hospital i in an optimal solution. The optimal solution of our two-stage stochastic equilibrium problem will be given in detail below.

From the above discussion, we can formulate a two-stage stochastic equilibrium problem as follows. The total cost includes the acquisition and holding costs, as well as the transportation costs, the government purchasing costs and the penalty costs for unmet demands. These costs reside in two phases: the storage phase and the distribution phase. In the storage phase, we denote by $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_{ij}$ the acquisition and holding costs, and denote by $\sum_{i=1}^{m} E[\vartheta_i(x^i, x^{-i^*}, \xi)]$ the sum of the costs of all hospitals from the second stage. Let $\sum_{j=1}^{n}\sum_{t=1}^{l} b_{ijt}\big(y_{ijt}(\xi) + z_{ijt}(\xi)\big)$ be the costs of transporting all the medical supplies from government distribution and social non-targeted donation for hospital i over all time horizons, and $\sum_{j=1}^{n}\sum_{t=1}^{l} p_{jt}(\xi)y_{ijt}(\xi)$ be the government purchasing costs of all the medical supplies for hospital i over all time horizons in the distribution phase. Noting that medical supplies are classified into non-reusable and reusable medical supplies, we use $M_{ijt}(\xi)$ to represent the amount of unmet demands of hospital i for medical supply j on day t, which is defined as

$$M_{ijt}(\xi) = \begin{cases} d_{ijt}(\xi_{it}) - y_{ijt}(\xi) - z_{ijt}(\xi) - s_{ijt}(\xi), & \text{if medical supply } j \text{ is non-reusable}, \\ d_{ijt}(\xi_{it}) - \sum\limits_{o=1}^{t}\big[y_{ijo}(\xi) + z_{ijo}(\xi) + s_{ijo}(\xi)\big], & \text{if medical supply } j \text{ is reusable}. \end{cases} \tag{1}$$

Here, we assume that reusable medical supplies do not expire within the time horizons considered. A penalty is added into the model to reduce a huge loss of life and wealth after an epidemic. The term of penalty costs is defined as $\sum_{j=1}^{n}\sum_{t=1}^{l} c_{ij}\big(M_{ijt}(\xi)\big)_{+}$ for hospital i, where the notation $(\rho)_{+} := \max(\rho, 0)$ for any $\rho \in R$.

Based on the above description, we obtain the following objective function for the first stage

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_{ij} + \sum_{i=1}^{m} E\Big[\vartheta_i\Big(x^{i}, x^{-i^{*}}, \xi\Big)\Big], \tag{2}$$

where for each $i \in \mathcal{I}$, $\vartheta_i(x^i, x^{-i^*}, \xi)$ is the optimal value of the second-stage equilibrium problem. The objective function of the second-stage equilibrium problem is given in (3) and the constraints are described in (6)-(8) and (10) below.

$$\begin{aligned} &F_i(y^i(\xi), z^i(\xi), s^i(\xi)) := \\ &\quad \sum_{j=1}^{n}\sum_{t=1}^{l} b_{ijt}(y_{ijt}(\xi) + z_{ijt}(\xi)) + \sum_{j=1}^{n}\sum_{t=1}^{l} p_{jt}(\xi)y_{ijt}(\xi) + \sum_{j=1}^{n}\sum_{t=1}^{l} c_{ij}\Big(M_{ijt}(\xi)\Big)_{+}. \end{aligned} \tag{3}$$

The objective function (2) of the first stage includes the acquisition and holding costs, as well as the expected sum of optimal values of the second stage. The objective function (3) of the second-stage problem (for every hospital i) includes the transportation costs, the government purchasing costs and the penalty costs for unmet demands. The government purchasing cost $p_{jt}(\xi)$ per unit for medical supply j on day t changes as the demands change. The higher the demands, the higher the price.

The storage amount of each medical supply should not exceed the budget of the government. Also, the storage amount of each medical supply of each hospital should not exceed its capacity. The medical supplies distributed by government and obtained from social non-targeted donation should not exceed the upper limits. Constraint (4) ensures that the stored medical supplies are limited by budgets. Constraint (5) ensures that the stored medical supply j in hospital i is limited by its warehouse capacity.

$$\sum_{i=1}^{m} x_{ij} \le x_{j\max}, \ \forall j \in \mathcal{J}, \tag{4}$$

$$x_{ij} \le x_{ij\max}, \ \forall i \in \mathcal{I}, \ j \in \mathcal{J}. \tag{5}$$

Constraints (6) and (7) limit the amounts of medical supplies distributed by government and obtained from social non-targeted donation for almost every (a.e.) realization of $\xi \in \Xi$, respectively. That is, for a.e. $\xi \in \Xi$,

$$y_{ijt}(\xi) + \sum_{\gamma \ne i}^{m} y^{*}_{\gamma jt}(\xi) \le y_{jt\max}, \ \forall j \in \mathcal{J}, \ t \in \mathcal{T}, \tag{6}$$

$$z_{ijt}(\xi) + \sum_{\gamma \neq i}^{m} z^{*}_{\gamma jt}(\xi) \leq z_{jt\,\max}, \ \ \forall j \in \mathcal{J}, \ t \in \mathcal{T}. \tag{7}$$

Constraint (8) ensures that the amount of medical supply j from storage used by hospital i over all time horizons does not exceed the total amount of storage. That is, for any $i \in \mathcal{I}$, and for a.e. $\xi \in \Xi$,

$$\sum_{t=1}^{l} s_{ijt}(\xi) \leq x_{ij}, \ \ \forall j \in \mathcal{J}. \tag{8}$$

Constraints (9) and (10) are nonnegativity conditions, since the amount of storage and the amount of distribution cannot be negative. That is,

$$x_{ij} \geq 0, \ \ \forall i \in \mathcal{I}, \ \ j \in \mathcal{J}, \tag{9}$$

and for any $i \in \mathcal{I}$, and for a.e. $\xi \in \Xi$,

$$y_{ijt}(\xi), \ \ z_{ijt}(\xi), \ \ s_{ijt}(\xi) \geq 0, \ \ \forall j \in \mathcal{J}, \ \ t \in \mathcal{T}. \tag{10}$$

We construct the two-stage stochastic equilibrium model in the following form

$$\begin{array}{ll} \min_x & \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} a_{ij}x_{ij} + \sum\limits_{i=1}^{m} E\left[\vartheta_i\left(x^i, x^{-i^*}, \xi\right)\right], \\ \text{s.t.} & (4), (5), (9), \end{array} \tag{11}$$

where $\vartheta_i\left(x^i, x^{-i^*}, \xi\right)$ is the optimal value of the second-stage equilibrium problem. That is, for every hospital $i \in \mathcal{I}$,

$$\begin{array}{ll} \min\limits_{y^i(\xi), z^i(\xi), s^i(\xi)} & F_i(y^i(\xi), z^i(\xi), s^i(\xi)) \\ \text{s.t.} & (6), (7), (8), (10). \end{array} \tag{12}$$

Here the objective function $F_i(y^i(\xi), z^i(\xi), s^i(\xi))$ is defined in (3).

The model (11)-(12) is a two-stage stochastic equilibrium model, where the second-stage problem is nonsmooth since the penalty in the objective function involves the max operator. The methods for solving the two-stage SP are not applicable for the two-stage stochastic equilibrium model. Hence it is computationally intractable. In the next section, we will replace the nonsmooth penalty by its smooth counterpart and transform the smooth approximation of the two-stage stochastic equilibrium problem to a monotone two-stage SVI problem that is computationally tractable.

4. Smooth approximation and transforming to monotone two-stage SVI

For simplicity, we always assume the support set Ξ being a finite set of "scenarios" ξ^k, $k = 1, 2, \ldots, K$, each having a nonzero probability τ_k such that $\sum_{k=1}^{K} \tau_k = 1$ at follows. In fact, if $\xi \in \Xi$ is continuously distributed, we can adopt the sample average approximation to obtain a discrete distribution that approximates the continuous distribution as in [28,29].

In order to construct the smooth approximation of $F_i(y^i(\xi), z^i(\xi), s^i(\xi))$ for any $\xi \in \Xi$, we need the following definition for smoothing function.

Definition 1. [37] Let $g : R^n \to R$ be a locally Lipschitz continuous function. We call $\tilde{g} : R^n \times R_{++} \to R$ a smoothing function of g if $\tilde{g}$ is continuously differentiable on R^n for any $\mu \in R_{++}$ and for any $x \in R^n$,

$$\lim_{z \to x, \ \mu \downarrow 0} \tilde{g}(z, \mu) = g(x). \tag{13}$$

We use the smoothing function $\tilde{q} : R \times R_{++} \to R$ as in [38,39]

$$\tilde{q}(\rho, \mu) = \mu \ln\left(e^{\rho/\mu} + 1\right), \tag{14}$$

to approximate the nonsmooth function ρ_+ for $\rho \in R$. It is easy to check that for any $\rho \in R$,

$$\lim_{\zeta \to \rho, \ \mu \downarrow 0} \tilde{q}(\zeta, \mu) = \rho_+,$$

and for each fixed smoothing parameter $\mu > 0$, the first-order derivative and the second-order derivative of $\tilde{q}$ with respect to ρ are

$$\tilde{q}'(\rho, \mu) = \frac{e^{\rho/\mu}}{e^{\rho/\mu} + 1}, \quad \tilde{q}''(\rho, \mu) = \frac{e^{\rho/\mu}}{\mu(e^{\rho/\mu} + 1)^2}. \tag{15}$$

Thus given a fixed smoothing parameter $\mu > 0$, we obtain the smooth approximation $\tilde{F}_i(y^i(\xi), z^i(\xi), s^i(\xi), \mu)$ for the objective function $F_i(y^i(\xi), z^i(\xi), s^i(\xi))$ in the second stage of the two-stage stochastic equilibrium model for any hospital $i \in \mathcal{I}$ as

$$\tilde{F}_i(y^i(\xi), z^i(\xi), s^i(\xi), \mu) := \tag{16}$$

$$\sum_{j=1}^{n}\sum_{t=1}^{l} b_{ijt}(y_{ijt}(\xi)+z_{ijt}(\xi))+\sum_{j=1}^{n}\sum_{t=1}^{l} p_{jt}(\xi)y_{ijt}(\xi)+\sum_{j=1}^{n}\sum_{t=1}^{l} c_{ij}\tilde{q}\big(M_{ijt}(\xi),\mu\big).$$

For each fixed $\mu > 0$, since the second-order derivative of $\tilde{q}(\cdot,\mu)$ in (15) is greater than zero, it follows that $\tilde{q}(\cdot,\mu)$ is a convex function. Moreover, for each $\xi \in \Xi$, we know that $M_{ijt}(\xi)$ defined in (1) is an affine function with respect to $(y^i(\xi)^T, z^i(\xi)^T, s^i(\xi)^T)^T$. According to Theorem 3.1.6 of [40], the composition of a convex function and an affine function is again a convex function. Thus $\tilde{q}\big(M_{ijt}(\xi),\mu\big)$ is a convex function with respect to $(y^i(\xi)^T, z^i(\xi)^T, s^i(\xi)^T)^T$. This, together with the fact that the first two terms in $\tilde{F}_i(y^i(\xi), z^i(\xi), s^i(\xi), \mu)$ are linear functions, and $c_{ij} > 0$ for all i, j, yields that $\tilde{F}_i(y^i(\xi), z^i(\xi), s^i(\xi), \mu)$ is a convex function with respect to $(y^i(\xi)^T, z^i(\xi)^T, s^i(\xi)^T)^T$.

The smooth approximation of the two-stage stochastic equilibrium model (11)-(12) is

$$\begin{array}{ll}\min_x & \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_{ij}+\sum_{i=1}^{m}\sum_{k=1}^{K}\tau_k\big[\tilde{\vartheta}_i\big(x^i, x^{-i*}, \xi^k, \mu\big)\big],\\ \text{s.t.} & (4), (5), (9),\end{array} \tag{17}$$

where $\tilde{\vartheta}_i\big(x^i, x^{-i^*}, \xi^k, \mu\big)$ is the optimal value of the smooth approximation of the second-stage equilibrium problem. That is, for every hospital $i \in \mathcal{I}$, it solves the following smooth minimization problem

$$\begin{array}{ll}\min_{y^i(\xi^k), z^i(\xi^k), s^i(\xi^k)} & \tilde{F}_i(y^i(\xi^k), z^i(\xi^k), s^i(\xi^k), \mu)\\ \text{s.t.} & (6), (7), (8), (10).\end{array} \tag{18}$$

The second-stage problem (18) for hospital i involves other hospitals' optimal strategies, and hence it is not solvable itself. We thus consider the following model defined in (19). More details on the equivalency of (17)-(18) and (19) have been provided in Subsection 4.1.

$$\begin{array}{ll}\min_{\substack{x, y(\xi^k), z(\xi^k), s(\xi^k),\\ k=1,\ldots,K}} & \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_{ij}+\sum_{i=1}^{m}\sum_{k=1}^{K}\tau_k\tilde{F}_i(y^i(\xi^k), z^i(\xi^k), s^i(\xi^k), \mu)\\ \text{s.t.} & (4), (5), (9),\\ & \sum_{i=1}^{m} y_{ijt}(\xi^k) \le y_{jt\max}, \ \forall j \in \mathcal{J}, \ t \in \mathcal{T}, \ k = 1, \ldots, K,\\ & \sum_{i=1}^{m} z_{ijt}(\xi^k) \le z_{jt\max}, \ \forall j \in \mathcal{J}, \ t \in \mathcal{T}, \ k = 1, \ldots, K,\\ & (8), (10) \text{ for } \xi = \xi^k, \quad k = 1, \ldots, K.\end{array} \tag{19}$$

We have already shown before that $\tilde{F}_i(y^i(\xi^k), z^i(\xi^k), s^i(\xi^k), \mu)$ is a convex function. The above model is a convex model with linear constraints. Moreover, the feasible region defined by the linear constraints is compact. Since a continuous function is sure to obtain its minimum in a nonempty and compact set, the existence of solutions of the convex model (19) is guaranteed. The linear constraints also guarantee that any solution of the convex model (19) is a KKT point.

