Taylor's law of fluctuation scaling for semivariances and higher moments of heavy-tailed data

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We generalize Taylor's law for the variance of light-tailed distributions to many sample statistics of heavy-tailed distributions with tail index α in (0, 1), which have infinite mean. We show that, as the sample size increases, the sample upper and lower semivariances, the sample higher moments, the skewness, and the kurtosis of a random sample from such a law increase asymptotically in direct proportion to a power of the sample mean. Specifically, the lower sample semivariance asymptotically scales in proportion to the sample mean raised to the power 2, while the upper sample semivariance asymptotically scales in proportion to the sample mean raised to the power $(2-\alpha)/(1-\alpha) > 2$. The local upper sample semivariance (counting only observations that exceed the sample mean) asymptotically scales in proportion to the sample mean raised to the power $(2-\alpha^2)/(1-\alpha)$. These and additional scaling laws characterize the asymptotic behavior of commonly used measures of the risk-adjusted performance of investments, such as the Sortino ratio, the Sharpe ratio, the Omega index, the upside potential ratio, and the Farinelli–Tibiletti ratio, when returns follow a heavy-tailed nonnegative distribution. Such power-law scaling relationships are known in ecology as Taylor's law and in physics as fluctuation scaling. We find the asymptotic distribution and moments of the number of observations exceeding the sample mean. We propose estimators of α based on these scaling laws and the number of observations exceeding the sample mean and compare these estimators with some prior estimators of α.

stable law | semivariance | Pareto | Taylor's law | power law

Heavy-tailed nonnegative random variables with infinite moments, such as nonnegative stable laws with index α in (0,1), have theoretical and practical importance [e.g., Carmona (1), Feller (2), Resnick (3), and Samorodnitsky and Taqqu (4)]. Heavy-tailed nonnegative random variables with some or all infinite moments have been claimed to arise empirically in finance [operational risks in Nešlehová et al. (5)], economics [income distributions in Campolieti (6) and Schluter (7); returns to technological innovations in Scherer et al. (8) and Silverberg and Verspagen (9)], demography [city sizes in Cen (10)], linguistics [word frequencies in Bérubé et al. (11)], and insurance [economic losses from earthquakes in Embrechts et al. (12) and Ibragimov et al. (13)]. Partial reviews are in Carmona (1) and Ibragimov (14).

Brown et al. (15) (hereafter BCD) showed that when a random sample is drawn from a nonnegative stable law with index $\alpha \in (0,1)$, the sample variance is asymptotically (as the sample size n goes to ∞) proportional to the sample mean raised to a power that is an explicit function of α (Eqs. **11** and **13**). This relationship generalizes to stable laws with infinite moments a widely observed power-law relationship between the variance and the mean in families of distributions with finite population mean and finite population variance. This power-law relationship is commonly known as Taylor's law in ecology [Taylor (16, 17)] and as fluctuation scaling in physics [Eisler et al. (18)].

To the two ingredients combined by BCD (nonnegative stable laws with infinite moments and Taylor's law), this paper adds two more ingredients. We establish scaling relationships that generalize the usual Taylor's law, for light-tailed distributions, to many functions of the sample in addition to the variance, including all positive absolute and central moments, upper and lower semivariances, and several measures of risk-adjusted investment performance such as the Sortino, Sharpe, and Farinelli–Tibiletti ratios. In addition, based on these scaling relationships, we propose several estimators of the index α of a nonnegative stable law with infinite first moment.

Section 1 defines most of the sample functions studied here. Section 2 gives background on Taylor's law, semivariances, and nonnegative stable laws, including key prior results from BCD. Section 3 establishes that the lower sample semivariance, the upper sample semivariance, the local lower sample semivariance, and the local upper sample semivariance are asymptotically each a power of the sample mean with explicitly given exponents. These results are the core of the paper. When investment returns obey a nonnegative heavy-tailed law with index $\alpha \in (0,1)$, these results reveal the asymptotic behavior of the Sharpe ratio, the Sortino ratio, and the Farinelli–Tibiletti ratio. Section 4 extends these results to higher central and noncentral moments and various indices of volatility. Section 5 analyzes the number of observations from a stable law or an approximately stable (i.e., regularly varying) law that exceed the sample mean.

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Significance

Many quantities are extremely large extremely rarely. Examples include income, wealth, financial returns, insurance losses, firm size, and city population size; earthquake magnitude, hurricane energy, tornado outbreaks, precipitation, and flooding; and pest outbreaks, infectious epidemics, and forest fires. When such a quantity is modeled as a nonnegative random variable with a heavy upper tail, the probability of an observation larger than some threshold falls as a small power (the "tail index") of the threshold. When the tail index is small enough, the mean and all higher moments of the random quantity are infinite. Surprisingly, the sample mean and the sample higher moments obey orderly scaling laws, which we prove and apply to estimating the tail index.

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Section 6 proposes and compares estimators of α by simulation. *SI Appendix* gives all proofs of results stated in the text and additional numerical simulations.

1. Preliminary

Let $\xrightarrow{d}$ mean "converges in distribution to." Let $\xrightarrow{p}$ mean "converges in probability to." Let $\xrightarrow{\text{a.s.}}$ mean "converges almost surely to."

Let X be a real-valued nonnegative random variable. Let n be a positive integer and assume that $n > 1$. For $i = 1, \ldots, n$, let X_i be independent and identically distributed as X. For any real $h \geq 0$, the hth (raw) sample moment is defined as

$$M'_h := \frac{1}{n}\sum_{i=1}^{n} X_i^h. \quad [1]$$

Thus, M'_1 is the sample mean. For any nonnegative integer h, the hth sample central moment is defined as

$$M_h := \frac{1}{n}\sum_{i=1}^{n}(X_i - M'_1)^h. \quad [2]$$

Clearly, $M_1 = 0$, and M_2 is the sample variance normalized by n. The sample variance normalized by $n-1$ is defined as

$$v_n := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - M'_1)^2. \quad [3]$$

Obviously, $v_n = M_2 n/(n-1)$ and $v_n/M_2 \xrightarrow{\text{a.s.}} 1$ as $n \to \infty$.

The lower sample semivariance and the upper sample semivariance are defined as

$$v_n^- := \frac{1}{n-1}\sum_{i:X_i \leq M'_1}(X_i - M'_1)^2,$$
$$v_n^+ := \frac{1}{n-1}\sum_{i:X_i > M'_1}(X_i - M'_1)^2, \quad [4]$$

so that $v_n = v_n^- + v_n^+$. Define N_n^- as the number of values of X_i that do not exceed the sample mean and N_n^+ as the number of values of X_i that (strictly) exceed the sample mean:

$$N_n^- := \#\{i : X_i \leq M'_1\}, \quad N_n^+ := \#\{i : X_i > M'_1\}. \quad [5]$$

Then, $N_n^- N_n^+ > 0$ unless $X_i = M'_1$ for all $i = 1, \ldots, n$. The local lower sample semivariance and the local upper sample semivariance are defined only when $N_n^- > 0$ and $N_n^+ > 0$, respectively, as

$$v_n^{-*} := \frac{1}{N_n^-}\sum_{i:X_i \leq M'_1}(X_i - M'_1)^2,$$
$$v_n^{+*} := \frac{1}{N_n^+}\sum_{i:X_i > M'_1}(X_i - M'_1)^2. \quad [6]$$

The local upper sample semivariance v_n^{+*} is the more mathematically challenging sequence to analyze because it depends on the asymptotic behavior of the number of observations that exceed the sample mean. Our result, *Theorem 9*, may be of independent interest in the study of heavy-tailed distributions.

