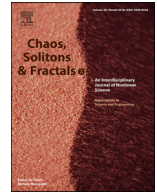




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Modeling the epidemic control measures in overcoming COVID-19 outbreaks: A fractional-order derivative approach

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ABSTRACT

Novel coronavirus named SARS-CoV-2 is one of the global threats and uncertain challenges worldwide faced at present. It has stroke rapidly around the globe due to viral transmissibility, new variants (strains), and human unconsciousness. Lack of adequate and reliable vaccination and proper treatment, control measures such as self-protection, physical distancing, lockdown, quarantine, and isolation policy plays an essential role in controlling and reducing the pandemic. Decisions on enforcing various control measures should be determined based on a theoretical framework and real-data evidence. We deliberate a general mathematical control measures epidemic model consisting of lockdown, self-protection, physical distancing, quarantine, and isolation compartments. Then, we investigate the proposed model through Caputo fractional order derivative. Fixed point theory has been used to analyze the Caputo fractional-order derivative model's existence and uniqueness solutions, whereas the Adams-Bashforth-Moulton numerical scheme was applied for numerical simulation. Driven by extensive theoretical analysis and numerical simulation, this work further illuminates the substantial impact of various control measures.

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1. Introduction

COVID-19, a transmissible respiratory disease, first time reported in Wuhan of Hubei Province, Republic of China [1] in 2019. Furthermore, it has rapidly spread internationally; thus, WHO declared that COVID-19 is a global pandemic [2]. According to health experts, it has found a variety of COVID-19 symptoms [3–6]. Enormous mainstream people experienced mild to moderate respiratory illnesses [7]. However, some of individuals would improve complexities of respiratory disappointment or severe respiratory suffering signs. Further, few studies emphasize that more than 80% of the individuals are asymptomatic infection carriers like they perceive no or mild side effects [8]. Thus, the recognition and control of SARS-CoV-2 disease become considerably more sophisticated. To this, various COVID-19 alleviation procedures have been adjusted so far, for example, self-protection, lockdown, quarantine, or isolation, with the end goal of decreasing community transmission of the disease.

One of the most commonly adopted mathematical epidemiological models is the SIR (Susceptible-Infected-Recovered) model. It characterizes the epidemic dynamics, predicts possible contagion scenarios, and simulates the time-histories of an epidemic phenomenon. The people who still can not seem to be contaminated by the virus represented by compartment S (susceptible). Infected individuals who showed symptoms and can spread the virus to the susceptible compartment. Finally, R (recovered), individuals who have recovered—beside, expected to have an immune acknowledgment to the virus [9–14]. But SARS-CoV-2 is a novel virus as well as we have exceptionally constrained information about this disease. Many scholars of the entire world investigated this pandemic's control measures [15–18]. The readers are requested to read some work of covid-19 [19–25]. Lacking proper treatment and vaccination, computational simulation with self-protection, physical distance, the lockdown situation, a great deal of testing, quarantine, and isolation would be played a significant role in analyzing and controlling the current pandemic. Considering such cases, we modified the usual SIR model to the SLTIA_dI_{snd}I_{sd}QJL_tRDP model.

Furthermore, many researchers from various disciplines have recently given deep concentration to the theory of fractional calculus and fractional differential equations [26–32]. As a measure of fact, it mentioned that fractional derivatives are beneficial for

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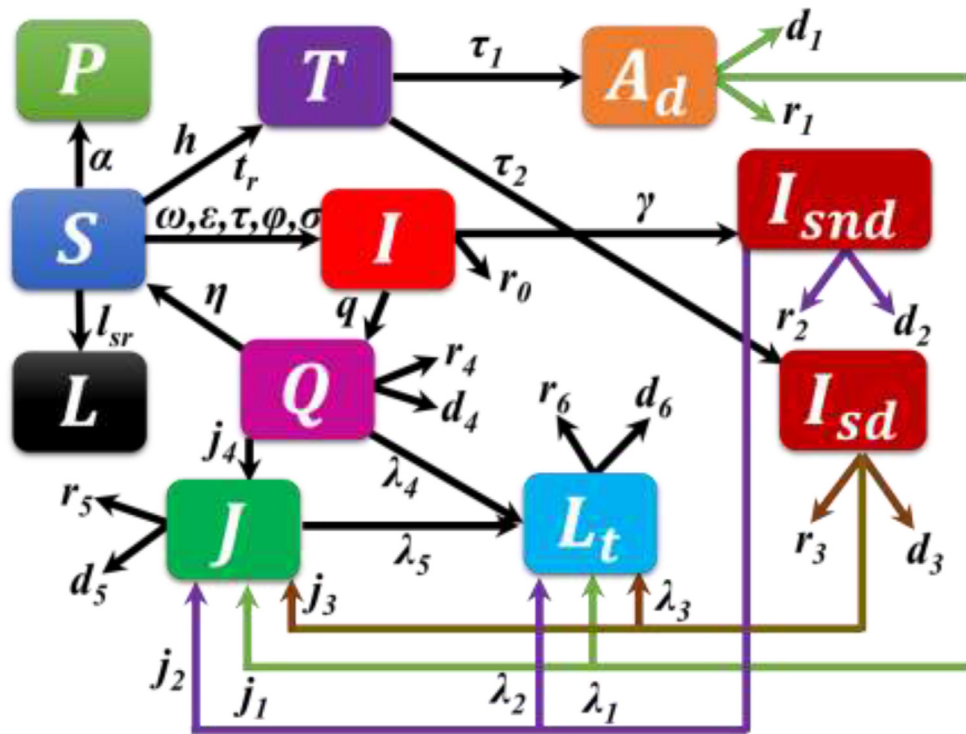