In order to express the formulas of the KKT systems involving in (17)-(18) as well as (19) in a concise way, we use the vectors and matrices for the objective functions and the constraints, as well as the corresponding Lagrange multipliers for the constraints.

For given positive integers α and β, we denote by I_α the $\alpha \times \alpha$ identity matrix, by $I_\alpha^{(\beta)}$ the $\alpha \times \alpha\beta$ matrix constituted by β blocks of I_α, i.e., $I_\alpha^{(\beta)} = (I_\alpha, \ldots, I_\alpha)$, by $\mathrm{diag}(\hat{v})$ the diagonal matrix with its (i,i) entry to be the ith component of the vector $\hat{v}$, by $\mathrm{Diag}(D, \varsigma)$ the block diagonal matrix with ς blocks D, by $e^{(\beta)}$ the β-dimensional column vector with all entries one, by $0_{\alpha\times\beta}$ the $\alpha \times \beta$ matrix with all entries zero. Denote

$$\tilde{F}(y(\xi^k), z(\xi^k), s(\xi^k), \mu) = (\tilde{F}_i(y^i(\xi^k), z^i(\xi^k), s^i(\xi^k), \mu), \ \forall i)^T,$$
$$h = \left(x_{\max}^{\text{budg}^T}, x_{\max}^{\text{hosp}^T}\right)^T, \ A = \left(I_n^{(m)^T}, I_{mn}\right)^T, \ p(\xi^k) = I_{nl}^{(m)^T} p^i(\xi^k),$$
$$B = \mathrm{Diag}(e^{(l)^T}, n), \ \hat{B} = \mathrm{Diag}(B, m).$$

Now assume that the first n_1 medical supply types are non-reusable and the remaining n_2 medical supply types are reusable, i.e., $n_1 + n_2 = n$. Let $H_1 \in R^{l\times l}$ be an upper triangular matrix whose upper diagonal entries are all 1, $H_2 = \mathrm{Diag}(H_1, n_2)$, $M_{ijt}^{\xi^k\mu} = \frac{1}{e^{M_{ijt}(\xi^k)/\mu}+1} - 1$, $H = \begin{pmatrix} I_{n_1 l} & 0_{n_1 l\times n_2 l}\\ 0_{n_2 l\times n_1 l} & H_2\end{pmatrix}$, $\hat{H} = \mathrm{Diag}(H, m)$, and

$$Q_{yzs}^{\xi^k\mu} = \hat{H}\Big(c_{ij}M_{ijt}^{\xi^k\mu}, \ \forall i, j, t\Big)^T. \tag{20}$$

The convex model (19) can be rewritten in the form

$$\begin{array}{lll} \min_{\substack{x,y(\xi^k),z(\xi^k),s(\xi^k),\\ k=1,\dots,K}} & a^Tx+\sum_{k=1}^{K}\tau_k {e^{(m)}}^T\tilde{F}(y(\xi^k),z(\xi^k),s(\xi^k),\mu) & \\ \text{s.t.} & Ax\le h, & \\ & I_{nl}^{(m)}y(\xi^k)\le y_{\max}^{\text{dist}},\ \ k=1,\dots,K, & \\ & I_{nl}^{(m)}z(\xi^k)\le z_{\max}^{\text{soci}},\ \ k=1,\dots,K, & \\ & \hat{B}s(\xi^k)-x\le 0,\ \ k=1,\dots,K, & \\ & x,y(\xi^k),z(\xi^k),s(\xi^k)\ge 0,\ \ k=1,\dots,K. & \end{array} \tag{21}$$

We denote by $\lambda\in R_+^{mn+n}$, $\pi_y(\xi^k)\in R_+^{nl}$, $\pi_z(\xi^k)\in R_+^{nl}$, and $\pi_s(\xi^k)\in R_+^{mn}$ the vectors of Lagrange multipliers of the first four groups of functional constraints in (21), respectively. Using the above notations, the KKT system of (21) can be written as the following two-stage SVI model

$$\begin{array}{lllll} 0\le & a+A^T\lambda-\sum_{k=1}^{K}\tau_k\pi_s(\xi^k) & \perp\ x & \ge 0, & \\ 0\le & h-Ax & \perp\ \lambda & \ge 0, & \\ 0\le & b+p(\xi^k)+{I_{nl}^{(m)}}^T\pi_y(\xi^k)+Q_{yzs}^{\xi^k\mu} & \perp\ y(\xi^k) & \ge 0, & k=1,\dots,K,\\ 0\le & b+{I_{nl}^{(m)}}^T\pi_z(\xi)+Q_{yzs}^{\xi^k\mu} & \perp\ z(\xi^k) & \ge 0, & k=1,\dots,K,\\ 0\le & \hat{B}^T\pi_s(\xi^k)+Q_{yzs}^{\xi^k\mu} & \perp\ s(\xi^k) & \ge 0, & k=1,\dots,K,\\ 0\le & y_{\max}^{\text{dist}}-I_{nl}^{(m)}y(\xi^k) & \perp\ \pi_y(\xi^k) & \ge 0, & k=1,\dots,K,\\ 0\le & z_{\max}^{\text{soci}}-I_{nl}^{(m)}z(\xi^k) & \perp\ \pi_z(\xi^k) & \ge 0, & k=1,\dots,K,\\ 0\le & x-\hat{B}s(\xi^k) & \perp\ \pi_s(\xi^k) & \ge 0, & k=1,\dots,K. \end{array} \tag{22}$$

Here $0\le u\perp v\ge 0$ means that $u\ge 0$, $v\ge 0$, and $u^Tv=0$, for two vectors u and v of the same dimension. The vectors x and λ are the first-stage decision vectors of the above two-stage SVI model, which should be given before the scenario ξ^k really happens. That is, x and λ satisfy the "nonanticipativity" property as mentioned in [13]. The vectors $y(\xi^k)$, $z(\xi^k)$, $s(\xi^k)$, $\pi_y(\xi^k)$, $\pi_z(\xi^k)$, $\pi_s(\xi^k)$ are the second-stage decision vectors which should be given after we know the realization of ξ^k. It is worth mentioning that (22) is a two-stage stochastic nonlinear complementarity problem (NCP), which is a special type of two-stage SVIs. The nonlinearity comes from $M_{ijt}^{\xi^k\mu}$ in defining $Q_{yzs}^{\xi^k\mu}$.

Proposition 1. *The two-stage SVI (22) is of maximal monotone type.*

Proof. See A.1 for the proof. □

4.1. Relation of two models

In this subsection, we discuss the relation of the two-stage stochastic equilibrium model (17)-(18) and the two-stage SP model (19).

The two-stage stochastic equilibrium model (17)-(18) can be written as

$$\begin{array}{ll} \min_x & a^Tx+\sum_{i=1}^{m}\sum_{k=1}^{K}\tau_k\tilde{\vartheta}_i\big(x^i,x^{-i*},\xi^k,\mu\big)\\ \text{s.t.} & Ax\le h,\\ & x\ge 0, \end{array} \tag{23}$$

where $\tilde{\vartheta}_i\big(x^i,x^{-i*},\xi^k,\mu\big)$ is the optimal value of the second-stage equilibrium problem. That is, for every hospital $i\in\mathcal{I}$, it solves the following smooth minimization problem

$$\begin{array}{ll} \min_{y^i(\xi^k),z^i(\xi^k),s^i(\xi^k)} & \tilde{F}_i(y^i(\xi^k),z^i(\xi^k),s^i(\xi^k),\mu)\\ \text{s.t.} & y^i(\xi^k)+\sum_{\gamma\ne i}^{m}y^{\gamma *}(\xi^k)\le y_{\max}^{\text{dist}},\\ & z^i(\xi^k)+\sum_{\gamma\ne i}^{m}z^{\gamma *}(\xi^k)\le z_{\max}^{\text{soci}},\\ & Bs^i(\xi^k)-x^i\le 0,\\ & y^i(\xi^k),z^i(\xi^k),s^i(\xi^k)\ge 0. \end{array} \tag{24}$$

Note that constraint (8) in the second stage of the two-stage stochastic equilibrium model involves the decision vector x of the first stage. Let us denote by $S_2(x)$ the solution set of the second-stage equilibrium problem (18) for a given decision vector x of the first stage. For each fixed x, a vector

$$\big(y(\xi^k)^T,z(\xi^k)^T,s(\xi^k)^T,\ k=1,\dots,K\big)^T\in S_2(x),$$

if there exist vectors

$$\hat{\pi}_y(\xi^k) \in R_+^{mnl},\ \hat{\pi}_z(\xi^k) \in R_+^{mnl},\ \hat{\pi}_s(\xi^k) \in R_+^{mn},\ k = 1, \dots, K, \tag{25}$$

together with the vector $\left(y(\xi^k)^T, z(\xi^k)^T, s(\xi^k)^T,\ k = 1, \dots, K\right)^T$, satisfy the following NCP.

$$\begin{array}{ll} 0 \le & b + p(\xi^k) + Q_{yzs}^{\xi^k\mu} + \hat{\pi}_y(\xi^k) \ \perp\ y(\xi^k) \ \ge 0, \quad k = 1, \dots, K, \\ 0 \le & b + Q_{yzs}^{\xi^k\mu} + \hat{\pi}_z(\xi^k) \ \perp\ z(\xi^k) \ \ge 0, \quad k = 1, \dots, K, \\ 0 \le & Q_{yzs}^{\xi^k\mu} + \hat{B}^T\hat{\pi}_s(\xi^k) \ \perp\ s(\xi^k) \ \ge 0, \quad k = 1, \dots, K, \\ 0 \le & {I_{nl}^{(m)}}^T y_{\max}^{\text{dist}} - {I_{nl}^{(m)}}^T I_{nl}^{(m)} y(\xi^k) \ \perp\ \hat{\pi}_y(\xi^k) \ \ge 0, \quad k = 1, \dots, K, \\ 0 \le & {I_{nl}^{(m)}}^T z_{\max}^{\text{soci}} - {I_{nl}^{(m)}}^T I_{nl}^{(m)} z(\xi^k) \ \perp\ \hat{\pi}_z(\xi^k) \ \ge 0, \quad k = 1, \dots, K, \\ 0 \le & x - \hat{B}s(\xi^k) \ \perp\ \hat{\pi}_s(\xi^k) \ \ge 0, \quad k = 1, \dots, K. \end{array} \tag{26}$$

Proposition 2. *For any feasible solution $x \in R^{mn}$ of (17), the solution set $S_2(x)$ of the second-stage equilibrium problem (18) is nonempty, convex, and compact.*

Proof. See A.2 for the proof. □

There may exist many choices of decisions in $S_2(x)$ for a given $x \in R^{mn}$. It is reasonable to assume that the government as the leader for the two-stage equilibrium model is allowed to select the decision vector

$$\left(\hat{y}(\xi^k)^T, \hat{z}(\xi^k)^T, \hat{s}(\xi^k)^T,\ k = 1, \dots, K\right)^T \in S_2(x)$$

such that the total cost involved in the second stage is minimized. That is,

$$\begin{aligned} \sum_{i=1}^{m}\sum_{k=1}^{K} \tau_k \tilde{\vartheta}_i\left(x^i, x^{-i*}, \xi^k, \mu\right) &= \sum_{k=1}^{K} \tau_k {e^{(m)}}^T \tilde{F}(\hat{y}(\xi^k), \hat{z}(\xi^k), \hat{s}(\xi^k), \mu) \\ &\le \sum_{k=1}^{K} \tau_k {e^{(m)}}^T \tilde{F}(y(\xi^k), z(\xi^k), s(\xi^k), \mu), \end{aligned} \tag{27}$$

for all $\left(y(\xi^k)^T, z(\xi^k)^T, s(\xi^k)^T,\ k = 1, \dots, K\right)^T \in S_2(x)$. By Proposition 2, such vector $\left(\hat{y}(\xi^k)^T, \hat{z}(\xi^k)^T, \hat{s}(\xi^k)^T,\ k = 1, \dots, K\right)^T \in S_2(x)$ exists.