For the remainder of this article, we assume two restrictions on X without further restatement. First, we assume that X takes only nonnegative values. Second, to assure that $\mathbb{P}(N_n^+ = 0) = 0$, we assume that X is not atomic [i.e., for all real a, we assume that $\mathbb{P}(X = a) = 0$]. Then, $\mathbb{P}(N_n^+ = 0) = 0$ and conversely; for otherwise, if $\mathbb{P}(X = a) > 0$ for some a, then $\mathbb{P}(N_n^+ = 0) \geq \{\mathbb{P}(X = a)\}^n > 0$. Under the assumption that X is not atomic, $\mathbb{P}(N_n^- N_n^+ > 0) = 1$, and v_n^{-*} and v_n^{+*} are well defined almost surely (a.s.); also, $v_n^- = N_n^- v_n^{-*}/(n-1)$, $v_n^+ = N_n^+ v_n^{+*}/(n-1)$, and $v_n = (N_n^- v_n^{-*} + N_n^+ v_n^{+*})/(n-1)$ a.s. The assumption that X is not atomic also plays an important role in *Theorems 5* and *8(3), Remark 2,* and *Corollaries 6(3)* and *8*.

Alternatively, we could assume that X is not constant (i.e., not a degenerate random variable with all probability mass concentrated at a single value). If X is atomic but not a constant, then $\mathbb{P}(N_n^- N_n^+ > 0) \to 1$ as $n \to \infty$, but $\mathbb{P}(N_n^- N_n^+ > 0) \neq 1$. Nevertheless, similar asymptotic results could still be proved.

The infinite sequences of random variables defined in Eqs. **1** to **6** (one random variable for each $n = 1, 2, \ldots$) exist a.s., whether or not X has any finite moments. Our goal here is to show that, if X is a stable distribution (or an approximately stable distribution under *Definition 1*) with support $(0, \infty)$ and index $\alpha \in (0, 1)$, then as $n \to \infty$, the quantities in Eqs. **1** to **6** and other related quantities defined in section 3, when divided by some power b of the sample mean M'_1, converge in distribution, in probability or almost surely, depending on the case. Here, b may depend on α and on which quantity is being examined.

2. Background and Prior Results

Taylor's law [Taylor (16)] says that the sample variance v_n scales approximately in direct proportion to a nonzero power b (positive or negative) of the sample mean M'_1. Taylor's law is a widely confirmed empirical pattern in ecology and other sciences [Taylor (17)], nearly always with $b > 0$ and often with $b \in (1, 2)$. Taylor's law holds also for the mean and variance of some single-parameter probability distributions, in addition to holding for the sample mean and sample variance. For example, for varying values of the population mean μ, the population variance σ^2 varies according to Taylor's law $\sigma^2 = a\mu^b$ with $a = 1$, $b = 1$ for the Poisson distribution and $a = 1$, $b = 2$ for the exponential distribution.

The semivariances, especially the lower, have important applications in agricultural and financial economics [Berck and Hihn (19), Bond and Satchell (20), Hogan and Warren (21), Jin et al. (22), Liagkouras and Metaxiotis (23), Nantell and Price (24), Porter (25), Turvey and Nayak (26), and van de Beek et al. (27)]. We know no prior proofs that the sample semivariances of a nonnegative stable law satisfy Taylor's law.

Higher moments include skewness and kurtosis in statistics and the Farinelli–Tibiletti ratio in finance. Power-law scaling relationships for moments other than the sample variance are generalized Taylor's laws [Giometto et al. (28)]. Generalized Taylor's laws are less widely studied empirically or theoretically.

Every stable random variable X with support $(0, \infty)$ has Laplace transform [Feller (2), pp. 448–449]

$$\mathscr{L}(s) := \mathbb{E}(e^{-sX}) = e^{-(cs)^\alpha}, \quad [7]$$

for $s \geq 0$, $0 < \alpha < 1$, and $c > 0$. We say that $X \stackrel{d}{=} F(c, \alpha)$ when the distribution of X has Laplace transform Eq. **7**, and then we say that X has index α. We have $X \stackrel{d}{=} F(c, \alpha) \stackrel{d}{=} cF(1, \alpha)$. Such a heavy-tailed distribution has an infinite mean. Consequently, the sample mean, sample variance, sample semivariances, and sample higher moments are not estimators of population moments, and the normal central limit theorem does not apply.

If $X \stackrel{d}{=} F(c, \alpha)$ for some $0 < \alpha < 1$, $c > 0$, the survival function of X evaluated at $t \in (0, \infty)$ is defined as $\overline{F}(c, \alpha)(t) := 1 - F(c, \alpha)(t)$. By Feller (2, p. 448), if $0 < \alpha < 1$ and $c > 0$, then as $t \to \infty$,

$$\overline{F}(c, \alpha)(t) \Big/ \frac{c^\alpha t^{-\alpha}}{\Gamma(1-\alpha)} \to 1. \quad [8]$$

Many distributions on $(0, \infty)$ satisfy Eq. **8** but are not of the special form $F(c, \alpha)$ in Eq. **7**.

Definition 1. $X \overset{d}{\approx} F(c,\alpha)$ and $F_X \overset{d}{\approx} F(c,\alpha)$ both mean that a nonnegative random variable X has a distribution function F_X that satisfies Eq. **8**: that is, as $t \to \infty$,

$$\{1 - F_X(t)\} \Big/ \frac{c^\alpha t^{-\alpha}}{\Gamma(1-\alpha)} \to 1. \qquad [9]$$

When Eq. **9** holds, we say that X is approximately stable.

For $\alpha \in (0,1)$ and real $g > \alpha$, $h > \alpha$, define

$$\alpha(g,h) := \frac{g-\alpha}{h-\alpha}, \qquad \alpha^* := \alpha(2,1) = \frac{2-\alpha}{1-\alpha}. \qquad [10]$$

If $g > h$, then $\alpha(g,h) > g/h$. Consequently, $\alpha^* > 2$. If $g < h$, then $\alpha(g,h) < g/h < 1$. Thus if, as we shall prove below, $\alpha(g,h)$ is the exponent b in Taylor's law for a stable nonnegative law with index $\alpha \in (0,1)$ and if $g \geq 2h$ or $g < h$, then the exponent b must fall outside the interval $(1,2)$ that is commonly (although not universally) observed in many ecological applications [Cohen et al. (29, 30)].

Among other results, BCD (ref. 15, p. 663, proposition 2) showed that if $X \overset{d}{=} F(1,\alpha)$, then as $n \to \infty$,

$$W_n := \frac{v_n}{M_1'^{\alpha^*}} \overset{d}{\to} W, \qquad [11]$$

where $\mathbb{E}(W_n) = 1-\alpha$, $Var(W_n) = \{\mathbb{E}(W_n)\}^2\{1 + 2\alpha/(n-1)\}$, and the limiting random variable W has $\mathbb{P}(0 < W < \infty) = 1$. W has a finite mean and a finite SD, both of which equal $1-\alpha$. Moreover, for all $h = 1,2,\ldots$, $\mathbb{E}(W_n^h) \to \mathbb{E}(W^h)$. The second and third moments of W are

$$\mathbb{E}(W^2) = 2\{\mathbb{E}(W)\}^2, \quad \mathbb{E}(W^3) = \left(6 - \frac{\alpha}{(5-2\alpha)}\right)\{\mathbb{E}(W)\}^3, \qquad [12]$$

while for an exponentially distributed random variable Y, $\mathbb{E}(Y^3) = 6\{\mathbb{E}(Y)\}^3$ (ref. 15, p. 666).