Fig. 1. Schematic flow diagram for the epidemic SARS-CoV-2 models. In our epidemic model, we presumed total of 13 compartments as follows: susceptible $S(t)$, lockdown $L(t)$, Tested state $T(t)$, infected $I(t)$ (asymptomatic, mild-symptomatic-infected, minor or moderate infection but not detected), asymptomatic infected with detected $A_d(t)$, symptomatic infected with not detected $I_{snd}(t)$, symptomatic infected with detected $I_{sd}(t)$, quarantine $Q(t)$, isolated or hospitalized $J(t)$, life-threatening condition $L_t(t)$, recovered $R(t)$, death $D(t)$, and self-protected $P(t)$. [The proposed model has a lot of recovery and death parameters, and their determination is more complicated. As a result, the recovered and death compartments are absent in the Schematic flow diagram.]

modeling many real-worlds problems due to memory and the universal properties [33–36]. As a result, the importance and potential application enlarged day by day [37–39]. The fractional-order differential equations supplement new dimensions in the investigation of epidemiological models. Yadav and Rene’s first time developed the Caputo–Fabrizio fractional derivative model of COVID-19 [40]. Subsequently, many studies [38,41–44] address the COVID-19 fractional-order differential model. In this work, we introduce Caputo fractional derivative [45] approach to our proposed epidemic model.

This research aims to model and analyze a modified SIR mathematical epidemic model by considering all possible control measures. Beside, we represent the proposed model through a Caputo fractional order derivative. Fixed point theory has been used to analyze the Caputo fractional-order derivative model’s existence and uniqueness solutions. Also, for numerical simulation, we applied the Adams-Bashforth-Moulton numerical scheme. The analysis of thirteen compartments and the concentration of COVID-19 in the surrounding circumstances concerning time for several fractional-order derivative values have been theoretically investigated and graphically manifested.

The development of this work is as follows. The formulation of the model is elaborately discussed in Section 2. In Section 3, we present a fractional model using Caputo fractional derivatives, where the fractional order of differentiation is p . Calibration of the epidemic model is given in Section 4. In Section 5, we offer some numerical results through the graphs. The concluding words are given in Section 6.

2. Mathematical model formulation

The proposed model displays the dynamics of thirteen compartments (Fig. 1), namely susceptible $S(t)$, lockdown $L(t)$, tested

state $T(t)$, Infected $I(t)$ (asymptomatic, mild-symptomatic-infected, minor or moderate infection but not detected), asymptomatic infected with detected $A_d(t)$, symptomatic infected with not detected $I_{snd}(t)$, symptomatic infected with detected $I_{sd}(t)$, quarantine $Q(t)$, isolated or hospitalized $J(t)$, life-threatening condition $L_t(t)$, recovered $R(t)$, death $D(t)$, and self-protected $P(t)$ individuals. The mean-field mathematical epidemiological dynamics are then modeled by the following system of ordinary differential equations.

$$\frac{dS(t)}{dt} = -p_d \beta S(t) \{ \omega I(t) + \varepsilon A_d(t) + \tau I_{snd}(t) + I_{sd}(t) + \phi Q(t) + \sigma J(t) \} - l_{sr} S(t) + l_d L(t) - t_r S(t) T(t) + h T(t) + \eta Q(t) - \alpha S(t), \quad (1.1)$$

$$\frac{dL(t)}{dt} = l_{sr} S(t) - l_d L(t), \quad (1.2)$$

$$\frac{dT(t)}{dt} = t_r S(t) T(t) - (h + \tau_1 + \tau_2) T(t), \quad (1.3)$$

$$\frac{dI(t)}{dt} = p_d \beta S(t) \{ \omega I(t) + \varepsilon A_d(t) + \tau I_{snd}(t) + I_{sd}(t) + \phi Q(t) + \sigma J(t) \} - (\gamma + q + r_0) I(t) \quad (1.4)$$

$$\frac{dA_d(t)}{dt} = \tau_1 T(t) - (j_1 + \lambda_1 + r_1 + d_1) A_d(t), \quad (1.5)$$

$$\frac{dI_{snd}(t)}{dt} = \gamma I(t) - (j_2 + \lambda_2 + r_2 + d_2) I_{snd}(t), \quad (1.6)$$

$$\frac{dI_{sd}(t)}{dt} = \tau_2 T(t) - (j_3 + \lambda_3 + r_3 + d_3) I_{sd}(t), \quad (1.7)$$

$$\frac{dQ(t)}{dt} = qI(t) - (\eta + j_4 + \lambda_4 + r_4 + d_4)Q(t), \tag{1.8}$$

$$\frac{dJ(t)}{dt} = j_1A_d(t) + j_2I_{snd}(t) + j_3I_{sd}(t) + j_4Q(t) - (\lambda_5 + r_5 + d_5)J(t), \tag{1.9}$$

$$\frac{dL_t(t)}{dt} = \lambda_1A_d(t) + \lambda_2I_{snd}(t) + \lambda_3I_{sd}(t) + \lambda_4Q(t) + \lambda_5J(t) - (r_6 + d_6)L_t(t), \tag{1.10}$$

$$\frac{dR(t)}{dt} = r_0I(t) + r_1A_d(t) + r_2I_{snd}(t) + r_3I_{sd}(t) + r_4Q(t) + r_5J(t) + r_6L_t(t), \tag{1.11}$$

$$\frac{dD(t)}{dt} = d_1A_d(t) + d_2I_{snd}(t) + d_3I_{sd}(t) + d_4Q(t) + d_5J(t) + d_6L_t(t), \tag{1.12}$$

$$\frac{dP(t)}{dt} = \alpha S(t), \tag{1.13}$$

The total population

$$N(t) = S(t) + L(t) + T(t) + I(t) + A_d(t) + I_{snd}(t) + I_{sd}(t) + Q(t) + J(t) + L_t(t) + R(t) + D(t) + P(t). \tag{1.14}$$

Now, if the vector of the state variable is,

$$x = (S(t), L(t), T(t), I(t), A_d(t), I_{snd}(t), I_{sd}(t), Q(t), J(t), L_t(t), R(t), D(t), P(t))'$$

and $f : R^{13} \rightarrow R^{13}$.