Similar as bilevel programming [41], by assuming (27), we indeed consider the *optimistic version* of the two-stage stochastic equilibrium model (17)-(18)

$$\begin{array}{ll} \min_{\substack{x, y(\xi^k), z(\xi^k), s(\xi^k), \\ k=1,\dots,K}} & a^T x + \sum_{k=1}^{K} \tau_k {e^{(m)}}^T \tilde{F}(y(\xi^k), z(\xi^k), s(\xi^k), \mu) \\ \text{s.t.} & Ax \le h, \\ & x \ge 0, \\ & \left(y(\xi^k)^T, z(\xi^k)^T, s(\xi^k)^T,\ k = 1, \dots, K\right)^T \in S_2(x). \end{array} \tag{28}$$

The solution set of (28) is nonempty, since the feasible set of (28) is nonempty, convex, and compact, and the objective function is continuous.

Proposition 3. *The two-stage SP model (19) and the optimistic version (28) of the two-stage equilibrium model (17)-(18) are equivalent in the sense that they have the same solution set.*

Proof. See A.3 for the proof. □

The result of this proposition is very interesting and justifies the two-stage SP model (19), although the government becomes the only decision maker to medical supplies for an epidemic. The two-stage SP model (19) can be built directly from the view of optimality for the society and the government is the only decision maker. While the optimistic version (28) of the two-stage stochastic equilibrium model (17)-(18) is built by considering the competitions in the second stage for the dynamic distribution, and assume that the government is able to select an optimal decision for the whole society from the possible choices competed by the hospitals. The two models indeed provide the same solution sets. Thus we conclude that in medical supplies for an epidemic, a powerful and smart government can make decisions from the view of social optimality that are also acceptable for all hospitals in the sense that no hospital can decrease its own costs by unilaterally changing its strategy.

4.2. Further extensions of the model

First of all, the nonsmooth term $(M_{ijt}(\xi))_+$ in the penalty term of (3) can be replaced by a more general nonsmooth function $\mathcal{R}((M_{ijt}(\xi))_+)$. For instance, the out-layer function $\mathcal{R}(\varphi)$ can be quadratic, or cubic, or piecewise linear as

$$\mathcal{R}(\varphi)=\begin{cases}\delta_1\varphi, & \text{if } \frac{\varphi}{d_{ijt}(\xi_{it})}\in[0,\sigma_1],\\ \delta_2\varphi-(\delta_2-\delta_1)\sigma_1 d_{ijt}(\xi_{it}), & \text{if } \frac{\varphi}{d_{ijt}(\xi_{it})}\in[\sigma_1,\sigma_2],\\ \delta_3\varphi-[(\delta_3-\delta_2)\sigma_2+(\delta_2-\delta_1)\sigma_1]d_{ijt}(\xi_{it}), & \text{if } \frac{\varphi}{d_{ijt}(\xi_{it})}\in[\sigma_2,1],\end{cases} \tag{29}$$

where $0<\sigma_1<\sigma_2<1$ and $0<\delta_1<\delta_2<\delta_3$ are given parameters.

One can substitute $(M_{ijt}(\xi))_+$ by a new nonnegative variable $\theta_{ijt}(\xi)$ in the penalty term of (3), and add the following constraints to the two-stage stochastic equilibrium model: for any $i\in\mathcal{I}$, $j\in\mathcal{J}$, and for a.e. $\xi\in\Xi$,

$$\begin{aligned}\theta_{ijt}(\xi)&\geq M_{ijt}(\xi),\\ \theta_{ijt}(\xi)&\geq 0.\end{aligned} \tag{30}$$

The two-stage stochastic equilibrium model can be transformed to the smooth two-stage SP model. In case $\Xi=\{\xi^1,\dots,\xi^K\}$, this strategy will increase $mnlK$ functional constraints and $mnlK$ nonnegativity constraints to the original model. The strategy of this type is beneficial for $mnlK$ of moderate size, but may increase computational cost dramatically when $mnlK$ is huge, or facing more complex nonsmooth penalty functions such as the piecewise function $\mathcal{R}((M_{ijt}(\xi))_+)$ in (29). The smoothing strategy we use in this paper can be easily extended to more complex nonsmooth penalty functions such as (29), and only enrolls a smoothing parameter $\mu\in R$ instead.

Secondly, it is very interesting to consider the case when the quantity of a type of non-reusable medical supply such as drugs delivered to hospital i on day t can be greater than the demands. Motivated by [2], for such non-reusable medical supply j, we can modify the amount of unmet demands of hospital i for medical supply j on day t as

$$M_{ijt}(\xi)=\sum_{o=1}^{t}[d_{ijo}(\xi_{io})-y_{ijo}(\xi)-z_{ijo}(\xi)-s_{ijo}(\xi)]. \tag{31}$$

Note that if $M_{ijt}(\xi)$ is defined in this way, in case that $M_{ijt}(\xi)<0$, i.e., the quantity of medical supply j that can be used on day t is more than the demands of hospital i, then the excess quantity will be delivered to the next day as part of the quantity that can be used. In case that $M_{ijt}(\xi)>0$, i.e., there exist unmet demands on day t, the unmet demands will be penalized each day if they are still not be covered in the following days. It is more reasonable to assign a time-variant penalty coefficient that increases with the delay time, e.g.,

$$c_{ij}\mathcal{Q}(\frac{t}{l}), \tag{32}$$

where c_{ij} is the unit penalty coefficient, t represents the day t, and l is the total number of days we consider. Here $\mathcal{Q}\left(\frac{t}{l}\right)$ monotonically increases with $\frac{t}{l}$. In [2], $\mathcal{Q}(\frac{t}{l})=(\frac{t}{l})^3$ is employed.

For the extensions mentioned above, the analysis also holds for the nonsmooth two-stage stochastic equilibrium model, its smooth approximation, as well as the corresponding two-stage SP and the two-stage SVI model. The PHM outlined in the next section is also applicable to solve the two-stage SVI models for extensions, with a slight modification of the function value and the Jacobian matrix of the function involved in the NCP.

5. PHM and strategy for updating smoothing parameter

The discretized two-stage SVI (22) is a deterministic variational inequality (VI) which may be solved by any existing solvers. However, when K is large, it is more efficient to solve (22) using the PHM [14] that can exploit the two-stage structure. As shown in [14], when (22) is of maximal monotone type, the PHM has solid convergence. The PHM has very clear structure and is easy to code. The efficient semismooth Newton method [42,43] can be enrolled in the PHM for the two-stage SVI, and the efficiency of the PHM has been demonstrated in [31,35]. Moreover, the PHM is attractive for dealing with further extensions of the proposed model, because it does not require the objective function to be linear or quadratic as some existing efficient solvers for the two-stage SP do, including the "linprog" in Matlab and the Benders decomposition method [44,45]. The PHM performs well when facing large number of scenarios, by using parallel computation.

We use the notation $y_k=y(\xi^k)$, $z_k=z(\xi^k)$, $s_k=s(\xi^k)$, $\pi_{yk}=\pi_y(\xi^k)$, $\pi_{zk}=\pi_z(\xi^k)$, $\pi_{sk}=\pi_s(\xi^k)$ and $p_k=p(\xi^k)$ for simplicity.

Algorithm 1 (PHM).

Given initial points $x^0 \in R^{mn}$, $\lambda^0 \in R^{mn+n}$,let $y_k^0,\ z_k^0,\ s_k^0 \in R^{mnl}$, $\pi_{yk}^0,\ \pi_{zk}^0 \in R^{nl}$, $\pi_{sk}^0 \in R^{mn}$ and $u_k^0 \in R^{mn}$, $v_k^0 \in R^{mn+n}$, for $k = 1,\dots,K$, such that $\sum_{k=1}^K \tau_k u_k^0 = 0$, $\sum_{k=1}^K \tau_k v_k^0 = 0$. Choose a step size $r > 0$ and a smoothing parameter $\mu > 0$. Set $\nu = 0$.

Step 1. For $k = 1, 2, \dots, K$, solve the NCP

$$\begin{cases} 0 \le & a + A^T\lambda_k - \pi_{sk} + r(x_k - x^\nu) + u_k^\nu \perp x_k \ge 0, \\ 0 \le & h - Ax_k + r(\lambda_k - \lambda^\nu) + v_k^\nu \perp \lambda_k \ge 0, \\ 0 \le & b + p_k + {I_{nl}^{(m)}}^T \pi_{yk} + Q_{yzs}^{\xi^k\mu} + r(y_k - y_k^\nu) \perp y_k \ge 0, \\ 0 \le & b + {I_{nl}^{(m)}}^T \pi_{zk} + Q_{yzs}^{\xi^k\mu} + r(z_k - z_k^\nu) \perp z_k \ge 0, \\ 0 \le & \hat{B}^T\pi_{sk} + Q_{yzs}^{\xi^k\mu} + r(s_k - s_k^\nu) \perp s_k \ge 0, \\ 0 \le & y_{\max}^{\text{dist}} - I_{nl}^{(m)} y_k + r(\pi_{yk} - \pi_{yk}^\nu) \perp \pi_{yk} \ge 0, \\ 0 \le & z_{\max}^{\text{soci}} - I_{nl}^{(m)} z_k + r(\pi_{zk} - \pi_{zk}^\nu) \perp \pi_{zk} \ge 0, \\ 0 \le & x_k - \hat{B}s_k + r(\pi_{sk} - \pi_{sk}^\nu) \perp \pi_{sk} \ge 0, \end{cases} \tag{33}$$

using the semismooth Newton method and obtain a solution $(\hat{x}_k^\nu, \hat{\lambda}_k^\nu, \hat{y}_k^\nu, \hat{z}_k^\nu, \hat{s}_k^\nu, \hat{\pi}_{yk}^\nu, \hat{\pi}_{zk}^\nu, \hat{\pi}_{sk}^\nu)$, $k = 1, 2, \dots, K$.

Step 2. For $k = 1, 2, \dots, K$, let

$$\begin{array}{ll} x^{\nu+1} = \sum_{k=1}^K \tau_k \hat{x}_k^\nu, & \lambda^{\nu+1} = \sum_{k=1}^K \tau_k \hat{\lambda}_k^\nu, \\ y_k^{\nu+1} = \hat{y}_k^\nu, \quad z_k^{\nu+1} = \hat{z}_k^\nu, \quad s_k^{\nu+1} = \hat{s}_k^\nu, & \pi_{yk}^{\nu+1} = \hat{\pi}_{yk}^\nu, \quad \pi_{zk}^{\nu+1} = \hat{\pi}_{zk}^\nu, \quad \pi_{sk}^{\nu+1} = \hat{\pi}_{sk}^\nu, \\ u_k^{\nu+1} = u_k^\nu + r(\hat{x}_k^\nu - x^{\nu+1}), & v_k^{\nu+1} = v_k^\nu + r(\hat{\lambda}_k^\nu - \lambda^{\nu+1}). \end{array}$$

Set $\nu = \nu + 1$, and go to Step 1.

Note that the proximal term ensures the existence of a unique solution to the subproblem in (33). The elements u_k^ν and v_k^ν have the role of Lagrange multipliers corresponding to the nonanticipativity constraints for the vectors x and λ. We use the efficient semismooth Newton method [43] to solve the subproblems in (33) for $k = 1,\dots,K$.

Strategy for updating μ and stopping criteria

Denote

$$w^{(1)} = \begin{pmatrix} x \\ \lambda \end{pmatrix}, \quad \bar{G}(w^{(1)}) = \begin{pmatrix} a + A^T\lambda - \sum_{k=1}^K \tau_k \pi_{sk} \\ h - Ax \end{pmatrix}.$$

For any ξ^k, $k = 1,\dots,K$, let us denote

$$w_k^{(2)} = \begin{pmatrix} y_k \\ z_k \\ s_k \\ \pi_{yk} \\ \pi_{zk} \\ \pi_{sk} \end{pmatrix}, \quad \hat{G}(w_k^{(2)}, \mu) = \begin{pmatrix} b + p_k + {I_{nl}^{(m)}}^T \pi_{yk} + Q_{yzs}^{\xi^k\mu} \\ b + {I_{nl}^{(m)}}^T \pi_{zk} + Q_{yzs}^{\xi^k\mu} \\ \hat{B}^T\pi_{sk} + Q_{yzs}^{\xi^k\mu} \\ y_{\max}^{\text{dist}} - I_{nl}^{(m)} y_k \\ z_{\max}^{\text{soci}} - I_{nl}^{(m)} z_k \\ x_k - \hat{B}s_k \end{pmatrix}.$$

Let

$$\begin{aligned} \eta_1 &= \frac{\left\| \min\left\{ {w^{(1)}}^{\nu+1}, \bar{G}^{\nu+1}({w^{(1)}}^{\nu+1}) \right\} \right\|}{1 + \left\| {w^{(1)}}^{\nu+1} \right\|}, \\ \eta_2(k,\mu) &= \frac{\left\| \min\left\{ {w_k^{(2)}}^{\nu+1}, \hat{G}^{\nu+1}({w_k^{(2)}}^{\nu+1}, \mu) \right\} \right\|}{1 + \left\| {w_k^{(2)}}^{\nu+1} \right\|}, \\ \eta(\mu) &= \max\{\eta_1, \max\{\eta_2(k,\mu),\ k = 1, 2, \dots, K\}\}. \end{aligned} \tag{34}$$

We adopt a strategy for updating μ as similar to that used in the smoothing methods for solving constrained nonsmooth optimization [37,38,46,47]. This strategy is helpful for accelerating the computational speed compared with that using a fixed μ, and is promising to get an approximate solution of the original nonsmooth two-stage SP. To be specific, we do the following procedures.