For general $c > 0$ in Eq. **7**, BCD showed that $v_n/M_1'^{\alpha^*} \overset{d}{\to} c^{-\frac{\alpha}{1-\alpha}} W$, where W is the limiting random variable in Eq. **11**. Consequently, for any $c > 0$, BCD showed that as $n \to \infty$,

$$\frac{\log v_n}{\log M_1'} \overset{p}{\to} \alpha^*. \qquad [13]$$

Thus, for large n, with arbitrarily high probability, $(\log v_n)/(\log M_1')$ will be close to α^*, regardless of $c > 0$. This scaling relationship is an asymptotic form of Taylor's law with exponent $b = \alpha^* > 2$. BCD further argued without detailed proofs that $X \overset{d}{\approx} F(c,\alpha)$ satisfies Eq. **13**.

A common sample statistic used to compare the effectiveness of investments is the well-known Sharpe ratio [Sharpe (31)] $(M_1' - r_f)/v_n^{1/2}$ for the period rates of return of a security, where r_f is a zero-risk reference: for example, the London interbank offered rate. In signal processing, the Sharpe ratio (with $r_f = 0$) is a useful but biased estimator of the signal-to-noise ratio [Miller and Gehr (32)]. In statistics, the reciprocal of the Sharpe ratio (with $r_f = 0$) is called the coefficient of variation. If the period rate of return has a distribution $X \overset{d}{\approx} F(c,\alpha)$, where $0 < c < \infty$ and $0 < \alpha < 1$, then the Sharpe ratio converges in probability to zero as $n \to \infty$. Why? Eq. **11** implies that, as $n \to \infty$, $M_1'^{\alpha^*}/v_n \overset{d}{\to} 1/W$, so $M_1'^{\alpha^*/2}/v_n^{1/2} \overset{d}{\to} 1/W^{1/2}$. However, $M_1'^{\alpha^*/2} = M_1' \times M_1'^{(\alpha^*/2)-1}$, and because $\alpha^* > 2$ (as noted just after Eq. **10**), the second factor $M_1'^{(\alpha^*/2)-1}$ goes a.s. to ∞. Therefore, the Sharpe ratio $(M_1' - r_f)/v_n^{1/2}$ must converge in probability to zero. Asymptotically, for large n, the Sharpe ratio reveals no information about the distribution.

Inspired by Taylor's law in Eq. **13**, one may consider $\log(M_1' - r_f)/\log v_n$ as a modified financial ratio, which converges to $1/\alpha^* = (1-\alpha)/(2-\alpha)$ in probability. Because $(1-\alpha)/(2-\alpha)$ is decreasing in α over $(0,1)$, the smaller α is, the heavier the distribution, so the larger the risk. The original Sharpe ratio is quasiconcave, scale invariant, and distribution based [Eling et al. (33)]. The modified ratio is also distribution based and reveals the tail index α for large-enough n. Because of the logarithmic transformation, the modified ratio is not scale invariant. However, both numerator and denominator diverge to infinity. The effect of finite scaling becomes negligible for large sample sizes, and hence, the ratio is F_α-asymptotically scale invariant.* In other words, when $X \overset{d}{\approx} F(c,\alpha)$, the modified ratio is asymptotically invariant with respect to c. The modified Sharpe ratio is F_α-asymptotically quasiconcave.† The proof is in *SI Appendix*. Thus, asymptotically with large sample size n, the modified Sharpe ratio inherits all the properties of the original Sharpe ratio. We discuss this using semivariances and partial moments for the financial ratios in the following sections.

3. Taylor's Laws for Semivariances

A. Lower Semivariances and Sortino Ratio. The lower semivariance of any nonnegative random variable with infinite expectation is almost surely asymptotic to the square of the sample mean.

Theorem 1 (Taylor's law for the lower semivariance). *Let X be a nonnegative random variable with $\mathbb{E}(X) = \infty$. Then, as $n \to \infty$,*

$$\frac{v_n^-}{M_1'^2} \overset{a.s.}{\longrightarrow} 1. \qquad [14]$$

This theorem does not assume X is stable or approximately stable.

The Sortino ratio [Sortino and Price (34)] is another sample statistic used to compare the risks and rewards in some period of a set of investments such as individual equities, mutual funds, trading systems, or investment managers. It is defined as $(M_1' - r_f)/s_d$, where M_1' is the sample mean of the period rate of return X, r_f is a threshold or reference point or target return, the zero-risk rate of return or minimal acceptable return, which we take to be zero, and $s_d := (v_n^-)^{1/2}$ is the downside risk, equal to the square root of the lower sample semivariance v_n^- of the period rate of return [e.g., Sortino and Price (34) and Rollinger and Hoffman (35)]. Under our assumption that $\mathbb{P}(0 < X < \infty) = 1$, one might interpret X as the ratio of final price to initial price, so that $0 < X < 1$ would represent a loss, while $X > 1$ would represent a gain. The possible use of n instead of $n-1$ in the denominator of Eq. **4** is immaterial for large samples. Eq. **14** shows that if the period rate of return X is a nonnegative random variable with an infinite mean, then the Sortino ratio converges a.s. to one as $n \to \infty$. When the mean is infinite, asymptotically, for large n, the Sortino ratio reveals no information about the distribution.

Similar to our modified Sharpe ratio for heavy-tailed distributions, for the Sortino ratio, we consider the ratio between the logarithm of the sample mean minus r_f and the logarithm of the sample lower semivariance, namely $\log(M_1' - r_f)/\log v_n^-$. *Theorem 1* and Slutsky's theorem imply that a power law with exponent 2 relates the lower semivariance to the sample mean. So Taylor's law holds between the sample mean and the lower semivariance.

*F_α-asymptotic scale invariance is defined in *SI Appendix*, section D.

†F_α-asymptotic quasiconcavity is defined in *SI Appendix*, section D.

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Corollary 1. *Let X be a nonnegative random variable with $\mathbb{E}(X)=\infty$. As $n\to\infty$,*

$$\frac{\log v_n^-}{\log M_1'} \xrightarrow{a.s.} 2. \quad \textbf{[15]}$$

The modified Sortino ratio is F_α-asymptotically quasiconcave and F_α-asymptotically scale invariant, like the original Sortino ratio; proofs are in *SI Appendix*. However, from *Corollary 1*, the limiting value of the modified Sortino ratio is independent of the tail index α.

We now extend Taylor's law to the local lower semivariance v_n^{-*}. The local lower semivariance differs from the lower semivariance by a factor equal to the ratio N_n^-/n. We show that $N_n^-/n\to 1$ almost surely if $\mathbb{E}(X)=\infty$.

Lemma 1. *Let X be a nonnegative random variable with $\mathbb{E}(X)=\infty$. Then, with N_n^- defined in Eq. **5**, as $n\to\infty$,*

$$\frac{N_n^-}{n} \xrightarrow{a.s.} 1. \quad \textbf{[16]}$$

Corollary 1 and *Lemma 1* imply that a power law with exponent 2 relates the local lower semivariance to the sample mean.