Then the right side of the proposed model (Eqs. (1.1)–(1.13)) is a continuously differentiable function on R^{13} . Necessarily, a novel clarification of (1.1–1.14) exists in Ω for any initial condition and remains for its maximal existence interval [46]. Therefore, the proposed model is well-defined in biological meaning. Also, according to Nabi et al. [43], the model's solution is positive for all $t \geq 0$ and bounded by the total population $N(t)$ (Eq. (1.14)). Thus, at any time, each compartment is considered to be in one of the following thirteen possible states.

(i) *Susceptible individuals, S(t)*: Initially, the susceptible is the fraction of the total population subject to the infected individuals (Eq. (1.1)). The suspected susceptible population is increased by the net inflow of people from quarantine and other compartments and diminished by self-protected and natural death rates. The susceptible populations likewise decrease the following disease, obtained by contact between susceptible and infected people, who might be contacting asymptomatic, mild-symptomatic infected, minor or moderate infection but not detected, asymptomatic infected (detected), symptomatic infected (not detected), symptomatic infected (detected) and isolated individuals. The transmission coefficient for these classes of infected individuals is $\beta\omega, \beta\varepsilon, \beta\tau, \beta, \beta\phi,$ and $\beta\sigma$ respectively. Here, the primary transmission coefficient of infectious and contact rates is β . The change parameter ϕ represents different levels of hygiene precautionary measures during quarantine.

(ii) *Lockdown individuals, L(t)*: These are the people who have followed lockdown policies. The lockdown compartment refers to susceptible individuals staying at home and staying safe from the virus. Here, we quantify the lockdown open and close mechanism by using the Heaviside function.

$$l_{sr}[d_1, d_2] = \begin{cases} 0, & t \notin [d_1, d_2] \\ 1, & t \in [d_1, d_2] \end{cases} \tag{2.1}$$

where, $d_1 =$ lockdown starting time and $d_2 =$ lockdown ending time.

(iii) *Tested individuals, T(t)*: One of the powerful tactics to control the spread of the disease is testing the susceptible population. Lack of plethora of testing undetected infected people generously the asymptomatic individual who sustains the environment and spreads the epidemic. The people of this compartment that is healthy again is susceptible, and detected individuals tested positive at the rate τ_1 and τ_2 .

(iv) *Asymptomatic infected, mild-symptomatic infected, and minor or moderate infected but not detected individuals, I(t)*: The asymptomatic, mild-symptomatic infected, minor or moderate infection but not detected individuals are the entire populations. They are infected by the SARS-CoV-2 virus but have no apparent substantial clinical side effects yet. This period is known as the latent phase, and at this juncture, a disease can be infectious or partially infectious. This compartment population is lessened by an infected (symptomatic and asymptomatic, which is identified by the test), symptomatic infected (not detected), quarantine and recover at the rates γ, q and r_0 .

(v) *Asymptomatic infected, detected individuals, A_d(t)*: It is one of the most hazardous components of any transmissible disease. Generally, people are not apparent by the clinical symptoms of COVID-19. As a result, the disease spreads smoothly. Furthermore, one is confirmed by testing at the rate τ_1 . Finally, the isolation rate j_1 , life-threatening λ_1 , recovery rate r_2 , and disease-induced death d_1 decreased this compartment population.

(vi) *Symptomatic infected, not detected individuals, I_{snd}(t)*: These are the people of symptomatic infected but not detected. They have mild clinical symptoms of COVID-19 after the latent period. But did not detect due to scarcity of testing, financial crisis, and lack of knowledge of the disease. The isolation rate j_2 , life-threatening λ_2 , recovery rate r_2 , and disease-induced death d_2 decreased this compartment population.

(vii) *Symptomatic with detected individuals, I_{sd}(t)*: These are the people who have been apparent the growth of clinical symptoms of COVID 19 after the latent period and confirmed by testing at the rate τ_2 . The isolation rate j_3 , life-threatening λ_3 , recovery rate r_3 , and disease-induced death d_3 decreased this compartment population.

(viii) *Quarantined individuals, Q(t)*: These are the people who have been contracting with a source of SARS-CoV-2 virus at rates q asymptomatic, mild-symptomatic infected, minor or moderate infection but not detected. The practical reality is that sometimes a few

$$\begin{aligned}
 a_{22} &= \frac{p_d \beta \omega}{v_2} + \frac{p_d \beta \tau \gamma}{v_2 v_4} + \frac{p_d \beta \phi q}{v_2 v_6} + \frac{p_d \beta \sigma (j_4 q v_1 v_3 v_4 v_5 v_8 + \gamma j_2 v_1 v_3 v_5 v_6 v_8)}{v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8} \\
 a_{23} &= \frac{p_d \beta \varepsilon}{v_3} + \frac{p_d \beta \sigma j_1}{v_3 v_7} \\
 a_{24} &= \frac{p_d \beta \tau}{v_4} + \frac{p_d \beta \sigma j_2}{v_4 v_7} \\
 a_{25} &= \frac{p_d \beta}{v_5} + \frac{p_d \beta \sigma j_3}{v_5 v_7} \\
 a_{26} &= \frac{p_d \beta \phi}{v_6} + \frac{p_d \beta \sigma j_4}{v_6 v_7} \\
 a_{27} &= \frac{p_d \beta \sigma}{v_7}
 \end{aligned}$$

According to the next-generation matrix approach, the basic reproduction number R_0 is the largest Eigenvalue of $|FV^{-1}|$. Thus, $R_0 = \rho(FV^{-1})$, where ρ is the spectral radius of FV^{-1} .

$$R_0 = \frac{p_d \beta \omega}{v_2} + \frac{p_d \beta \tau \gamma}{v_2 v_4} + \frac{p_d \beta \phi q}{v_2 v_6} + \frac{p_d \beta \sigma (j_4 q v_1 v_3 v_4 v_5 v_8 + \gamma j_2 v_1 v_3 v_5 v_6 v_8)}{v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8} \tag{2.5}$$

2.2. Strength number

The concept of the reproduction number discussed above has been widely used in epidemiology for assessing whether or not the spread will exist. However, a reproduction number could not specify whether a model has formed the waves or not. To determine the complications in the epidemic spread and assist in detecting the waves, we consider a new number approach termed the strength number, derived using the next-generation matrix by taking the second derivative of infectious classes.