- Choose a relatively large initial smoothing parameter μ_0 from the beginning, an acceptable level of the final smoothing parameter $0 < l_\mu < \mu_0$, and constants $\gamma > 0$, $\delta_\mu \in (0, 1)$.
- For $k \ge 0$, we use the iterates of the PHM (Algorithm 1) to approximately solve the two-stage SVI with μ_ν. If $\eta(\mu_\nu) < \gamma\mu_\nu$ and $\mu_\nu > l_\mu$, that is, the two-stage SVI with μ_ν is approximately solved but μ_ν is not acceptable as the final smoothing parameter, we then set $\mu_{\nu+1} = \delta_\mu \mu_\nu$, otherwise, we set $\mu_{\nu+1} = \mu_\nu$.

For $k = 1, \ldots, K$, define

$$w_k = \begin{pmatrix} x_k \\ \lambda_k \\ y_k \\ z_k \\ s_k \\ \pi_{yk} \\ \pi_{zk} \\ \pi_{sk} \end{pmatrix}, \quad G(w_k) = \begin{pmatrix} a + A^T\lambda_k - \pi_{sk} + r(x_k - x^\nu) + u_k^\nu \\ h - Ax_k + r(\lambda_k - \lambda^\nu) + v_k^\nu \\ b + p_k + I_{nl}^{(m)^T}\pi_{yk} + Q_{yzs}^{\xi^k\mu} + r(y_k - y_k^\nu) \\ b + I_{nl}^{(m)^T}\pi_{zk} + Q_{yzs}^{\xi^k\mu} + r(z_k - z_k^\nu) \\ \hat{B}^T\pi_{sk} + Q_{yzs}^{\xi^k\mu} + r(s_k - s_k^\nu) \\ y_{\max}^{\text{dist}} - I_{nl}^{(m)}y_k + r(\pi_{yk} - \pi_{yk}^\nu) \\ z_{\max}^{\text{soci}} - I_{nl}^{(m)}z_k + r(\pi_{zk} - \pi_{zk}^\nu) \\ x_k - \hat{B}s_k + r(\pi_{sk} - \pi_{sk}^\nu) \end{pmatrix}.$$

The implementation of the semismooth Newton method requires the Jacobian matrix J_k of $G(w_k)$. We give the form of the Jacobian matrix J_k of $G(w_k)$ below.

Denote

$$U_\mu^{\xi^k} = \left(\frac{c_{ij}e^{M_{ijt}(\xi^k)/\mu}}{\mu(e^{M_{ijt}(\xi^k)/\mu} + 1)^2}, \ \forall i, j, t \right)^T, \quad C_{yzs}^{\xi^k\mu} = \hat{H}\mathrm{diag}(U_\mu^{\xi^k})\hat{H}^T.$$

The Jacobian matrix of $G(w_k)$ is

$$J_k = \bar{J}_k + rI_{3mnl+2nl+3mn+n}, \tag{35}$$

where

$$\bar{J}_k = \begin{pmatrix} 0 & A^T & 0 & 0 & 0 & 0 & 0 & -I \\ -A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{yzs}^{\xi^k\mu} & C_{yzs}^{\xi^k\mu} & C_{yzs}^{\xi^k\mu} & I_{nl}^{(m)^T} & 0 & 0 \\ 0 & 0 & C_{yzs}^{\xi^k\mu} & C_{yzs}^{\xi^k\mu} & C_{yzs}^{\xi^k\mu} & 0 & I_{nl}^{(m)^T} & 0 \\ 0 & 0 & C_{yzs}^{\xi^k\mu} & C_{yzs}^{\xi^k\mu} & C_{yzs}^{\xi^k\mu} & 0 & 0 & \hat{B}^T \\ 0 & 0 & -I_{nl}^{(m)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{nl}^{(m)} & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & -\hat{B} & 0 & 0 & 0 \end{pmatrix}.$$

The semismooth Newton method is efficient, since the Jacobian matrix J_k is sparse and has special structure.

6. Case study: COVID-19 pandemic in Wuhan

In this section, we focus on the storage and dynamic distribution of medical supplies in the city of Wuhan in China that suffered from COVID-19. All experiments are performed in Windows 10 on an Intel Core 10 CPU at 3.70 GHZ with 64 GB of RAM, using MATLAB R2021a.

6.1. Problem description

Wuhan is located in Hubei province, China, which includes 13 districts with a population of approximately 11 million people and with a size of approximately 8500 square kilometers. The sudden outbreak of COVID-19 caused tension in the city for months. At the beginning, medical supplies were extremely in short, while the number of infected people dramatically increased.

We consider 7 types of medical supplies in 7 hospitals over a period of 14 days. The hospitals are Jinyintan Hospital, Tongji Medical College of Huazhong University of Science and Technology, Union Hospital Tongji Medical College Huazhong University of Science and Technology, Renmin Hospital of Wuhan University Hubei General Hospital, Huoshenshan Hospital, Leishenshan Hospital, Wuhan No.1 Hospital, which are selected because they are the main hospitals for the COVID-19 treatment. Jinyintan Hospital (H1), Huoshenshan Hospital (H5) and Leishenshan Hospital (H6) mainly treated patients with mild infection and the remaining four hospitals mainly treated patients with severe infection. According to the list of key supplies for epidemic prevention and control (medical emergency) on the official website of the Ministry of Industry and Information Technology, China, we consider the following seven types of medical supplies: drugs, inspection and testing articles, protective articles, disinfection supplies, vehicle equipments, disinfection equipments, and electronic instruments. The first four types of medical supplies are non-reusable, and the last three are reusable. We consider the 14 days from February 10th, 2020 to February 23rd, 2020, which was the period of the greatest scarcity of medical supplies. We assume that there is only one warehouse from which medical supplies are transported to hospitals in the second stage. The locations of the hospitals and the warehouse are marked on the map given in Fig. 1.

Based on the volume and weight, shelf life, acquisition cost, warehouse rent, holding cost of each medical supply and the average number of years that of such epidemics occur, we estimate the unit reserve price of each medical supply for each

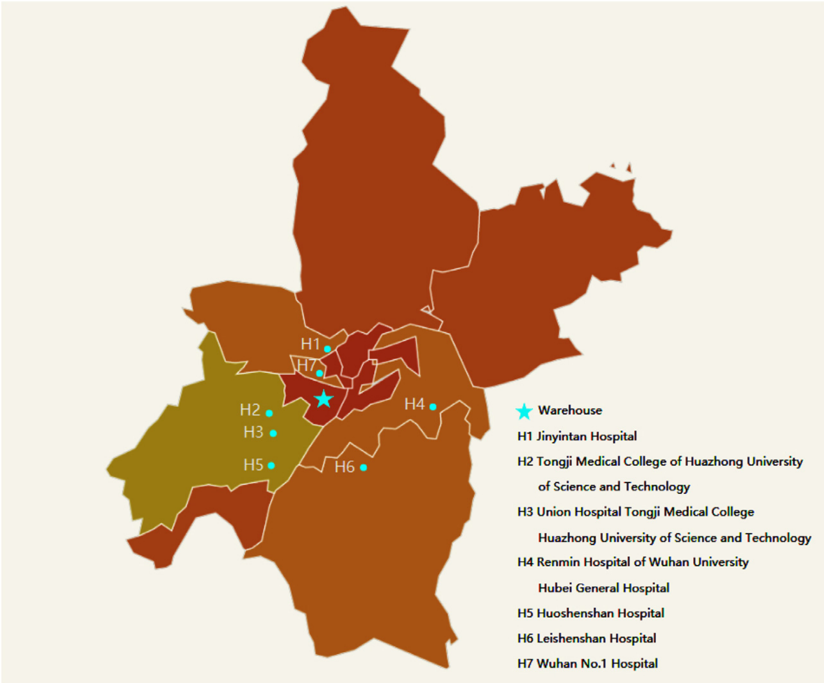


Fig. 1. Wuhan Map: hospitals and the warehouse.

Table 1
Unit reserve prices of medical supplies in each hospital.

MS	a_{ij}(CNY/unit)						
	H1	H2	H3	H4	H5	H6	H7
1	2600	2610	2610	2630	2610	2620	2640
2	21	22	22	24	22	23	25
3	220	225	225	235	225	230	240
4	22	23	23	25	23	24	26
5	402000	402500	402500	403500	402500	403000	404000
6	3100	3120	3120	3160	3120	3140	3180
7	10100	10150	10150	10250	10150	10200	10300

MS=Medical supply, 1=Drugs, 2=Inspection and testing articles, 3=Protective articles, 4=Disinfection supplies, 5=Vehicle equipments, 6=Disinfection equipments, 7=Electronic instruments.

Table 2
Unit transport price, demand ratio, and upper limit.

MS	b_j(CNY/unit·km)	d_j(/patient·day)	$x_{j\max}$	$y_{j\max}$(/day)	$z_{j\max}$(/day)	$x_{ij\max}$
1	1	2	1000	10000	50	400
2	0.2	1	0	5700	100	400
3	0.5	6	1000	32800	1000	400
4	0.5	1	1000	4700	100	300
5	10	0.02	15	80	8	10
6	1	0.05	50	180	20	20
7	1.5	0.1	80	400	20	20

hospital in Table 1. More specifically, the unit reserve price a_{ij} is a rough estimation?> of the current market price from Baidu's B2B platform (https://b2b.baidu.com) for medical supply 5 – vehicle equipments, and JD mall (https://www.jd.com/) for the other medical supplies. Because the housing prices (https://www.anjuke.com/fangjia/wuhan2020/) of different districts in Wuhan are different, we mainly determine the unit reserve prices according to the housing prices of the districts where the hospitals are located in. Based on the preservation requirements and the volume sizes during the transportation, we give the unit transport prices for all the medical supplies, which are roughly estimated according to the transport company's charge standard (https://www.deppon.com/newwebsite/mail/price). The transportation costs for distance less than 50km will be charged as the costs of 50km. Each patient's daily needs for each medical supply is a rough estimation. As-

Table 3
Number of infected people.

Hospital	ξ_{it}													
	D1	D2	D3	D4	D5	D6	D7	D8	D9	D10	D11	D12	D13	D14
H1	831	807	793	822	811	822	788	688	647	592	589	592	569	552
H2	1040	1040	1047	1048	1047	1045	1023	1022	993	1008	1013	974	946	937
H3	787	778	791	791	778	777	770	736	749	741	745	750	729	714
H4	800	774	794	772	774	788	785	780	760	730	732	701	691	669
H5	1013	1000	1014	1014	1010	1007	1021	1020	1002	1005	1008	966	1008	964
H6	123	159	483	473	567	602	746	818	980	977	962	1059	1052	1042
H7	445	614	629	1052	1052	1082	1093	1089	1074	1056	1049	1047	1029	1020

D1=The first day of the time horizons, followed by the second, and so on.

Table 4
Unit prices of medical supplies.