Corollary 2. *Let X be a nonnegative random variable with $\mathbb{E}(X)=\infty$. Then, as $n\to\infty$,*

$$\frac{\log v_n^{-*}}{\log M_1'} \xrightarrow{a.s.} 2. \quad \textbf{[17]}$$

If X is approximately stable with infinite expectation, then *Lemma 1* and *Corollaries 1* and *2* imply further results that will be useful later for studying the local upper semivariance and upper semivariance.

Corollary 3. *Let $X \overset{d}{\approx} F(1,\alpha)$, $0<\alpha<1$. Let $\alpha^* := (2-\alpha)/(1-\alpha)$ as defined in Eq. **10**. Then, as $n\to\infty$,*

$$\frac{v_n^-}{M_1'^{\alpha^*}} \xrightarrow{a.s.} 0 \quad \text{and} \quad \frac{v_n^{-*}}{M_1'^{\alpha^*}} \xrightarrow{a.s.} 0. \quad \textbf{[18]}$$

B. Upper Semivariances. Although the asymptotic values of the ratios in Eqs. **15** and **17** are both two, which is independent of α, if one replaces the lower or local lower semivariances by the upper or local upper semivariances, respectively, Taylor's law continues to hold, and it depends on α.

Theorem 2. *Let $X \overset{d}{\approx} F(1,\alpha)$, $0<\alpha<1$. Then, as $n\to\infty$,*

$$\frac{\log v_n^+}{\log M_1'} \xrightarrow{p} \alpha^* \quad \text{and} \quad \frac{\log v_n^{+*}}{\log M_1'} \xrightarrow{p} \alpha^*+\alpha = \frac{2-\alpha^2}{1-\alpha}. \quad \textbf{[19]}$$

Inspired by Taylor's law in Eq. **19**, one may consider ratios between the logarithm of the sample mean minus r_f and the logarithm of either the sample upper or local upper semivariances, namely $\log(M_1'-r_f)/\log v_n^+$ and $\log(M_1'-r_f)/\log v_n^{+*}$, respectively, which converge in probability to $1/\alpha^*=(1-\alpha)/(2-\alpha)$ and $(1-\alpha)/(2-\alpha^2)$, respectively. Because $(1-\alpha)/(2-\alpha)$ and $(1-\alpha)/(2-\alpha^2)$ are both decreasing in α, the smaller α is, the heavier the distribution is, and the larger these ratios are asymptotically. The asymptotic properties and proofs are in *SI Appendix*, *Proposition* D.3.

4. Fluctuation Scaling for Higher Moments

In this section, we show that the sample higher moments are proportional to a power of the sample mean. These relations imply power-law relations between sample higher moments used in financial ratios such as the Farinelli–Tibiletti ratio (36).

A. Higher Sample Moments, Skewness, and Kurtosis.

Theorem 3. *If $X \overset{d}{\approx} F(1,\alpha)$, $0<\alpha<1$, and $h>\alpha$, then, as $n\to\infty$,*

$$\frac{M_h'}{(M_1')^{\alpha(h,1)}} \xrightarrow{d} \{\Gamma(1-\alpha)\}^{\frac{h-1}{1-\alpha}}\frac{U_h}{V^{\alpha(h,1)}},$$

where the random vector (U_h, V) has the joint Laplace transform

$$\mathbb{E}(e^{-sU_h-tV}) = \exp\left\{-\int_0^\infty \{r_h(y,s,t)\}^{-\alpha}e^{-y}\,dy\right\},$$

for $s,t,y>0$, and $r_h(y,s,t)$ is the unique positive root of the equation $sx^h+tx-y=0$.

The ratio in *Theorem 3* may not be a practically useful financial ratio since α is usually unknown. However, the following *Theorem 4* and its corollaries heavily depend on it. The following remark uses the joint moment-generating function to give the marginal distributions of U_h and V.

Remark 1. In the joint Laplace transform defined in *Theorem 3*, if we set $t=0$, then $r_h(y,s,0)=(y/s)^{1/h}$ and

$$\mathbb{E}(e^{-sU_h}) = \exp\left\{-\int_0^\infty \left\{(y/s)^{1/h}\right\}^{-\alpha}e^{-y}\,dy\right\}.$$

Hence, U_h follows the distribution $F(\{\Gamma(1-\alpha/h)\}^{h/\alpha},\alpha/h)$. On the other hand, if we set $s=0$, then $r_h(y,0,t)=y/t$ and

$$\mathbb{E}(e^{-tV}) = \exp\left\{-\int_0^\infty \{(y/t)\}^{-\alpha}e^{-y}\,dy\right\}.$$

Hence, V follows the distribution $F(\{\Gamma(1-\alpha)\}^{1/\alpha},\alpha)$.

These results follow Albrecher et al. (ref. 37, remark 2.1) by the arguments in their proof. The following theorem shows that Taylor's law holds for raw moments.

Theorem 4. *If $X \overset{d}{\approx} F(1,\alpha)$, $0<\alpha<1$, $h_1>\alpha$, and $h_2>\alpha$, then as $n\to\infty$,*

$$\frac{\log M_{h_2}'}{\log M_{h_1}'} \xrightarrow{p} \alpha(h_2,h_1).$$

In particular, for $h>\alpha$, as $n\to\infty$,

$$\frac{\log M_h'}{\log M_1'} \xrightarrow{p} \alpha(h,1).$$

For a positive integer $h>1$, the ratio between the central moment M_h and the $\alpha(h,1)$ power of the sample mean M_1' converges to a distribution given in *Corollary 4*.

Corollary 4. *If $X \overset{d}{\approx} F(1,\alpha)$, $0<\alpha<1$, and $h>1$ is a positive integer, then as $n\to\infty$,*

$$\frac{M_h}{(M_1')^{\alpha(h,1)}} \xrightarrow{d} \{\Gamma(1-\alpha)\}^{\frac{h-1}{1-\alpha}}\frac{U_h}{V^{\alpha(h,1)}},$$

where the random vector (U_h, V) is specified in Theorem 3.

Theorem 5. *If $X \overset{d}{\approx} F(1,\alpha)$, $0<\alpha<1$, and $h>1$ is a positive integer, then as $n\to\infty$,*

$$\frac{\log|M_h|}{\log M_1'} \xrightarrow{p} \alpha(h,1).$$

For any positive integers $h_1>1$ and $h_2>1$, as $n\to\infty$,

$$\frac{\log|M_{h_2}|}{\log|M_{h_1}|} \xrightarrow{p} \alpha(h_2,h_1).$$

For the raw moments, we have generalized *Theorem 3* for the ratio of two raw moments with orders both larger than α.