For determining the strength number (SN) of the proposed model, we assume the total population is finite (N). Then the mass and standard action incidence have no difference.

Thus,

$$p_d \beta \omega SI = \frac{p_d \beta \omega SI}{N} \tag{2.6}$$

Now, according to Atangana et al. [48,49]

$$\begin{aligned}
 \frac{\partial^2}{\partial I^2} \left(\frac{p_d \beta \omega SI}{N} \right) &= p_d \beta \omega S \frac{\partial}{\partial I} \left(\frac{N - NI}{N^2} \right) \\
 &= -\frac{p_d \beta \omega S}{N^2}
 \end{aligned} \tag{2.7}$$

Similarly,

$$\frac{\partial^2}{\partial A_d^2} \left(\frac{p_d \beta \varepsilon SA_d}{N} \right) = -\frac{p_d \beta \varepsilon S}{N^2} \tag{2.8}$$

$$\frac{\partial^2}{\partial I_{snd}^2} \left(\frac{p_d \beta \tau SI_{snd}}{N} \right) = -\frac{p_d \beta \tau S}{N^2} \tag{2.9}$$

$$\frac{\partial^2}{\partial I_{sd}^2} \left(\frac{p_d \beta SI_{sd}}{N} \right) = -\frac{p_d \beta S}{N^2} \tag{2.10}$$

$$\frac{\partial^2}{\partial Q^2} \left(\frac{p_d \beta \phi SQ}{N} \right) = -\frac{p_d \beta \phi S}{N^2} \tag{2.11}$$

$$\frac{\partial^2}{\partial J^2} \left(\frac{p_d \beta \sigma SQ}{N} \right) = -\frac{p_d \beta \sigma S}{N^2} \tag{2.12}$$

In this case,

$$F_{SN} = \begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{p_d \beta \omega}{N^2} & -\frac{p_d \beta \varepsilon}{N^2} & -\frac{p_d \beta \tau}{N^2} & -\frac{p_d \beta \phi}{N^2} & -\frac{p_d \beta \phi}{N^2} & -\frac{p_d \beta \sigma}{N^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.13}$$

Then

$$\text{Det}(F_{SN}V^{-1} - \lambda I_8) = 0 \tag{2.14}$$

Therefore, strength number,

$$SN = - \left[\frac{p_d \beta \omega}{N^2 v_2} + \frac{p_d \beta \tau \gamma}{N^2 v_2 v_4} + \frac{p_d \beta \phi q}{N^2 v_2 v_6} + \frac{p_d \beta \sigma (j_4 q v_1 v_3 v_4 v_5 v_8 + \gamma j_2 v_1 v_3 v_5 v_6 v_8)}{N^2 v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8} \right] < 0 \tag{2.15}$$

$SN = 0$ indicates that the spread will not renew and will hence have a single magnitude and die out. $SN > 0$ suggests sufficient strength to initiate the renewal phase, implying that the spread will have more than one wave. On the other hand, biologists will offer a clear explanation of the quantity, as mentioned earlier, which will be proven when the second derivative of infectious classes is studied.

2.3. The first derivative of the Lyapunov function

The endemic equilibrium E^* for the endemic Lyapunov function is $\{S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P\}$, $L_f < 0$.

Theorem 1. The endemic equilibrium points E^* in the SLTIA_dI_{snd}I_{sd}QJL_tRDP model are globally asymptotically stable when the reproductive number $R_0 > 1$.

Proof. For proof of the theorem, the Lyapunov function can be expressed as follows

$$\begin{aligned} L_f(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P) = & \left(S - S^* - S^* \log \frac{S}{S^*} \right) + \left(L - L^* - L^* \log \frac{L}{L^*} \right) \\ & + \left(T - T^* - T^* \log \frac{T}{T^*} \right) + \left(I - I^* - I^* \log \frac{I}{I^*} \right) + \left(A_d - A_d^* - A_d^* \log \frac{A_d}{A_d^*} \right) \\ & + \left(I_{snd} - I_{snd}^* - I_{snd}^* \log \frac{I_{snd}}{I_{snd}^*} \right) + \left(I_{sd} - I_{sd}^* - I_{sd}^* \log \frac{I_{sd}}{I_{sd}^*} \right) \\ & + \left(Q - Q^* - Q^* \log \frac{Q}{Q^*} \right) + \left(J - J^* - J^* \log \frac{J}{J^*} \right) + \left(L_t - L_t^* - L_t^* \log \frac{L_t}{L_t^*} \right) \\ & + \left(R - R^* - R^* \log \frac{R}{R^*} \right) + \left(D - D^* - D^* \log \frac{D}{D^*} \right) + \left(P - P^* - P^* \log \frac{P}{P^*} \right) \end{aligned} \tag{2.16}$$

Differentiating both sides concerning t yields

$$\begin{aligned} \frac{dL_f}{dt} = \dot{L}_f = & \left(\frac{S - S^*}{S} \right) \dot{S} + \left(\frac{L - L^*}{L} \right) \dot{L} + \left(\frac{T - T^*}{T} \right) \dot{T} + \left(\frac{I - I^*}{I} \right) \dot{I} + \left(\frac{A_d - A_d^*}{A_d} \right) \dot{A}_d \\ & + \left(\frac{I_{snd} - I_{snd}^*}{I_{snd}} \right) \dot{I}_{snd} + \left(\frac{I_{sd} - I_{sd}^*}{I_{sd}} \right) \dot{I}_{sd} + \left(\frac{Q - Q^*}{Q} \right) \dot{Q} + \left(\frac{J - J^*}{J} \right) \dot{J} + \left(\frac{L_t - L_t^*}{L_t} \right) \dot{L}_t \\ & + \left(\frac{R - R^*}{R} \right) \dot{R} + \left(\frac{D - D^*}{D} \right) \dot{D} + \left(\frac{P - P^*}{P} \right) \dot{P} \end{aligned} \tag{2.17}$$