MS	p_{jt}(CNY10^3/unit)													
	D1	D2	D3	D4	D5	D6	D7	D8	D9	D10	D11	D12	D13	D14
1	2.55	3	3.5	3.6	3.8	4.5	4.8	4.8	5	5.2	5.5	6	6	6
2	0.02	0.025	0.029	0.03	0.03	0.032	0.035	0.035	0.035	0.04	0.045	0.05	0.05	0.05
3	0.2	0.3	0.45	0.55	0.6	0.7	0.8	0.8	0.8	0.8	0.9	1	1	1
4	0.02	0.03	0.04	0.05	0.06	0.08	0.09	0.11	0.15	0.18	0.2	0.26	0.26	0.26
5	400	405	410	410	415	420	425	425	430	440	450	455	455	455
6	3	3.5	4	4.1	4.2	4.3	4.5	4.5	4.5	4.6	4.6	4.8	4.8	4.8
7	10	12	14	15	16	16.5	18	18	18	20	22	22	22	22

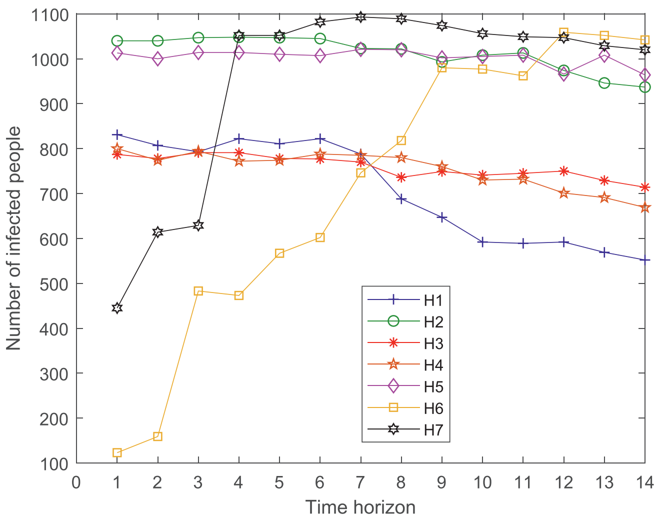


Fig. 2. Number of infected people in each hospital.

sume that the daily allocation upper limits of government and social non-targeted donation are the same every day. Based on the government budgets and the daily production capacities of medical supplies, we give the upper bound vector for budgets $x_{\max}^{\mathrm{budg}}$, the upper bound vector from government distribution $y_{\max}^{\mathrm{dist}}$ and the upper bound vector by social non-targeted donation $z_{\max}^{\mathrm{soci}}$. For simplicity, we give each hospital the same storage ceiling $x_{\max}^{\mathrm{hosp}}$. Table 2 lists the specific values of the above parameters. We make samples based on the daily hospital infection data from February 10th, 2020 to February 23rd, 2020, published by Wuhan Municipal Health Commission, as shown in Table 3 and plotted in Fig. 2. Wuhan suffered from the largest number of patients and the least medical supplies in these 14 days. The vector of prices $p(\xi)$ of medical supplies changes over time, and the trend has a positive relation with the total number of infected people each day, as shown in Table 4. In the tables, "MS" refers to medical supply.

Unlike the other parameters that have clear rules for estimating, the penalty coefficient c_{ij} for unmet demands may follow different rules guided from the government to reflect fairness. We consider three different cases for the penalty coefficients described below. The specific vectors of penalty coefficients are listed in Table 5.

- **Case 1**: The vectors of penalty coefficients for the hospitals are the same.

Table 5
Penalty coefficient c_{ij} for each of the three cases (CNY10^2/unit).

c_{ij}	Case	Hospital	MS						
			1	2	3	4	5	6	7
	Case 1	H1–H7	1020	1	60	6	8000	240	1500
	Case 2	H1,H5,H6	1020	1	60	6	8000	240	1500
		H2,H3,H4,H7	1025	6	65	11	8005	245	1505
	Case 3	H1	1020	1	60	6	8000	240	1500
		H2	1026	7	66	12	8006	246	1506
		H3	1024	5	64	10	8004	244	1504
		H4	1023	4	63	9	8003	243	1503
		H5	1022	3	62	8	8002	242	1502
		H6	1021	2	61	7	8001	241	1501
		H7	1025	6	65	11	8005	245	1505

Table 6
Problem size.

	SP	SVI
decision variables	49 + 2058 $\times$ sn	105 + 2303 $\times$ sn
functional constraints	56 + 245 $\times$ sn	-
nonnegativity constraints	49 + 2058 $\times$ sn	-

Table 7
Elapsed time while sn increases.

	Time (s)			
sn	20	200	500	1000
PHM	130.5	618.2	1169.2	1997.2
linprog	0.9	34.6	412.4	3572.7
Benders	100.9	1268.3	3651.3	-

Table 8
Elapsed time.

	VI			SVI		
Case	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
Time (s)	16.8	33.6	29.9	618.2	1166.4	1224.2

- **Case 2**: Two different vectors of penalty coefficients are adopted. One with lower values of components is for H1, H5 and H6 that mainly treated patients with mild infection, and the other is for the remaining four hospitals that mainly treated patients with severe infection.
- **Case 3**: Each hospital is assigned a distinct vector of penalty coefficients. Not only the mild/severe degree of patients is addressed, but also the numbers of patients in hospitals are taken into consideration.

Let "sn" be the number of scenarios. We randomly take sn scenarios within a band of a 10% fluctuation of the actual number of infected people as the possible scenarios of ξ with equal probability. We use the strategy in Section 5 for updating the smoothing parameter, where we set $\mu_0 = 20$, $l_\mu = 0.7, \gamma = 0.001$, and $\alpha = 0.5$ for all the tests. The computed solutions satisfy $\eta(\mu_\nu) < 7 \times 10^{-4}$ and $\mu_\nu < l_\mu$ for all the different cases. The parameter r in the PHM in each experiment is set to be 1.

Table 6 shows the dimensions "$d_1 + d_2\times$ sn" of the two-stage SP model and the transformed two-stage SVI model, where d_1 refers to the dimension corresponding to the first stage that does not change with sn, and d_2 refers to the dimension corresponding to the second stage for one scenario.

It is clear that when sn is large, the dimensions of the two-stage SP and the two-stage SVI become very large.

6.2. Comparison with existing solvers

In this subsection, we compare our PHM using updating rule for μ, with the Matlab code "linprog" for linear programming, and the Benders decomposition method for two-stage stochastic linear programming (the Matlab code written by Jeonghun Song of Seoul National University downloaded from website[1]), with different sn ranging from 20 to 1000 under

[1] https://www.mathworks.com/matlabcentral/fileexchange/69060-benders-decomposition-for-stochastic-linear-programming.

Table 9
Costs using different models and solutions.

Case	Cost list	VI (CNY)	SVI (CNY)	SVI using solution of VI (CNY)	Percentage of improvement
Case 1	Total cost	3,643,553,093	3,671,411,266	3,855,724,611	4.78%
	Acquisition and holding cost	9,797,453	9,870,622	9,797,453	
	Transportation cost	21,296,679	21,299,820	21,296,679	
	Government purchasing cost	1,012,934,477	1,013,526,136	1,012,934,477	
	Penalty cost	2,599,524,482	2,626,714,685	2,811,695,999	
Case 2	Total cost	3,647,823,189	3,675,413,216	3,881,660,478	5.31%
	Acquisition and holding cost	9,802,026	9,875,669	9,802,026	
	Transportation cost	21,296,870	21,295,726	21,296,870	
	Government purchasing cost	1,012,936,255	1,013,307,167	1,012,936,255	
	Penalty cost	2,603,788,037	2,630,934,652	2,837,625,326	
Case 3	Total cost	3,651,239,735	3,678,874,951	3,886,157,568	5.33%
	Acquisition and holding cost	9,800,353	9,875,745	9,800,353	
	Transportation cost	21,296,909	21,295,860	21,296,909	
	Government purchasing cost	1,012,936,323	1,013,311,363	1,012,936,323	
	Penalty cost	2,607,206,149	2,634,391,981	2,842,123,981	

Table 10
Ten different storage upper limits of $x_{j\max}$ from budgets.

MS	$x_{j\max}$									
	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
1	0	1e3	1e4	1.4e4	6e4	1.4e5	3e5	6e5	1.2e6	1e9
2	0	0	0	0	0	0	0	0	0	0
3	0	1e3	1e4	1.4e4	6e4	4.592e5	9e5	1.8e6	3.6e6	1e9
4	0	1e3	1e4	1.4e4	6e4	6.58e4	1.2e5	2.4e5	4.8e5	1e9
5	0	1.5e1	1.5e3	2e3	3e3	7.5e3	1e4	2e4	4e4	1e9
6	0	5e1	5e2	8e2	1e3	2.5e3	5e3	1e4	5e4	1e9
7	0	8e1	8e2	9e2	1e3	4e3	8e3	1.6e4	3.2e4	1e9

Table 11
Optimal solution of storage by stochastic model.

x^*_{ij}	Case	Hospital	MS						
			1	2	3	4	5	6	7
	Case 1	H1	400	0	400	300	3	15	15
		H2	202	0	220	263	3	12	19
		H3	198	0	153	174	2	10	13
		H4	0	0	0	0	3	0	13
		H5	201	0	227	263	3	12	18
		H6	0	0	0	0	0	0	2
		H7	0	0	0	0	0	0	1
	Case 2	H1	400	0	400	300	3	15	15
		H2	343	0	347	300	3	12	19
		H3	256	0	247	289	2	10	14
		H4	0	0	0	0	3	0	12
		H5	1	0	6	111	3	12	18
		H6	0	0	0	0	0	0	1
		H7	0	0	0	0	0	0	1
	Case 3	H1	0	0	278	0	3	15	15
		H2	327	0	276	300	3	13	19
		H3	282	0	129	300	2	10	14
		H4	0	0	0	0	3	0	12
		H5	391	0	318	300	3	12	19
		H6	0	0	0	100	0	0	1
		H7	0	0	0	0	0	0	0
	$x_{j\max}$		1000	0	1000	1000	15	50	80

Case 1 in Table 7. To be specific, we use the strategy to assign an auxiliary vector $\theta_{ijt}(\xi)$ as in (30) and transform the non-smooth two-stage stochastic equilibrium model to a smooth two-stage SP. Then the linear programming (the Matlab code “linprog”) and the Benders decomposition method can be employed.

Because the solution set of the linear programming (corresponding to the two-stage SVI model) is not a singleton, we find that the Matlab code “linprog” by setting options to use ‘interior-point-legacy’ algorithm performs stable and the fairness of the solution is much better than that obtained by setting options to use ‘dual-simplex’ algorithm, although they have

Table 12
Optimal solution of storage for the same a_{ij} and c_{ij}.

x^*_{ij}	Hospital	MS						
		1	2	3	4	5	6	7
	H1	53	0	142	100	3	9	14
	H2	185	0	151	180	3	11	18
	H3	121	0	86	95	2	8	13
	H4	114	0	130	98	3	8	13
	H5	181	0	158	180	3	10	18
	H6	177	0	201	168	0	1	2
	H7	168	0	132	180	0	3	2
	$x_{j\max}$	1000	0	1000	1000	15	50	80

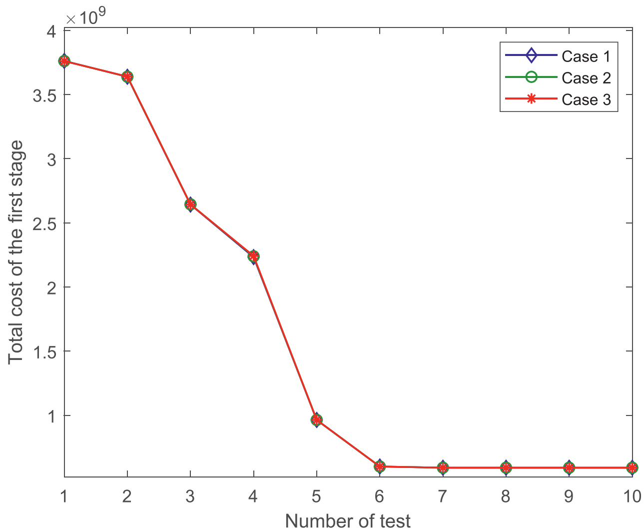
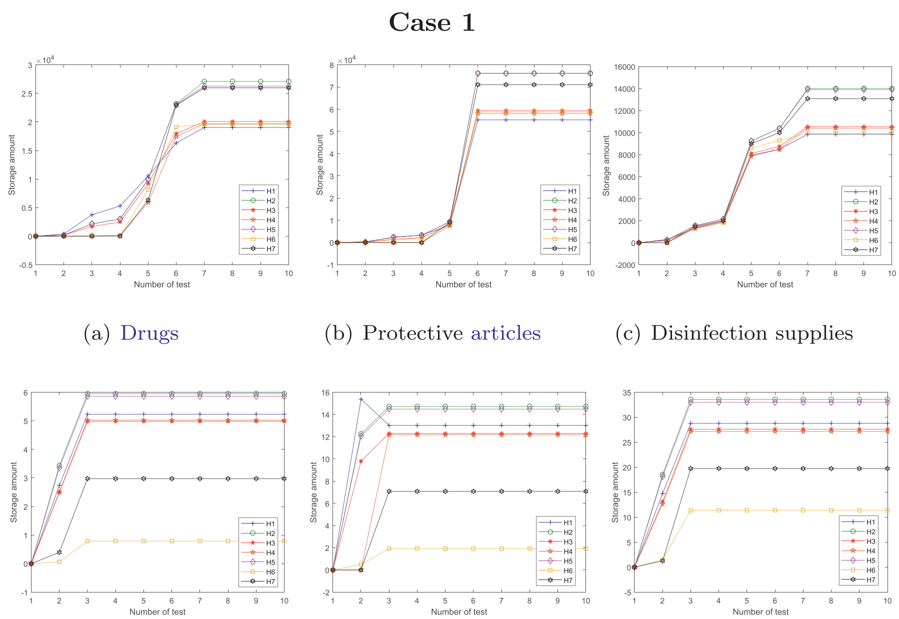


Fig. 3. Objective values in the first stage from no storage to sufficient upper limits of storage.