Theorem 6. *If $X \overset{d}{\approx} F(1,\alpha)$, $0<\alpha<1$, and both $h_1, h_2 > \alpha$, then as $n \to \infty$,*

$$\frac{M'_{h_2}}{(M'_{h_1})^{\alpha(h_2,h_1)}} \xrightarrow{d} \{\Gamma(1-\alpha)\}^{\frac{h_2-h_1}{h_1-\alpha}} \frac{U_{h_2}}{(U_{h_1})^{\alpha(h_2,h_1)}},$$

where (U_{h_1}, U_{h_2}) has the joint Laplace transform

$$\mathbb{E}(e^{-sU_{h_2}-tU_{h_1}}) = \exp\left\{-\int_0^\infty \{r_{h_2,h_1}(y,s,t)\}^{-\alpha} e^{-y}\, dy\right\},$$

with $y>0$, $s>0$, $t>0$, and $r_{h_2,h_1}(y,s,t)$ is the unique positive root x of $sx^{h_2}+tx^{h_1}-y=0$. Moreover, as $n\to\infty$,

$$\frac{\log M'_{h_2}}{\log M'_{h_1}} \xrightarrow{p} \alpha(h_2,h_1).$$

Corollary 5. *If $X \overset{d}{\approx} F(1,\alpha)$, $0<\alpha<1$, and $h_2 \geq h_1 > 1$ are positive integers, then as $n\to\infty$,*

$$n^{\frac{h_1-h_2}{h_1}} \frac{M_{h_2}}{(M_{h_1})^{h_2/h_1}} \xrightarrow{d} \frac{U_{h_2}}{(U_{h_1})^{h_2/h_1}},$$

where (U_{h_1}, U_{h_2}) is defined in Theorem 6.

Remark 2. From *Corollary 5*, it is clear that the skewness $M_3/(v_n)^{3/2}$ and the kurtosis $M_4/(v_n)^2$ diverge to infinity, yet the scaled skewness and the scaled kurtosis have distributions, asymptotically as $n\to\infty$,

$$\frac{M_3}{n^{1/2}(v_n)^{3/2}} \xrightarrow{d} \frac{U_3}{(U_2)^{3/2}} \text{ and } \frac{M_4}{n(v_n)^2} \xrightarrow{d} \frac{U_4}{(U_2)^2},$$

where the joint distributions of (U_2, U_3) and (U_2, U_4) are defined in *Theorem 6*. The limiting distribution of $M_4/\{n(v_n)^2\}$ matches the result derived in Cohen et al. (ref. 38, equation 3.9). Moreover, by Slutsky's theorem, as $n\to\infty$,

$$\frac{\log|M_3|}{\log[(v_n)^{3/2}]} \xrightarrow{p} \frac{2}{3}\alpha(3,2) \text{ and } \frac{\log M_4}{\log[(v_n)^2]} \xrightarrow{p} \frac{1}{2}\alpha(4,2).$$

B. Central Lower and Local Lower Partial Moments.

Definition 2. Define $c_+ := \max\{0, c\}$ for $c\in\mathbb{R}$. For $h>0$, define

$$M_h^- := \frac{1}{n}\sum_{i=1}^n [(M'_1 - X_i)_+]^h, \qquad M_h^{-*} := \frac{nM_h^-}{N_n^-}.$$

Theorem 7. *Let X be a nonnegative random variable with $\mathbb{E}(X)=\infty$, and let $h>0$. Then, as $n\to\infty$,*

$$M_h^-/(M'_1)^h \xrightarrow{a.s.} 1 \quad \text{and} \quad \log M_h^- - h\log M'_1 \xrightarrow{a.s.} 0.$$

Corollary 6. *Let X be a nonnegative random variable with $\mathbb{E}(X)=\infty$. Then, as $n\to\infty$,*

1) $M_1^-/M'_1 \xrightarrow{a.s.} 1$;
2) for $h>1$, $M_h^-/(M'_1)^{\alpha(h,1)} \xrightarrow{a.s.} 0$;
3) for $h>0$,

$$\frac{\log M_h^-}{\log M'_1} \xrightarrow{a.s.} h \quad \text{and} \quad \frac{\log M_h^{-*}}{\log M'_1} \xrightarrow{a.s.} h.$$

C. Central Upper Moments and Local Upper Moments.

Definition 3. For $h>0$, define the hth central upper moments and central local upper moments:

$$M_h^+ := \frac{1}{n}\sum_{i=1}^n [(X_i - M'_1)_+]^h, \qquad M_h^{+*} := \frac{nM_h^+}{N_n^+}.$$

Theorem 8 (central upper moments). *Let $X \overset{d}{\approx} F(1,\alpha)$, $0<\alpha<1$. Then, as $n\to\infty$,*

1) for $0<h<1$, $M_h^+/(M'_1)^h \xrightarrow{p} 0$;
2) for $h\geq 1$,

$$\frac{M_h^+}{(M'_1)^{\alpha(h,1)}} \xrightarrow{d} \{\Gamma(1-\alpha)\}^{\frac{h-1}{1-\alpha}} \frac{U_h}{V^{\alpha(h,1)}},$$

where the random vector (U_h, V) has the joint Laplace transform defined in Theorem 3;
3) for $h\geq 1$,

$$\frac{\log M_h^+}{\log M'_1} \xrightarrow{p} \alpha(h,1) \quad \text{and} \quad \frac{\log M_h^{+*}}{\log M'_1} \xrightarrow{p} \frac{h-\alpha^2}{1-\alpha}. \qquad [20]$$

D. Omega Index, Upside Potential Ratio, and Farinelli–Tibiletti Ratio.

Farinelli–Tibiletti (36) extended the Sharpe ratio to an index including asymmetrical information on the volatilities above and below the benchmark $r_f\in\mathbb{R}$. Their index Φ_{FT} is defined by

$$\Phi_{\mathrm{FT}}(r_f,p,q) := \frac{[\mathbb{E}[(X-r_f)_+]^p]^{1/p}}{[\mathbb{E}[(r_f-X)_+]^q]^{1/q}}.$$

The Omega index, introduced by Cascon et al. (39), is $\Phi_{\mathrm{FT}}(r_f,1,1)$ with $p=q=1$. The upside potential index, introduced by Sortino et al. (40), is $\Phi_{\mathrm{FT}}(r_f,1,2)$ with $p=1$ and $q=2$. The ratio $\Phi_{\mathrm{FT}}(r_f,p,q)$ may not be well defined since the expectations may not exist for the heavy-tailed distributions. However, one can define an empirical version of the Farinelli–Tibiletti ratio by

$$\Phi^n_{\mathrm{FT}}(r_f,p,q) := \frac{[\frac{1}{n}\sum_{i=1}^n[(X_i-r_f)_+]^p]^{1/p}}{[\frac{1}{n}\sum_{i=1}^n[(r_f-X_i)_+]^q]^{1/q}}.$$

The following corollary shows that both $\Phi^n_{\mathrm{FT}}(r_f,p,q)$ and $\Phi^n_{\mathrm{FT}}(M'_1,p,q)$ converge to ∞ in probability.

Corollary 7. *If $X \overset{d}{\approx} F(1,\alpha)$, $0<\alpha<1$, $r_f>0$, $p>1$, and $q>1$, then as $n\to\infty$, $\Phi^n_{\mathrm{FT}}(r_f,p,q) \xrightarrow{p} \infty$ and $\Phi^n_{\mathrm{FT}}(M'_1,p,q) \xrightarrow{p} \infty$.*

A modification of the usual Farinelli–Tibiletti ratio might have the ratio of the logarithm of the numerator to the logarithm of the denominator in $\Phi_{\mathrm{FT}}(r_f,p,q)$. However, for a fixed $r_f>0$, the numerator converges to infinity in probability, while the denominator is bounded above with probability one. Therefore, this ratio diverges to infinity.