Applying the values of $\dot{S}, \dot{L}, \dot{T}, \dot{I}, \dot{A}_d, \dot{I}_{snd}, \dot{I}_{sd}, \dot{Q}, \dot{J}, \dot{L}_t, \dot{R}, \dot{D}, \dot{P}$ in Eq. (2.17), then we get,

$$\begin{aligned} \frac{dL_f}{dt} = & \left(\frac{S - S^*}{S} \right) \{ -p_d \beta S(\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J) - l_{sr} S + l_d L - t_r ST + hT + \eta Q - \alpha S \} \\ & + \left(\frac{L - L^*}{L} \right) (l_{sr} S - l_d L) + \left(\frac{T - T^*}{T} \right) \{ t_r ST - (h + \tau_1 + \tau_2) T \} \\ & + \left(\frac{I - I^*}{I} \right) \{ p_d \beta S(\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J) - (\gamma + q + r_0) I \} \\ & + \left(\frac{A_d - A_d^*}{A_d} \right) \{ \tau_1 T - (j_1 + \lambda_1 + r_1 + d_1) A_d \} \\ & + \left(\frac{I_{snd} - I_{snd}^*}{I_{snd}} \right) \{ \gamma I - (j_2 + \lambda_2 + r_2 + d_2) I_{snd} \} \\ & + \left(\frac{I_{sd} - I_{sd}^*}{I_{sd}} \right) \{ \tau_2 T - (j_3 + \lambda_3 + r_3 + d_3) I_{sd} \} \\ & + \left(\frac{Q - Q^*}{Q} \right) \{ q I - (\eta + j_4 + \lambda_4 + r_4 + d_4) Q \} \\ & + \left(\frac{J - J^*}{J} \right) \{ j_1 A_d + j_2 I_{snd} + j_3 I_{sd} + j_4 Q - (\lambda_5 + r_5 + d_5) J \} \\ & + \left(\frac{L_t - L_t^*}{L_t} \right) \{ \lambda_1 A_d + \lambda_2 I_{snd} + \lambda_3 I_{sd} + \lambda_4 Q + \lambda_5 J - (r_6 + d_6) L_t \} \\ & + \left(\frac{R - R^*}{R} \right) (r_0 I + r_1 A_d + r_2 I_{snd} + r_3 I_{sd} + r_4 Q + r_5 J + r_6 L_t) \end{aligned}$$

$$+ \left(\frac{D-D^*}{D}\right)(d_1A_d + d_2I_{snd} + d_3I_{sd} + d_4Q + d_5J + d_6L_t) + \left(\frac{P-P^*}{P}\right)(\alpha S) \tag{2.18}$$

Putting $S = S - S^*, L = L - L^*, T = T - T^*, I = I - I^*, A_d = A_d - A_d^*, I_{snd} = I_{snd} - I_{snd}^*, I_{sd} = I_{sd} - I_{sd}^*, Q = Q - Q^*, J = J - J^*, L_t = L_t - L_t^*, R = R - R^*, D = D - D^*, P = P - P^*$ in Eq. (2.18), then

$$\begin{aligned} \frac{dL_f}{dt} = & \left(\frac{S-S^*}{S}\right) \left[-p_d\beta(S-S^*) \{ \omega(I-I^*) + \varepsilon(A_d - A_d^*) + \tau(I_{snd} - I_{snd}^*) + (I_{sd} - I_{sd}^*) + \phi(Q - Q^*) + \sigma(J - J^*) \} \right. \\ & \left. - l_{sr}(S - S^*) + l_d(L - L^*) - t_r(S - S^*)(T - T^*) + h(T - T^*) + \eta(Q - Q^*) - \alpha(S - S^*) \right] \\ & + \left(\frac{L-L^*}{L}\right) \{ l_{sr}(S - S^*) - l_d(L - L^*) \} \\ & + \left(\frac{T-T^*}{T}\right) \{ t_r(S - S^*)(T - T^*) - (h + \tau_1 + \tau_2)(T - T^*) \} \\ & + \left(\frac{I-I^*}{I}\right) \left[p_d\beta(S - S^*) \{ \omega(I - I^*) + \varepsilon(A_d - A_d^*) + \tau(I_{snd} - I_{snd}^*) + (I_{sd} - I_{sd}^*) + \phi(Q - Q^*) + \sigma(J - J^*) \} - (\gamma + q + r_0)(I - I^*) \right] \\ & + \left(\frac{A_d - A_d^*}{A_d}\right) \{ \tau_1(T - T^*) - (j_1 + \lambda_1 + r_1 + d_1)(A_d - A_d^*) \} \\ & + \left(\frac{I_{snd} - I_{snd}^*}{I_{snd}}\right) \{ \gamma(I - I^*) - (j_2 + \lambda_2 + r_2 + d_2)(I_{snd} - I_{snd}^*) \} \\ & + \left(\frac{I_{sd} - I_{sd}^*}{I_{sd}}\right) \{ \tau_2(T - T^*) - (j_3 + \lambda_3 + r_3 + d_3)(I_{sd} - I_{sd}^*) \} \\ & + \left(\frac{Q - Q^*}{Q}\right) \{ q(I - I^*) - (\eta + j_4 + \lambda_4 + r_4 + d_4)(Q - Q^*) \} \\ & + \left(\frac{J - J^*}{J}\right) \{ j_1(A_d - A_d^*) + j_2(I_{snd} - I_{snd}^*) + j_3(I_{sd} - I_{sd}^*) + j_4(Q - Q^*) - (\lambda_5 + r_5 + d_5)(J - J^*) \} \\ & + \left(\frac{L_t - L_t^*}{L_t}\right) \{ \lambda_1(A_d - A_d^*) + \lambda_2(I_{snd} - I_{snd}^*) + \lambda_3I_{sd} + \lambda_4(Q - Q^*) + \lambda_5(J - J^*) - (r_6 + d_6)(L_t - L_t^*) \} \\ & + \left(\frac{R - R^*}{R}\right) \{ r_0(I - I^*) + r_1(A_d - A_d^*) + r_2(I_{snd} - I_{snd}^*) + r_3(I_{sd} - I_{sd}^*) + r_4(Q - Q^*) + r_5(J - J^*) + r_6(L_t - L_t^*) \} \\ & + \left(\frac{D - D^*}{D}\right) \{ d_1(A_d - A_d^*) + d_2(I_{snd} - I_{snd}^*) + d_3(I_{sd} - I_{sd}^*) + d_4(Q - Q^*) + d_5(J - J^*) + d_6(L_t - L_t^*) \} \\ & + \left(\frac{P - P^*}{P}\right) \{ \alpha(S - S^*) \} \end{aligned} \tag{2.19}$$