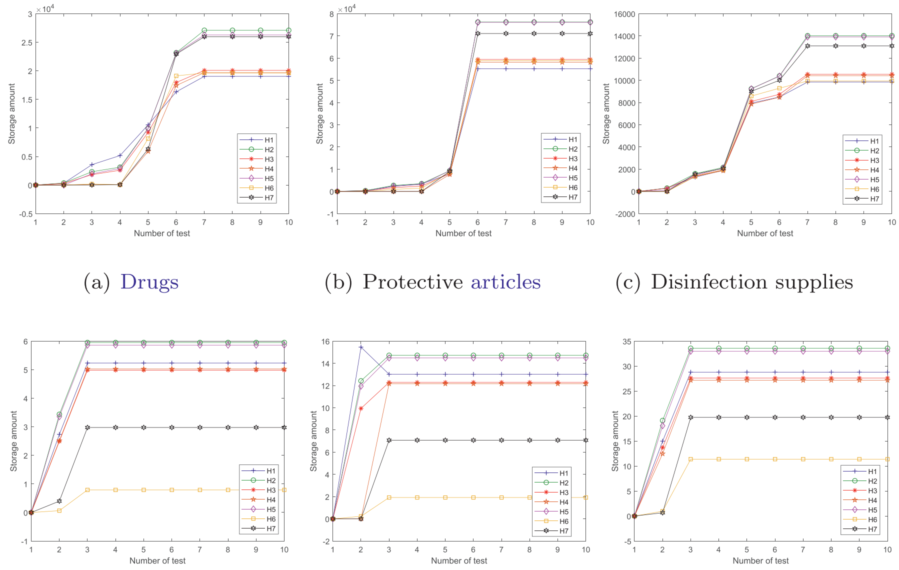


(a) Drugs (b) Protective articles (c) Disinfection supplies

(d) Vehicle equipments (e) Disinfection equipments (f) Electronic instruments

Fig. 4. Storage amounts of medical supplies to hospitals.

Case 2

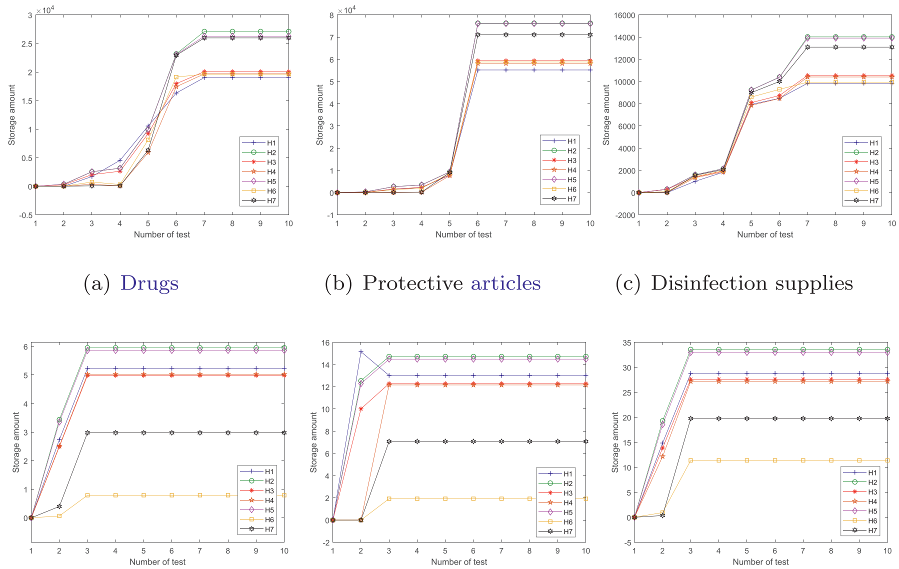


(a) Drugs (b) Protective articles (c) Disinfection supplies

(d) Vehicle equipments (e) Disinfection equipments (f) Electronic instruments

Fig. 5. Storage amounts of medical supplies to hospitals.

Case 3



(a) Drugs (b) Protective articles (c) Disinfection supplies

(d) Vehicle equipments (e) Disinfection equipments (f) Electronic instruments

Fig. 6. Storage amounts of medical supplies to hospitals.

approximately the same final objective values. Therefore, when implementing the "linprog" in Matlab and the Benders decomposition method that also involved the "linprog" code, we always set options to use 'interior-point-legacy' algorithm. The computed solutions obtained by using 'interior-point-legacy' are comparable to those using the PHM in terms of fairness.

For each sn, we record the elapsed time in seconds. The limitation for the elapsed time is set to be 5 hours. Algorithms have not reached the stopping criteria, or fail to find a feasible solution are considered to be failed. Note that the Benders decomposition method has not reached the stopping criteria after running for 5 hours at sn = 1000. We find that the

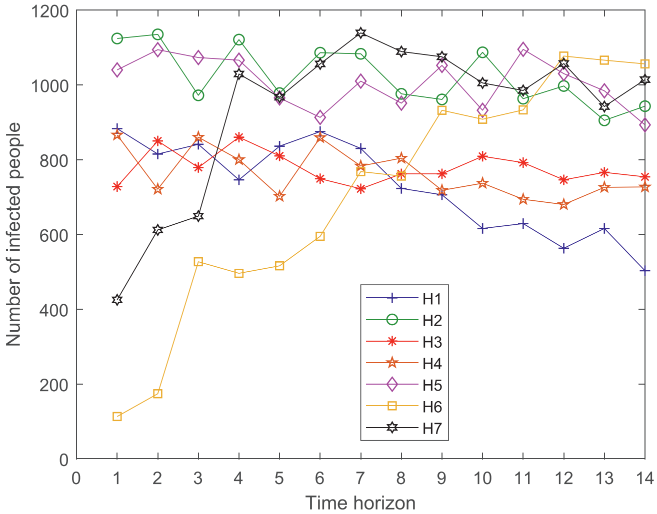
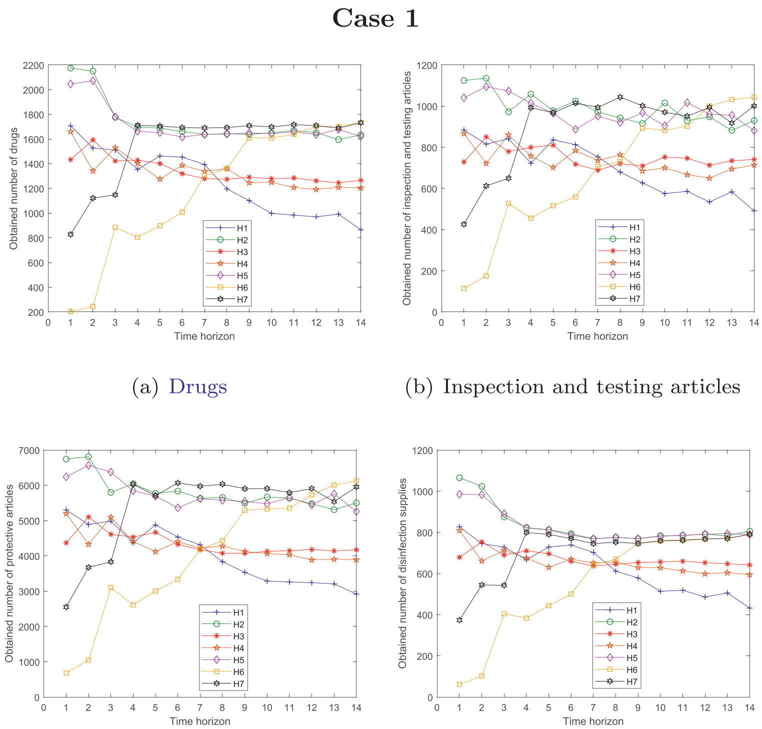


Fig. 7. Number of infected people in each hospital under scenario ξ^1.



(a) Drugs (b) Inspection and testing articles

(c) Protective articles (d) Disinfection supplies

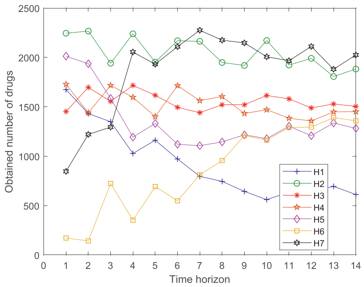
Fig. 8. Dynamic distributions of non-reusable medical supplies to hospitals for 14 days.

objective values obtained by the three algorithms are approximately the same for successful runs, and the PHM performs the best when sn = 1000, by using parallel computation.

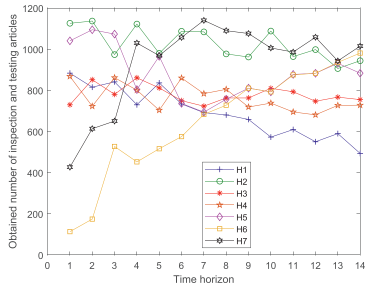
6.3. Stochastic VS. deterministic

We compare the two-stage SVI model with the deterministic VI model for the three cases to evaluate their performances in addressing demand uncertainty. We use sn = 200 for the two-stage SVI model, and use the number of real infected people for the deterministic VI model.

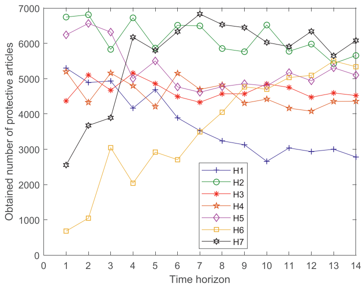
Case 2



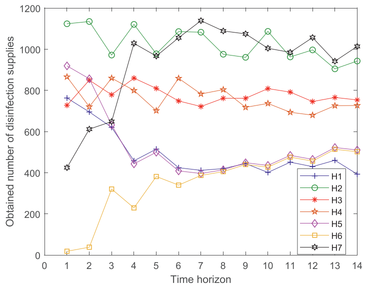
(a) Drugs



(b) Inspection and testing articles



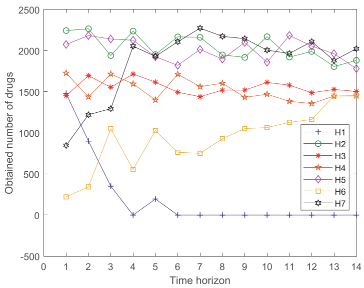
(c) Protective articles



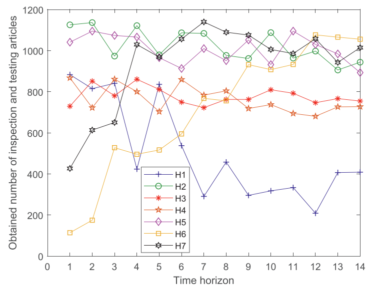
(d) Disinfection supplies

Fig. 9. Dynamic distributions of non-reusable medical supplies to hospitals for 14 days.

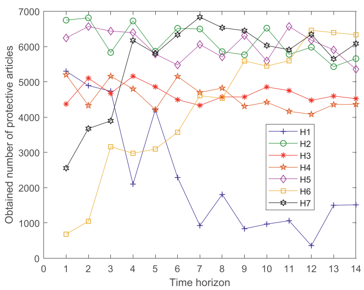
Case 3



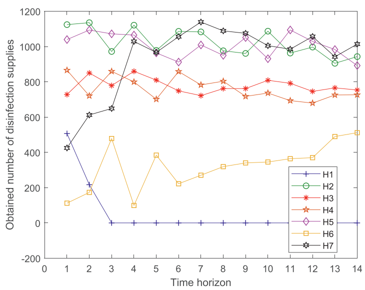
(a) Drugs



(b) Inspection and testing articles

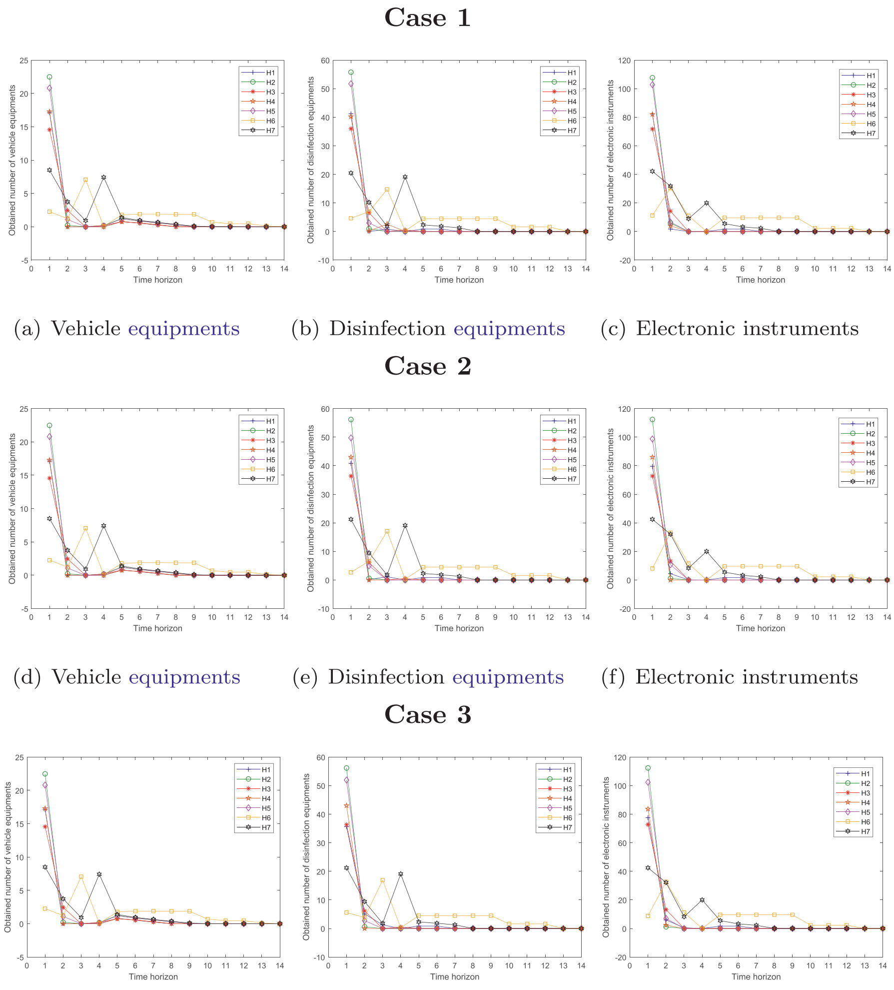


(c) Protective articles



(d) Disinfection supplies

Fig. 10. Dynamic distributions of non-reusable medical supplies to hospitals for 14 days.