We propose as an alternative to the Farinelli–Tibiletti ratio:

$$\Phi_{\mathrm{FTlog}}(p,q) := p\log M_q^-/(q\log M_p^+),$$

which is the ratio of the logarithm of the numerator to that of the denominator in $\Phi_{\mathrm{FT}}(M'_1,p,q)$. The following corollary describes generalized Taylor's laws for the ratio of the logarithm of the upper central moment to the logarithm of the lower central moment.

Corollary 8. *If $X \overset{d}{\approx} F(1,\alpha)$, $0<\alpha<1$, $p\geq 1$, and $q\geq 1$, then as $n\to\infty$,*

$$\frac{\log M_p^+}{\log M_q^-} \xrightarrow{p} \frac{p-\alpha}{q(1-\alpha)}.$$

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Corollary 8 implies that

$$\Phi_{\mathrm{FTlog}}(p, q) \xrightarrow{p} p(1-\alpha)/(p-\alpha),$$

which is decreasing in α for $p \geq 1, q \geq 1$. Therefore, the smaller α is, the heavier the distribution is, and the larger the risk is. Our modified Farinelli–Tibiletti ratio $\Phi_{\mathrm{FTlog}}(p, q)$ is asymptotically scale invariant and distribution based, like the original Farinelli–Tibiletti ratio, and satisfies F_α-asymptotic quasiconcavity (*SI Appendix*).

5. Number of Observations Exceeding Sample Mean of Stable Law

A. Asymptotic Distributions and Moments of N_n^+/n^α. In a sample of size *n* from an approximately stable law with index $\alpha \in (0, 1)$, asymptotically the number of observations above the sample mean scales as n^α and has a distribution given by *Theorem 9*. To prove this result, we use Einmahl (ref. 41, corollary 2.1) together with *SI Appendix*, *Lemma C.1*.

Theorem 9. *If* $X \overset{d}{\approx} F(1, \alpha)$, $0 < \alpha < 1$, *and* $U \overset{d}{=} F(1, \alpha)$, *then as* $n \to \infty$,

$$\frac{N_n^+}{n^\alpha} \xrightarrow{d} V := \frac{U^{-\alpha}}{\Gamma(1-\alpha)}.$$

The asymptotic moments of N_n^+/n^α are the moments of V defined in *Theorems 9* and *10*.

Theorem 10. *Let* $U \overset{d}{=} F(1, \alpha)$, $0 < \alpha < 1$, $V := U^{-\alpha}/\Gamma(1-\alpha)$, *and* $\varepsilon \overset{d}{=} \mathrm{Exp}(1)$ *(an exponential random variable with mean and parameter 1), where* ε *is independent of U.*

1) $U^{-\alpha}\varepsilon^\alpha \overset{d}{=} \mathrm{Exp}(1)$.

2) For integer $K > 0$,

$$\mathbb{E}[U^{-K\alpha}] = \frac{K!}{\Gamma(1+K\alpha)},$$

$$\mathbb{E}[V^K] = \frac{K!}{\Gamma(1+K\alpha)\{\Gamma(1-\alpha)\}^K}.$$

Specifically, when $K = 1$, *then* $\mathbb{E}[U^{-\alpha}] = \{\Gamma(1+\alpha)\}^{-1}$ *and* $\mathbb{E}[V] = \{\Gamma(1+\alpha)\Gamma(1-\alpha)\}^{-1}$; *when* $K = 2$, *then* $\mathbb{E}[U^{-2\alpha}] = 2\{\Gamma(1+2\alpha)\}^{-1}$, $\mathbb{E}[V^2] = 2\{\Gamma(1+2\alpha)\{\Gamma(1-\alpha)\}^2\}^{-1}$. *Hence*

$$\mathrm{Var}(U^{-\alpha}) = \frac{2}{\Gamma(1+2\alpha)} - \frac{1}{\{\Gamma(1+\alpha)\}^2},$$

$$\mathrm{Var}(V) = \frac{1}{\{\Gamma(1-\alpha)\}^2}\mathrm{Var}(U^{-\alpha}).$$

3) $\mathrm{SD}(V) < \mathbb{E}[V]$. *For example, when* $\alpha = 1/2$, $\mathbb{E}[V^2] = 2/\pi$, $\mathbb{E}[V] = 2/\pi$, $\mathrm{Var}(V) = \frac{2}{\pi}(1 - \frac{2}{\pi})$. *Numerically,* $\mathrm{SD}(V) \approx 0.48097$, $\mathbb{E}[V] \approx 0.63662$, *where here "≈ " means the numerical approximation is inexact.*

4) For $K \geq 2$, $\mathbb{E}[V^K] < K!(\mathbb{E}[V])^K$.

5) $V \leq_{st} \varepsilon$ *[i.e., by the definition of the stochastic ordering* $\leq_{st}$, $\mathbb{P}(V > t) \leq \mathbb{P}(\varepsilon > t)$ *for all* $t \in \mathbb{R}$*].*

Part 1 of *Theorem 10* is not well known. The moment results in part 2 of *Theorem 10* are derived using fractional calculus by Wolfe (42). Because the logarithm of the moment-generating function of a nonnegative random variable is a convex function of the moment (by Artin's theorem) [Marshall and Olkin (ref. 43, theorem B.8)], it follows that $\log \mathbb{E}(U^{-x\alpha}) = \log \Gamma(1+x) - \log \mathbb{E}(W^x)$ is concave in $x \in [1, \infty)$.

The distribution of $U^{-\alpha}$ approximates the standard exponential distribution $\mathrm{Exp}(1)$ when $\alpha \to 0$.

Corollary 9. *Let* $U \overset{d}{=} F(1, \alpha)$. *Then, as* $\alpha \to 0$,

$$U^{-\alpha} \xrightarrow{d} \mathrm{Exp}(1).$$

6. Numerical Experiments

A. Tail Estimators. The preceding results describe the asymptotic ratio of the logarithm of the sample mean to the logarithm of various forms of the sample variance, such as the ordinary sample variance v_n, the upper semivariance v_n^+, the local upper semivariance v_n^{+*}, and the lower semivariance v_n^- when a random sample is from an approximately stable $F(1, \alpha)$ satisfying Eq. **9**. Most of these ratios (apart from that for the lower semivariance) depend asymptotically only on α. Based on these results, we propose estimators of the index α. We define the ratios R_1, R_2, R_3, and R_L where

$$R_1 := \frac{\log v_n}{\log M_1'} \xrightarrow{p} \frac{2-\alpha}{1-\alpha}, \qquad R_2 := \frac{\log v_n^+}{\log M_1'} \xrightarrow{p} \frac{2-\alpha}{1-\alpha},$$

$$R_3 := \frac{\log v_n^{+*}}{\log M_1'} \xrightarrow{p} \frac{2-\alpha^2}{1-\alpha}, \qquad R_L := \frac{\log v_n^-}{\log M_1'} \xrightarrow{\text{a.s.}} 2.$$

The results generalize to $F(c, \alpha)$ for $c > 0$ because as noted after Eq. **9**, $X/c \overset{d}{\approx} F(1, \alpha)$ if and only if $X \overset{d}{\approx} F(c, \alpha)$ for $c > 0$. Applying the continuous mapping theorem to the above results for the variance, the upper semivariance, and the local upper semivariance yields three consistent estimators of α:

$$B_1 := \frac{2-R_1}{1-R_1}, \qquad B_2 := \frac{2-R_2}{1-R_2},$$

$$B_3 := \frac{R_3 - \sqrt{R_3^2 - 4(R_3-2)}}{2}.$$

The Hill estimator [Hill (44)] is a traditional tail-index estimator, which requires the largest k observations where $k \to \infty$ and $k/n \to 0$ as $n \to \infty$. However, k depends on the unknown parameters such as α and the series representation of the survival function [Hall (45)]. In practice, the number k is based on the "stable" point in the Hill plot, which may not always be available. Gomes and Guillou (46) give a comprehensive review.