For avoiding complexity, the above equation can be written in the following form

$$\frac{dL_f}{dt} = \mathbf{\Pi} - \mathbf{\Omega} \tag{2.20}$$

where

$$\begin{aligned} \mathbf{\Pi} = & p_d\beta(\omega I^* + \varepsilon A_d^* + \tau I_{snd}^* + I_{sd}^* + \phi Q^* + \sigma J^*) \frac{(S - S^*)^2}{S} + l_d L + l_d \frac{S^*}{S} L^* \\ & + t_r \frac{(S - S^*)^2}{S} T^* + hT + h \frac{S^*}{S} T^* + \eta Q + \eta \frac{S^*}{S} Q^* + l_{sr} S + l_{sr} \frac{L^*}{L} S^* + t_r \frac{(T - T^*)^2}{T} S \\ & + \left(p_d\beta S + p_d\beta \frac{I^*}{I} S^* \right) (\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J) + \tau_1 T \\ & + \tau_1 \frac{A_d^*}{A_d} T^* + \gamma I + \gamma \frac{I_{snd}^*}{I_{snd}} I^* + \tau_2 T + \tau_2 \frac{I_{sd}^*}{I_{sd}} T^* + qI + q \frac{Q^*}{Q} I^* + j_1 A_d + j_2 I_{snd} \\ & + j_3 I_{sd} + j_4 Q + j_1 \frac{J^*}{J} A_d^* + j_2 \frac{J^*}{J} I_{snd}^* + j_3 \frac{J^*}{J} I_{sd}^* + j_4 \frac{J^*}{J} Q^* + \lambda_1 A_d + \lambda_2 I_{snd} \\ & + \lambda_3 I_{sd} + \lambda_4 Q + \lambda_5 J + \lambda_1 \frac{L_t^*}{L_t} A_d^* + \lambda_2 \frac{L_t^*}{L_t} I_{snd}^* + \lambda_3 \frac{L_t^*}{L_t} I_{sd}^* + \lambda_4 \frac{L_t^*}{L_t} Q^* \\ & + \lambda_5 \frac{L_t^*}{L_t} J^* + r_0 I + r_1 A_d + r_2 I_{snd} + r_3 I_{sd} + r_4 Q + r_5 J \\ & + r_6 L_t + r_0 \frac{R^*}{R} I^* + r_1 \frac{R^*}{R} A_d^* + r_2 \frac{R^*}{R} I_{snd}^* + r_3 \frac{R^*}{R} I_{sd}^* + r_4 \frac{R^*}{R} Q^* + r_5 \frac{R^*}{R} J^* + r_6 \frac{R^*}{R} L_t^* + d_1 A_d + d_2 I_{snd} \\ & + d_3 I_{sd} + d_4 Q + d_5 J + d_6 L_t + d_1 \frac{D^*}{D} A_d^* + d_2 \frac{D^*}{D} I_{snd}^* + d_3 \frac{D^*}{D} I_{sd}^* + d_4 \frac{D^*}{D} Q^* \\ & + d_5 \frac{D^*}{D} J^* + d_6 \frac{D^*}{D} L_t^* + \alpha S + \frac{P^*}{P} \alpha S^* \end{aligned}$$

and

$$\begin{aligned} \Omega = & p_d \beta (\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J) \frac{(S - S^*)^2}{S} + l_{sr} \frac{(S - S^*)^2}{S} + l_d L^* + l_d \frac{S^*}{S} L \\ & + t_r \frac{(S - S^*)^2}{S} T + h T^* + h \frac{S^*}{S} T + \eta Q^* + \eta \frac{S^*}{S} Q + \alpha \frac{(S - S^*)^2}{S} + l_{sr} S^* + l_{sr} \frac{L^*}{L} S \\ & + l_d \frac{(L - L^*)^2}{L} + t_r \frac{(T - T^*)^2}{T} S^* + (h + \tau_1 + \tau_2) \frac{(T - T^*)^2}{T} \\ & + \left(p_d \beta S^* + p_d \beta \frac{I^*}{I} S \right) (\omega I^* + \varepsilon A_d^* + \tau I_{snd}^* + I_{sd}^* + \phi Q^* + \sigma J^*) + (\gamma + q + r_0) \frac{(I - I^*)^2}{I} \\ & + \tau_1 T^* + \tau_1 \frac{A_d^*}{A_d} T + (j_1 + \lambda_1 + r_1 + d_1) \frac{(A_d - A_d^*)^2}{A_d} + \gamma I^* + \gamma \frac{I_{snd}^*}{I_{snd}} I \\ & + (j_2 + \lambda_2 + r_2 + d_2) \frac{(I_{snd} - I_{snd}^*)^2}{I_{snd}} + \tau_2 T^* + \tau_2 \frac{I_{sd}^*}{I_{sd}} T \\ & + (j_3 + \lambda_3 + r_3 + d_3) \frac{(I_{sd} - I_{sd}^*)^2}{I_{sd}} + q I^* + q \frac{Q^*}{Q} I \\ & + (\eta + j_4 + \lambda_4 + r_4 + d_4) \frac{(Q - Q^*)^2}{Q} + j_1 A_d^* + j_2 I_{snd}^* + j_3 I_{sd}^* + j_4 Q^* \\ & + j_1 \frac{J^*}{J} A_d + j_2 \frac{J^*}{J} I_{snd} + j_3 \frac{J^*}{J} I_{sd} + j_4 \frac{J^*}{J} Q + (\lambda_5 + r_5 + d_5) \frac{(J - J^*)^2}{J} \\ & + \lambda_1 A_d^* + \lambda_2 I_{snd}^* + \lambda_3 I_{sd}^* + \lambda_4 Q^* + \lambda_5 J^* + \lambda_1 \frac{L_t^*}{L_t} A_d + \lambda_2 \frac{L_t^*}{L_t} I_{snd} \\ & + \lambda_3 \frac{L_t^*}{L_t} I_{sd} + \lambda_4 \frac{L_t^*}{L_t} Q + \lambda_5 \frac{L_t^*}{L_t} J + (r_6 + d_6) \frac{(L_t - L_t^*)^2}{L_t} + r_0 I^* + r_1 A_d^* \\ & + r_2 I_{snd}^* + r_3 I_{sd}^* + r_4 Q^* + r_5 J^* + r_6 L_t^* + r_0 \frac{R^*}{R} + r_1 \frac{R^*}{R} A_d + r_2 \frac{R^*}{R} I_{snd} \\ & + r_3 \frac{R^*}{R} I_{sd} + r_4 \frac{R^*}{R} Q + r_5 \frac{R^*}{R} J + r_6 \frac{R^*}{R} L_t + d_1 A_d^* + d_2 I_{snd}^* \\ & + d_3 I_{sd}^* + d_4 Q^* + d_5 J^* + d_6 L_t^* + d_1 \frac{D^*}{D} A_d + d_2 \frac{D^*}{D} I_{snd} \\ & + d_3 \frac{D^*}{D} I_{sd} + d_4 \frac{D^*}{D} Q + d_5 \frac{D^*}{D} J + d_6 \frac{D^*}{D} L_t + \alpha S^* + \frac{P^*}{P} \alpha S \end{aligned}$$