(a) Vehicle equipments (b) Disinfection equipments (c) Electronic instruments

(d) Vehicle equipments (e) Disinfection equipments (f) Electronic instruments

(g) Vehicle equipments (h) Disinfection equipments (i) Electronic instruments

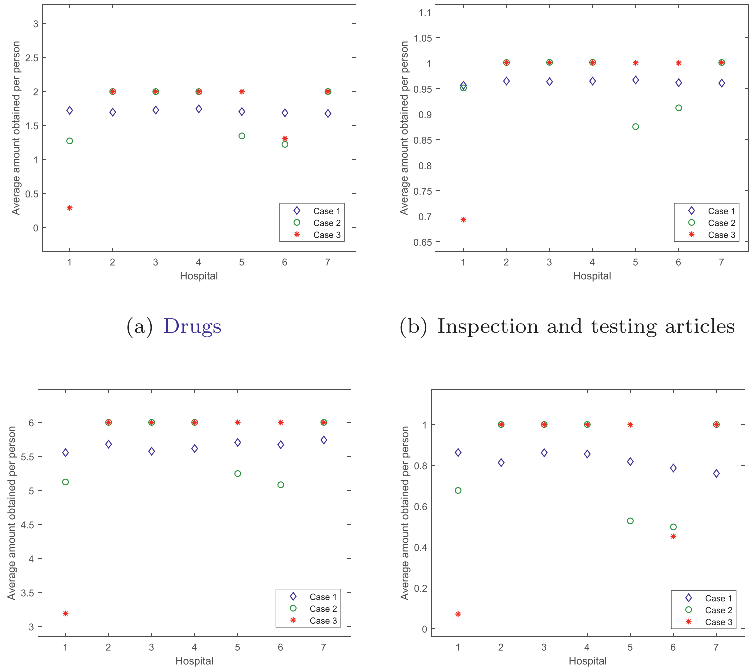
Fig. 11. Dynamic distributions of reusable medical supplies to hospitals for 14 days.

Table 8 lists the elapsed time. Table 9 lists the objective value (total cost), followed by specific costs including the acquisition and holding costs, the transportation costs, the government purchasing costs and the penalty costs, of the deterministic VI model and the two-stage SVI model, respectively, as well as the costs of substituting the solution of the deterministic VI model into the two-stage SVI model. The last column of Table 9 lists the percentage of improvement in the total cost achieved by using the two-stage SVI model instead of the deterministic VI model. The objective values of the deterministic VI model are approximately 3.64×10^9 CNY, 3.65×10^9 CNY, and 3.65×10^9 CNY for the three cases, respectively. The objective values of the two-stage SVI model are increased to 3.67×10^9 CNY, 3.68×10^9 CNY, and 3.68×10^9 CNY for the three cases, respectively. In reality, the demands of the deterministic VI model cannot be absolutely reliable. We will suffer when sticking to the nominal solution obtained by the deterministic VI model for all possible scenarios, by noting that the objective values of the two-stage SVI models at the solution of the deterministic VI models are much higher than those of using the optimal solutions of the two-stage SVI models for all the three cases.

6.4. Storage in the first stage

Below we focus on analyzing the numerical results obtained by our proposed two-stage SVI model and the PHM with updating strategy for μ. We take $\mathrm{sn} = 20$. We compare the optimal values in the first stage from no storage to sufficient upper limits of storage in all the three cases. The upper limit $x_{j\max}$ increases from u_1 to u_{10} as shown in Table 10, which is essentially determined by the budgets.

Because inspection and testing articles cannot be stored before an epidemic, the upper limit $x_{2\max}$ is always set to be 0. We set the upper limit of storage capacity $x_{ij\max}$ here sufficiently large.



(a) Drugs (b) Inspection and testing articles

(c) Protective articles (d) Disinfection supplies

Fig. 12. Average amounts of non-reusable medical supplies obtained per person.

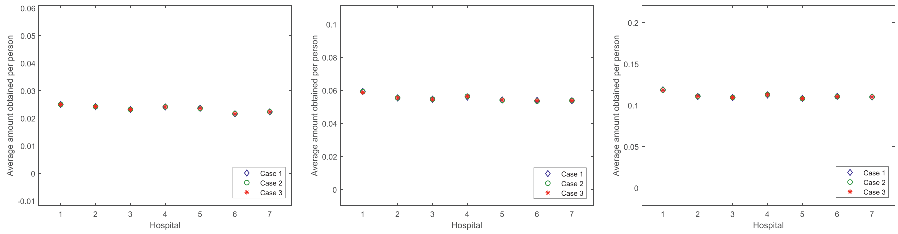
The objective values of the first stage corresponding to $u_1, \ldots, u_{10}$ are plotted in Fig. 3. For each case, the highest total cost is obtained at the level of u_1, i.e., without storage, around 3.76×10^9 CNY for all the three cases, and the global optimal values can be reached with sufficient upper limits of storage $u_7, \ldots, u_{10}$, approximately 5.92×10^8 CNY. Hence it is necessary to store medical supplies in advance. When budgets are large enough, e.g. $x_{j\,\max}$ is at the level of u_7, the optimal values are obtained. They are 591,767,466 CNY, 591,774,067 CNY and 591,774,405 CNY for the three cases, respectively. If we further increase the budget for storage, we will get no further improvement on saving the total cost.

We fix $x_{j\,\max}$ at the level of u_2, and record the computed solutions in Table 11 for the storage decision in the first stage of our proposed two-stage SVI model. We find that the sum of all the hospitals' reserves has reached the upper limit for each of the three cases. This indicates that if the budgets are not sufficient, storing medical supplies as much as possible before an epidemic is a good decision to save money and to meet the medical demands later on.

We can see from Table 11 that when the penalty coefficients are completely the same (Case 1), the model will give reserve priorities to the hospitals with low unit reserve price such as H1; when the two different vectors of penalty coefficients are adopted (Case 2), the hospitals with higher penalty coefficients and lower reserve coefficients may get reserve priorities. Compared with that of Case 1, the storage amounts of drugs for H2 and H3 in Case 2 increase by 141 and 58 respectively, while the storage amounts of H5 decrease by 200; when each hospital is assigned a distinct vector of penalty coefficients (Case 3), it is obvious that the hospitals with higher penalty coefficients and lower unit reserve prices are likely to get reserve priorities, by noting that the storage amounts of drugs for H2, H3 and H5 increase significantly in Case 3.

We show in Figs. 4–6 the storage amounts of the hospitals with the increase of upper limit $x_{j\,\max}$ for the three cases. Subfigures (a), (b), (c) are for non-reusable medical supplies, and subfigures (d), (e), (f) are for reusable medical supplies. We then find the storages of hospitals become stable if the budgets are sufficient, i.e., after u_3 for reusable medical supplies and after u_7 for non-reusable medical supplies. Our proposed two-stage SVI model determines the storage amounts of medical supplies pre-epidemic in order to minimize the total cost under the given budgets, by considering both the unit reserve prices and the penalty coefficients.

Because the housing prices of different geographical locations are different, we set the hospitals' unit reserve prices different in Table 1. In order to understand the effect of unit reserve prices on the solution, we do numerical experiments with the same unit reserve prices and the same penalty coefficients. Table 12 lists the optimal solution of storage for this case. By comparing with the solution of Case 1 in Table 11, we can find that the solution in Table 12 tends to be balance when the unit reserve prices are the same.



(a) Vehicle equipments (b) Disinfection equipments (c) Electronic instruments

Fig. 13. Average amounts of reusable medical supplies obtained per person.

6.5. Dynamic distribution in the second stage

In this subsection, we fix $x_{j\max}$ at the level of u_2 for the two-stage SVI model, and concentrate on the dynamic distribution in the second stage. The dynamic distribution is determined after the realization of scenario is grasped. Since the graphs under different scenarios are similar, we only present the graphs under the first scenario ξ^1.

Fig. 7 shows the number of infected people in each hospital over the 14 days under ξ^1. The total amounts of dynamic distribution are $y(\xi)+z(\xi)+s(\xi)$ in the second stage. Figs. 8–10 show the obtained amounts of non-reusable medical supplies of the 14 days in each hospital for the three cases, respectively. As we can see by contrasting Figs. 7 and 8, when the penalty coefficients are the same (Case 1), the trend of non-reusable medical supplies obtained by each hospital owns very similar pattern to that of the number of infected people in each hospital. From Figs. 9 and 10, when the penalty coefficients are different (Case 2 and Case 3), the hospitals with higher penalty coefficients obtain priorities for non-reusable medical supplies.

Fig. 11 shows the obtained amounts of reusable medical supplies of the 14 days in each hospital for the three cases. Because there are almost no shortage of reusable medical supplies, the distribution curves of reusable medical supplies for the three cases are basically the same. That is, in optimal solutions, the reusable medical supplies are offered as early as possible.

For each hospital i and each medical supply j, the average amount $\mathrm{Ave}\gamma_{ij}(\xi)$ of medical supplies obtained per person relates closely to the fairness, which can be computed as follows.

$$\mathrm{Ave}\gamma_{ij}(\xi)=\frac{\sum_{t=1}^{14}\Gamma_{ijt}(\xi)}{\sum_{t=1}^{14}\xi_{it}},$$

where

$$\Gamma(\xi)=\hat{H}^T(y(\xi)+z(\xi)+s(\xi)),\quad \text{with}\quad \Gamma(\xi)=(\Gamma_{ijt}(\xi),\ \forall i,j,t)^T.$$

We show in Fig. 12 the average amounts of non-reusable medical supplies obtained per person in each hospital for the three cases. It can be seen that for Case 1, the average amounts of non-reusable medical supplies obtained per person in 7 hospitals are very close; for Case 2, the average amounts of non-reusable medical supplies obtained per person in H1, H5, H6 are similar, and those in H2, H3, H4, H7 are similar, respectively; for Case 3, the average amounts of non-reusable medical supplies obtained per person in hospitals with lower penalty coefficients are relatively small.

The three cases actually reflect the different attitudes to fairness. In Case 1, we do not distinguish hospitals and set the penalty coefficients all the same. The results indeed reflect the fairness in the sense of near “equitability”. That is, all hospitals are serviced as approximately indistinguishable entities. We believe that the small differences of unit reserve prices among hospitals lead to the minor differences of the average amounts of medical supplies obtained. In Case 2, we divide the hospitals into two groups that treated patients with mild and severe infection, respectively. The four hospitals that treated severe patients receive similar higher average amounts of medical supplies, which indicates that they get priorities of medical supplies. The hospitals that treated mild patients receive similar lower average amounts of medical supplies. This decision also reflects the fairness in the sense of “balance” that hospitals are serviced with medical supplies regarding their severe/mild infections of patients. In Case 3, the results show that the higher the penalty coefficients are, the more the amounts of the medical supplies obtain. This is also the fairness in the sense of “balance” regarding not only the difference of mild/severe infections but also the total number of infected people in the hospitals as we desire. While in contrast, reusable medical supplies almost have no shortages. Hence they basically have met the demands of all the hospitals throughout the time horizons for the three cases as shown in Fig. 13.