Theorem 9 implies that N_n^+/n converges to zero in probability, which motivates the choice of $k = N_n^+ + 1$ in the Hill estimator:

$$\left(\frac{1}{k}\sum_{i=n-k+1}^{n} \log(X_{(i)}) - \log(X_{(n-k+1)})\right)^{-1},$$

where $X_{(i)}$ is the *i*th-order statistic, $1 \leq i \leq n$. We evaluate this choice of $k = N_n^+ + 1$ in the Hill estimator, denoted by HI.N, numerically. We also replace the smallest $(n-k)$ order statistics in the original Hill estimator by the sample mean M_1' to obtain a new Hill-type estimator:

$$\mathrm{HI.M} := \left(\frac{1}{N_n^+}\sum_{X_i > M_1'} \log(X_i/M_1')\right)^{-1}.$$

From Bergström (47), the survival function of the stable law for $0 < \alpha < 1$ is

$$\overline{F}(1,\alpha)(x) = \int_x^\infty -\frac{1}{\pi}\sum_{k=1}^\infty \frac{(-1)^k}{k!}(\sin \pi\alpha k)\frac{\Gamma(ak+1)}{t^{ak+1}}\,dt$$

$$= \frac{1}{\pi}\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!}(\sin \pi\alpha k)\frac{\Gamma(ak)}{x^{ak}}$$

$$= Cx^{-\alpha}[1 + Dx^{-\alpha} + o(x^{-\alpha})],$$

where $C > 0$ and $D \neq 0$. From Hall (45), it is optimal to choose k tending to infinity at a rate of order $n^{2\alpha/(2\alpha+\alpha)} = n^{2/3}$. We also consider this choice $k = n^{2/3}$ for another Hill-type estimator, denoted by HI.Opt, and we compare the behavior with other estimators.

Table 1. **Bias ($\times 10^3$; average of [estimate minus true α]) for tail-index estimators B_1, B_2, B_3, HI.N, HI.M, HI.Opt, and MHB3 with sample size $n = 10^4$ from $F(1, \alpha)$**

α	B1	B2	B3	HI.N	HI.M	HI.Opt	MHB3
0.1	−5.24	−3.87	−3.00	10.25	135.16	−0.92	−5.82
0.2	−11.96	−6.88	−3.79	−9.31	73.52	−1.73	−9.65
0.3	−19.43	−8.55	−2.38	−25.60	30.89	−2.05	−12.03
0.4	−27.72	−9.75	0.63	−32.82	4.87	−1.54	−13.44
0.5	−35.03	−8.91	5.96	−29.40	−5.30	1.42	−12.56
0.6	−43.76	−10.44	9.41	−24.21	−8.21	6.67	−10.26
0.7	−50.27	−11.28	12.19	−10.06	0.13	19.37	−3.26
0.8	−53.49	−12.55	11.48	31.58	37.82	51.80	7.30
0.9	−50.31	−13.69	5.46	204.26	208.27	153.44	5.45

Table 2. **MSE ($\times 10^3$) (mean squared [estimate minus true α]) for tail-index estimators B_1, B_2, B_3, HI.N, HI.M, HI.Opt, and MHB3 with sample size $n = 10^4$ from $F(1, \alpha)$**

α	B1	B2	B3	HI.N	HI.M	HI.Opt	MHB3
0.1	0.14	0.15	0.11	2.61	20.06	0.02	0.10
0.2	0.53	0.58	0.35	4.73	9.13	0.09	0.31
0.3	1.13	1.23	0.71	7.33	6.31	0.19	0.57
0.4	1.86	1.96	1.16	9.15	6.39	0.34	0.85
0.5	2.60	2.66	1.76	9.12	6.76	0.54	1.15
0.6	3.47	3.20	2.32	7.93	6.13	0.84	1.53
0.7	4.15	3.38	2.60	6.46	5.30	1.52	1.94
0.8	4.32	3.05	2.28	6.97	6.67	4.35	2.13
0.9	3.59	2.10	1.33	57.51	58.26	26.36	1.33

In our simulations, we generate 10^4 independent random samples, each with sample size n, from $F(1, \alpha)$ by using the rstable function from the R package stabledist with arguments for the tail-index parameter alpha $= \alpha$, the skewness parameter beta $= 1$, the scale parameter gamma $= |1 - i\tan(\pi\alpha/2)|^{-1/\alpha}$, the location parameter delta $= 0$, and parameterization pm $= 1$. Setting pm $= 1$ specifies that we use the parameterization of stable laws in Samorodnitsky and Taqqu (4). For each random sample, we calculate the six estimators B_1, B_2, B_3, HI.N, HI.M, and HI.Opt. Then, we estimate the bias as the average of the 10^4 differences between each estimator of α and the true α. We estimate the mean squared error (MSE) as the average of 10^4 squared differences between each estimator of α and the true α.

In Table 1 for bias and Table 2 for MSE, the sample size is $n = 10^4$. According to the bias estimates in Table 1, B_1 tends to underestimate α, while B_2 and B_3 reduce the bias from B_1 by introducing the upper semivariance, which focuses more on larger numbers. B_3 has smaller bias than B_2 for most of the α except $\alpha = 0.7$ and 0.8. In Table 2, B_3 has smaller MSE than B_1 and B_2. Estimators HI.N and HI.M do not perform as well as B_3.

The estimator HI.Opt with the optimal choice of $k = n^{2/3}$ for the Hill estimator has the smallest bias, when $\alpha \leq 0.6$, and MSE, when $\alpha \leq 0.7$. However, B_3 from Taylor's law of the local semivariance has better performance, especially much smaller bias, than HI.Opt for $\alpha \geq 0.8$. Since HI.Opt tends to overestimate α, especially when $\alpha \geq 0.7$, we defined the estimator MHB3 to be the minimum of B_3 and HI.Opt. This MHB3 not only reduces the bias dramatically but also improves the MSE of B_3 for α close to 1.

The advantages of B_3 and MHB3 gradually vanish when sample size increases because $k = n^{2/3}$ is an asymptotically optimal choice. However, for sample sizes smaller than 10^4, B_3 and MHB3 can improve HI.Opt even more. More comparisons are in *SI Appendix* for sample sizes $n = 10^2, 10^3$, and 10^5. On the other hand, although the behavior of B_1, B_2, and B_3 depends on c in $F(c, \alpha)$, one sees similar patterns in bias and MSE. B_3 and MHB3 still have better bias and MSE for $\alpha \geq 0.8$ for small sample sizes. More comparisons are in *SI Appendix* for $F(2, \alpha)$ and $F(0.5, \alpha)$.

Tables in *SI Appendix* also show that both bias and MSE decrease when sample size increases, as expected of consistent estimators and as proved in *Corollary 1*.