Taking everything into consideration, if $\Pi < \Omega$, then $\frac{dL_f}{dt} < 0$. Therefore, when $S = S^*, L = L^*, T = T^*, I = I^*, A_d = A_d^*, I_{snd} = I_{snd}^*, I_{sd} = I_{sd}^*, Q = Q^*, J = J^*, L_t = L_t^*, R = R^*, D = D^*, P = P^*$ Eq. (2.20) can be written as

$$0 = \Pi - \Omega$$

$$\Rightarrow \frac{dL_f}{dt} = 0 \tag{2.21}$$

Thus, the most prominent compact invariant set for the proposed model

$$\left\{ (S^*, L^*, T^*, I^*, A_d^*, I_{snd}^*, I_{sd}^*, Q^*, J^*, L_t^*, R^*, D^*, P^*) \in \Gamma : \frac{dL_f}{dt} = 0 \right\} \tag{2.22}$$

is the endemic equilibrium of the suggested model at the point $\{E^*\}$. It follows that E^* is globally asymptotically stable in Γ if $\Pi < \Omega$ according to Lasalle's invariance.

2.4. The second derivative of the Lyapunov function

The first derivative of the Lyapunov function is used to assess the global stability of endemic equilibrium points. The first derivative analysis provides essential information that the second derivative analysis may supplement without loss of generality. For example, the second derivative of these Lyapunov functions tells us the curvature according to its sign, but the first derivative offers us information on the disease's progress. We are confident that its second derivative will provide further insights.

$$\begin{aligned} \frac{d^2 L_f}{dt^2} = & \frac{d}{dt} \left\{ \left(1 - \frac{S^*}{S}\right) \dot{S} + \left(1 - \frac{L^*}{L}\right) \dot{L} + \left(1 - \frac{T^*}{T}\right) \dot{T} + \left(1 - \frac{I^*}{I}\right) \dot{I} + \left(1 - \frac{A_d^*}{A_d}\right) \dot{A}_d + \left(1 - \frac{I_{snd}^*}{I_{snd}}\right) \dot{I}_{snd} + \left(1 - \frac{I_{sd}^*}{I_{sd}}\right) \dot{I}_{sd} + \left(1 - \frac{Q^*}{Q}\right) \dot{Q} \right. \\ & \left. + \left(1 - \frac{J^*}{J}\right) \dot{J} + \left(1 - \frac{L_t^*}{L_t}\right) \dot{L}_t + \left(1 - \frac{R^*}{R}\right) \dot{R} + \left(1 - \frac{D^*}{D}\right) \dot{D} + \left(1 - \frac{P^*}{P}\right) \dot{P} \right\} \\ = & \left(\frac{\dot{S}}{S}\right)^2 S^* + \left(\frac{\dot{L}}{L}\right)^2 L^* + \left(\frac{\dot{T}}{T}\right)^2 T^* + \left(\frac{\dot{I}}{I}\right)^2 I^* + \left(\frac{\dot{A}_d}{A_d}\right)^2 A_d^* + \left(\frac{\dot{I}_{snd}}{I_{snd}}\right)^2 I_{snd}^* + \left(\frac{\dot{I}_{sd}}{I_{sd}}\right)^2 I_{sd}^* \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{\dot{Q}}{Q}\right)^2 Q^* + \left(\frac{\dot{J}}{J}\right)^2 J^* + \left(\frac{\dot{L}_t}{L_t}\right)^2 L_t^* + \left(\frac{\dot{R}}{R}\right)^2 R^* + \left(\frac{\dot{D}}{D}\right)^2 D^* + \left(\frac{\dot{P}}{P}\right)^2 P^* + \left(1 - \frac{S^*}{S}\right)\dot{S} \\
 & + \left(1 - \frac{I^*}{I}\right)\dot{I} + \left(1 - \frac{T^*}{T}\right)\dot{T} + \left(1 - \frac{I^*}{I}\right)\dot{I} + \left(1 - \frac{A_d^*}{A_d}\right)\dot{A}_d + \left(1 - \frac{I_{snd}^*}{I_{snd}}\right)\dot{I}_{snd} \\
 & + \left(1 - \frac{I_{sd}^*}{I_{sd}}\right)\dot{I}_{sd} + \left(1 - \frac{Q^*}{Q}\right)\dot{Q} + \left(1 - \frac{J^*}{J}\right)\dot{J} + \left(1 - \frac{L_t^*}{L_t}\right)\dot{L}_t \\
 & + \left(1 - \frac{R^*}{R}\right)\dot{R} + \left(1 - \frac{D^*}{D}\right)\dot{D} + \left(1 - \frac{P^*}{P}\right)\dot{P}
 \end{aligned} \tag{2.23}$$