We take into account all the medical supplies together and compare the average cost per patient over the 14 days of each hospital for the three cases, as shown in Fig. 14. For hospital i, the average cost can be computed by the formula below.

$$\mathrm{AveCost}^i(\xi)=\frac{F_i(y^i(\xi),z^i(\xi),s^i(\xi))}{\sum_{t=1}^{14}\xi_{it}}.$$

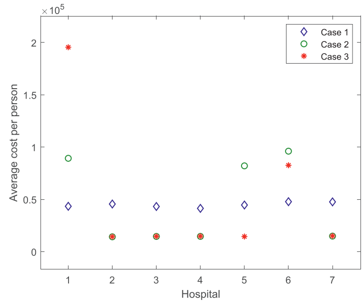
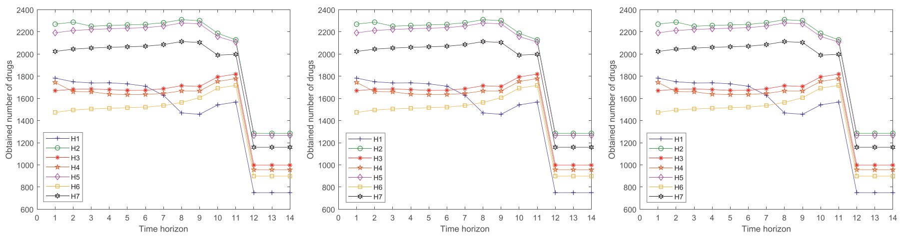


Fig. 14. Average cost per patient of each hospital.



(a) Case 1 (b) Case 2 (c) Case 3

Fig. 15. Dynamic distributions of drugs to hospitals for 14 days.

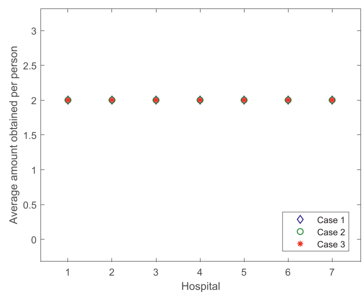


Fig. 16. Average amounts of drugs obtained per person for the extension of the model.

It can be seen that when the penalty coefficients are the same (Case 1), the average costs per patient of the 7 hospitals are very close; when the penalty coefficients are different (Case 2 and Case 3), the average costs per patient in the hospitals with the lower penalty coefficients are higher. This dynamic distribution decision coincides with the guidance of "fairness" we desire. The penalty term in the second-stage objective function makes it flexible to adopt different attitudes of fairness. The competitions among hospitals in the second stage automatically provide decision vectors with fairness.

Finally, we do preliminary numerical experiments for the extension of the model in Subsection 4.2 that the quantities of drugs delivered to hospital i on day t can be greater than the demands. We use formula (31) for the amount of unmet demands, and assign $\mathcal{Q}(\frac{t}{l}) = (\frac{t}{l})^3$ in (32) for medical supply 1 – drugs. If we still use the original $y_{1\,\max} = 10000$ in Table 2, there is no quantity of drugs delivered to hospital i on day t that is greater than the demand at the computed solution for each case.

We then increase $y_{1\,\max} = 13000$. Fig. 15 shows the obtained amounts of drugs in each hospital for 14 days for the three cases. Due to the sufficient supplies of drugs, almost all the hospitals choose to purchase the excess quantity at a low price in advance for future use, and consequently the distribution curves of drugs for the three cases are basically the same. Fig. 16 shows the average amounts of drugs obtained per person in each hospital for the three cases, which are approximately the same. This is reasonable since the demands of all the hospitals throughout the time horizons for the three cases have been met.

7. Conclusions

This paper proposes a monotone smooth two-stage SVI model for the storage and dynamic distribution of medical supplies in epidemic management, which originates from the nonsmooth two-stage stochastic equilibrium model. The solution set of the two-stage SVI model is guaranteed to be nonempty. The government decides the storage amounts in the first stage before an epidemic happens, and after the realization of the random vector has been known, the hospitals compete in the second stage to provide the possible choices of the dynamic distributions over all time horizons. The government is assumed to select the one that is optimal in view of minimizing the total cost in the second stage of all hospitals.

The PHM is presented to solve the monotone two-stage SVI model with solid convergence results. The updating rule for the smoothing parameter helps to accelerate the computational speed and is promising to provide an approximate solution of the original nonsmooth two-stage SP. The efficient semismooth Newton method is employed to solve the subproblems in the PHM. A case study in Wuhan, during the peak of the COVID-19 pandemic, uses our proposed two-stage SVI model, and is solved by our suggested PHM together with the semismooth Newton method, and the updating rule for smoothing parameter. Numerical results show that our model is flexible to obtain a solution of desired “fairness”, and our solution for the storage and dynamic distribution of medical supplies in Wuhan can clearly save money.

In the future, we will consider more hospitals involved in the second stage. With the increasement of hospitals, the dimensions of decision vectors will become much higher, which may lead to computational inefficiency when using the semismooth Newton method in the PHM. It is worthwhile to study whether there are strategies, and/or new methods to solve large-scale models efficiently. In addition, we would like to add the decision variables for locations of warehouses. This is an important issue to be addressed in the first stage, but usually introduces integer variables in the model. The computation would be more challenging.

Acknowlgedgments

The authors would like to thank the associate editor and the anonymous referees for valuable suggestions. The work was supported by the Natural Science Foundation of Beijing, China [grant number 1202021]; and the Fundamental Research Funds for the Central Universities [grant number 2018YJS184].

Appendix A. Proofs

A1. Proof of proposition 1

Proof. According to Theorem 3.5 of [13], the monotonicity of the SVI (22) is equivalent to the monotonicity of the VI

$$\begin{aligned}
0 \le \quad a + A^T\lambda - \pi_s(\xi^k) \quad &\perp \quad x \quad \ge 0,\\
0 \le \quad h - Ax \quad &\perp \quad \lambda \quad \ge 0,\\
0 \le \quad b + p(\xi^k) + {I_{nl}^{(m)}}^T \pi_y(\xi^k) + Q_{yzs}^{\xi^k\mu} \quad &\perp \quad y(\xi^k) \quad \ge 0,\\
0 \le \quad b + {I_{nl}^{(m)}}^T \pi_z(\xi) + Q_{yzs}^{\xi^k\mu} \quad &\perp \quad z(\xi^k) \quad \ge 0,\\
0 \le \quad \hat{B}^T \pi_s(\xi^k) + Q_{yzs}^{\xi^k\mu} \quad &\perp \quad s(\xi^k) \quad \ge 0,\\
0 \le \quad y_{\max}^{\text{dist}} - I_{nl}^{(m)} y(\xi^k) \quad &\perp \quad \pi_y(\xi^k) \quad \ge 0,\\
0 \le \quad z_{\max}^{\text{soci}} - I_{nl}^{(m)} z(\xi^k) \quad &\perp \quad \pi_z(\xi^k) \quad \ge 0,\\
0 \le \quad x - \hat{B}s(\xi^k) \quad &\perp \quad \pi_s(\xi^k) \quad \ge 0,
\end{aligned} \tag{A.1}$$

for each $k = 1, \ldots, K$. For each k, the above VI is monotone, since it is just the KKT system of the convex programming for each scenario ξ^k relating to (21) below

$$\begin{array}{ll}
\min_{x, y(\xi^k), z(\xi^k), s(\xi^k)} & a^T x + {e^{(m)}}^T \tilde{F}(y(\xi^k), z(\xi^k), s(\xi^k), \mu)\\
\text{s.t.} & Ax \le h,\\
& I_{nl}^{(m)} y(\xi^k) \le y_{\max}^{\text{dist}},\\
& I_{nl}^{(m)} z(\xi^k) \le z_{\max}^{\text{soci}},\\
& \hat{B}s(\xi^k) - x \le 0,\\
& x, y(\xi^k), z(\xi^k), s(\xi^k) \ge 0.
\end{array} \tag{A.2}$$

The monotonicity of the two-stage SVI, together with the fact the involved functions defined for the two-stage SVI are continuous, yields that (22) is of maximal monotone type according to Example 12.48 of [48]. □

A2. Proof of proposition 2

Proof. Note that the right-hand side vectors in (5)-(7) are positive vectors. There may exist components in the right-hand side vectors (4) to be zero, since some medical supplies are not available before an epidemic occurs. Let $\mathcal{J}_0 = \{j \in \mathcal{J} \ : x_{j\max} = 0\}$. For any vector

$$u_{yzs} := \left(y(\xi^k)^T, z(\xi^k)^T, s(\xi^k)^T,\ k = 1, \ldots, K\right)^T \in S_2(x),$$

it is obvious from constraint (4) that

$$x_{ij} = 0, \quad \forall i \in \mathcal{I},\ j \in \mathcal{J}_0, \tag{A.3}$$

and consequently from constraint (8)

$$s_{ijt}(\xi^k) = 0, \quad \forall t \in \mathcal{T},\ j \in \mathcal{J}_0,\ k = 1, \ldots, K. \tag{A.4}$$

Denote $\breve{x}$ the subvector of x, and $\breve{s}(\xi^k)$ the subvector of $s(\xi^k)$, and $\breve{\pi}(\xi^k)$ the subvector of $\hat{\pi}_s(\xi^k)$, by deleting all components that correspond to $j \in \mathcal{J}_0$.

We obtain a reduced two-stage stochastic equilibrium problem, by deleting the components of x and $s(\xi^k)$ in the objective functions (2) and (3) that correspond to $j \in \mathcal{J}_0$, the constraints in (4), (8), and (9), and the constraints $s_{ijt}(\xi^k) \geq 0$ in (10) for $j \in \mathcal{J}_0$. Let $\breve{S}_2(\breve{x})$ be the solution set of the reduced second-stage equilibrium problem. The reduced two-stage equilibrium problem is equivalent to the reduced NCP by deleting the complementarity conditions in (26) that correspond to the variables of $s(\xi^k)$ and $\hat{\pi}_s(\xi^k)$ for $j \in \mathcal{J}_0$, the constraints in (4), (8), and (9), and the constraints $s_{ijt}(\xi^k) \geq 0$ in (10) for $j \in \mathcal{J}_0$.

The vector

$$(y(\xi^k)^T, z(\xi^k)^T, \breve{s}(\xi^k)^T, k = 1, \ldots, K)^T \in \breve{S}_2(\breve{x}),$$

because this vector, together with the vectors $\hat{\pi}_y(\xi^k)$, $\hat{\pi}_z(\xi^k)$, and $\breve{\pi}_s(\xi^k)$, satisfies the reduced monotone NCP. It is clear that there is a strict interior point for this reduced monotone NCP, i.e., the involved function values in the NCP at this point are all positive. Thus according to Theorem 2.3.5 of [49], $\breve{S}_2(\breve{x})$ is nonempty, convex, and compact. By noting (A.3) and (A.4), we immediately know that $S_2(x)$ is nonempty, convex, and compact. □

A3. Proof of proposition 3

Proof. We denote for simplicity

$$u_{yzs} := \left(y(\xi^k)^T, z(\xi^k)^T, s(\xi^k)^T,\ k = 1, \ldots, K\right)^T.$$

Let (x^*, u^*_{yzs}) be a solution of the two-stage SP model (19). Then there exist the vectors of Lagrange multipliers λ^*, $\pi^*_y(\xi^k)$, $\pi^*_z(\xi^k)$, and $\pi^*_s(\xi^k)$ for $k = 1, \ldots, K$, together with x^* and u^*_{yzs}, that satisfy the KKT system (22). Thus u^*_{yzs}, together with the vectors of Lagrange multipliers

$$\hat{\pi}_y(\xi^k) = {I_{nl}^{(m)}}^T \pi^*_y(\xi^k),\ \ \hat{\pi}_z(\xi^k) = {I_{nl}^{(m)}}^T \pi^*_z(\xi^k),\ \ \hat{\pi}_s(\xi^k) = \pi^*_s(\xi^k),$$

for $k = 1, \ldots, K$, satisfies the system (26) with $x = x^*$. Hence $u^*_{yzs} \in S_2(x^*)$, and (x^*, u^*_{yzs}) is a feasible solution of the optimistic version (28) of the smooth two-stage equilibrium model (17)-(18).

Let $(\hat{x}, \hat{u}_{yzs})$ be a solution of the optimistic version (28) of the two-stage equilibrium model (17)-(18), which is clear to be a feasible solution of the two-stage SP model (19).

Because the objective value at an optimal solution is no more than the objective value at any feasible solution, we get from (28) and (19) that

$$\begin{aligned} & a^T\hat{x} + \sum_{k=1}^K \tau_k {e^{(m)}}^T \tilde{F}(\hat{y}(\xi^k), \hat{z}(\xi^k), \hat{s}(\xi^k), \mu) \\ & \quad \leq a^T x^* + \sum_{k=1}^K \tau_k {e^{(m)}}^T \tilde{F}(y^*(\xi^k), z^*(\xi^k), s^*(\xi^k), \mu) \\ & \quad \leq a^T\hat{x} + \sum_{k=1}^K \tau_k {e^{(m)}}^T \tilde{F}(\hat{y}(\xi^k), \hat{z}(\xi^k), \hat{s}(\xi^k), \mu). \end{aligned}$$

Consequently, the solution (x^*, u^*_{yzs}) of the two-stage SP model (19) is also a solution of the optimistic version (28) of the two-stage stochastic equilibrium model (17)-(18), and conversely the solution $(\hat{x}, \hat{u}_{yzs})$ of the optimistic version (28) is also a solution of the two-stage SP model (19). □

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