B. Asymptotic Distribution of N_n^+/n^α. To illustrate *Theorem 9*, we generate 10^3 independent random samples from $F(1, \alpha)$ with sample size $n = 10^6$ and calculate N_n^+/n^α for each random sample. We use the 10^3 values of N_n^+/n^α to estimate the distribution of N_n^+/n^α. To estimate the distribution of $U^{-\alpha}/\Gamma(1-\alpha)$, we generate 10^3 independent random values $U_1, \ldots, U_{10^3}$ from $F(1, \alpha)$ and calculate the corresponding $U_i^{-\alpha}/\Gamma(1-\alpha)$ for $i = 1, \ldots, 10^3$. Then, we use the 10^3 values of $U_i^{-\alpha}/\Gamma(1-\alpha)$ to estimate the distribution of $U^{-\alpha}/\Gamma(1-\alpha)$. The histograms and

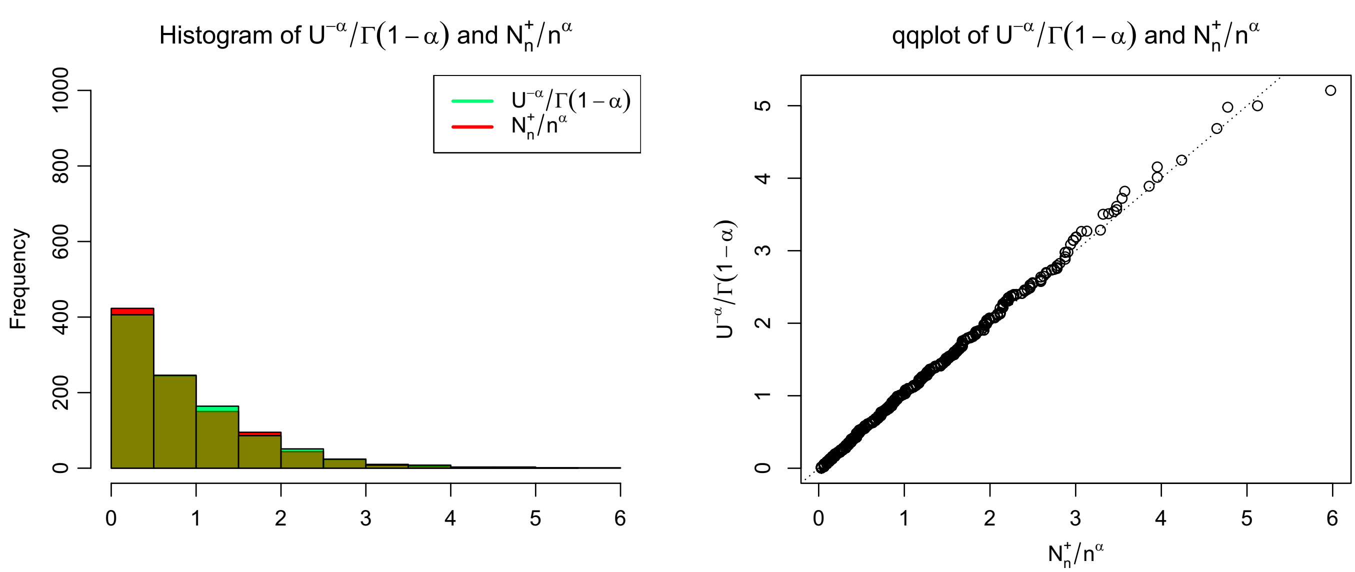

Fig. 1. Histogram and quantile–quantile plot of N_n^+/n^α and $U^{-\alpha}/\Gamma(1-\alpha)$ for $\alpha = 0.25$. The P value of the KS test is 0.1995.

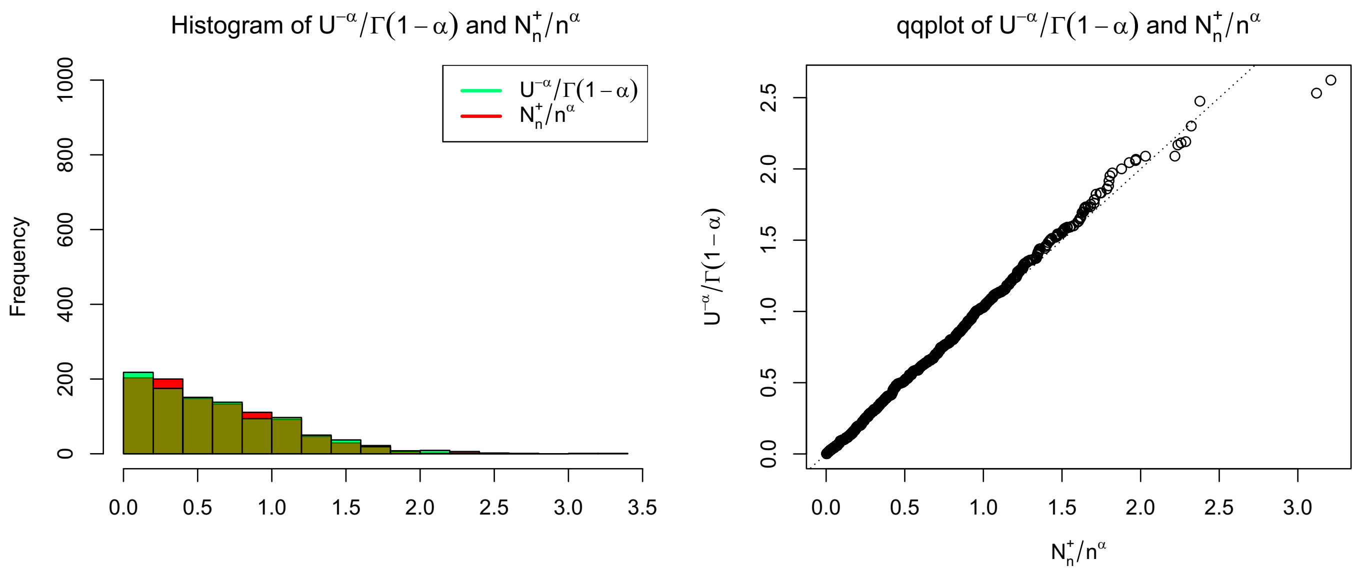

Fig. 2. Histogram and quantile–quantile plot of N_n^+/n^α and $U^{-\alpha}/\Gamma(1-\alpha)$ for $\alpha = 0.50$. The *P* value of the KS test is 0.9135.

quantile–quantile plots of N_n^+/n^α and $U^{-\alpha}/\Gamma(1-\alpha)$ with $\alpha = 0.25$ and $\alpha = 0.5$ are in Figs. 1 and 2, respectively. The histograms mostly overlap. The *P* values of the two-sample Kolmogorov–Smirnov (KS) test are 0.1995 and 0.9135, respectively. These observations support the convergence of N_n^+/n^α in distribution.

As expected, the speed of convergence of N_n^+/n^α in *Theorem 9* depends on α. Similarly, the speeds of convergence of the moment ratios in *Theorems 3* and *6* also depend on both α and the orders of the moments. We discuss the sample sizes required to see the convergence in distributions in *Theorems 3, 6,* and *9* in *SI Appendix*. From our simulation results, smaller α and higher-order moments result in faster convergence in distribution for the ratios of the moments.

Data Availability. Computer code has been deposited in GitHub (https://github.com/cftang9/TLHM). Readers can generate the tables and figures using the R code there.

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