Here,

$$\dot{S} = -p_d\beta\dot{S}(\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J) - p_d\beta S(\omega\dot{I} + \varepsilon\dot{A}_d + \tau\dot{I}_{snd} + \dot{I}_{sd} + \phi\dot{Q} + \sigma\dot{J}) - I_{sr}\dot{S} + I_d\dot{L} - t_r\dot{S}T - t_r\dot{S}\dot{T} + h\dot{T} + \eta\dot{Q} - \alpha\dot{S}$$

$$\dot{L} = I_{sr}\dot{S} - I_d\dot{L}$$

$$\dot{T} = t_r\dot{S}T + t_r\dot{S}\dot{T} - (h + \tau_1 + \tau_2)\dot{T}$$

$$\dot{I} = p_d\beta\dot{S}(\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J) + p_d\beta S(\omega\dot{I} + \varepsilon\dot{A}_d + \tau\dot{I}_{snd} + \dot{I}_{sd} + \phi\dot{Q} + \sigma\dot{J}) - (\gamma + q + r_0)\dot{I}$$

$$\dot{A}_d = \tau_1\dot{T} - (j_1 + \lambda_1 + r_1 + d_1)\dot{A}_d$$

$$\dot{I}_{snd} = \gamma\dot{I} - (j_2 + \lambda_2 + r_2 + d_2)\dot{I}_{snd}$$

$$\dot{I}_{sd} = \tau_2\dot{T} - (j_3 + \lambda_3 + r_3 + d_3)\dot{I}_{sd}$$

$$\dot{Q} = q\dot{I} - (\eta + j_4 + \lambda_4 + r_4 + d_4)\dot{Q}$$

$$\dot{J} = j_1\dot{A}_d + j_2\dot{I}_{snd} + j_3\dot{I}_{sd} + j_4\dot{Q} - (\lambda_5 + r_5 + d_5)\dot{J}$$

$$\dot{L}_t = \lambda_1\dot{A}_d + \lambda_2\dot{I}_{snd} + \lambda_3\dot{I}_{sd} + \lambda_4\dot{Q} + \lambda_5\dot{J} - (r_6 + d_6)\dot{L}_t$$

$$\dot{R} = r_0\dot{I} + r_1\dot{A}_d + r_2\dot{I}_{snd} + r_3\dot{I}_{sd} + r_4\dot{Q} + r_5\dot{J} + r_6\dot{L}_t$$

$$\dot{D} = d_1\dot{A}_d + d_2\dot{I}_{snd} + d_3\dot{I}_{sd} + d_4\dot{Q} + d_5\dot{J} + d_6\dot{L}_t$$

$$\dot{P} = \alpha\dot{S}$$

Thus,

$$\begin{aligned}
 \frac{dI_f}{dt} = & \left(\frac{\dot{S}}{S}\right)^2 S^* + \left(\frac{\dot{L}}{L}\right)^2 L^* + \left(\frac{\dot{T}}{T}\right)^2 T^* + \left(\frac{\dot{I}}{I}\right)^2 I^* + \left(\frac{\dot{A}_d}{A_d}\right)^2 A_d^* + \left(\frac{\dot{I}_{snd}}{I_{snd}}\right)^2 I_{snd}^* + \left(\frac{\dot{I}_{sd}}{I_{sd}}\right)^2 I_{sd}^* \\
 & + \left(\frac{\dot{Q}}{Q}\right)^2 Q^* + \left(\frac{\dot{J}}{J}\right)^2 J^* + \left(\frac{\dot{L}_t}{L_t}\right)^2 L_t^* + \left(\frac{\dot{R}}{R}\right)^2 R^* + \left(\frac{\dot{D}}{D}\right)^2 D^* + \left(\frac{\dot{P}}{P}\right)^2 P^* \\
 & + \left(1 - \frac{S^*}{S}\right)\{-p_d\beta\dot{S}(\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J) \\
 & - p_d\beta S(\omega\dot{I} + \varepsilon\dot{A}_d + \tau\dot{I}_{snd} + \dot{I}_{sd} + \phi\dot{Q} + \sigma\dot{J}) - I_{sr}\dot{S} + I_d\dot{L} - t_r\dot{S}T \\
 & - t_r\dot{S}\dot{T} + h\dot{T} + \eta\dot{Q} - \alpha\dot{S}\} + \left(1 - \frac{L^*}{L}\right)(I_{sr}\dot{S} - I_d\dot{L}) \\
 & + \left(1 - \frac{T^*}{T}\right)(t_r\dot{S}T + t_r\dot{S}\dot{T} - (h + \tau_1 + \tau_2)\dot{T}) \\
 & + \left(1 - \frac{I^*}{I}\right)\{p_d\beta\dot{S}(\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J) \\
 & + p_d\beta S(\omega\dot{I} + \varepsilon\dot{A}_d + \tau\dot{I}_{snd} + \dot{I}_{sd} + \phi\dot{Q} + \sigma\dot{J}) - (\gamma + q + r_0)\dot{I}\} \\
 & + \left(1 - \frac{A_d^*}{A_d}\right)\{\tau_1\dot{T} - (j_1 + \lambda_1 + r_1 + d_1)\dot{A}_d\}
 \end{aligned}$$

$$\begin{aligned}
 & + \left(1 - \frac{I_{snd}^*}{I_{snd}}\right) \{ \gamma \dot{I} - (j_2 + \lambda_2 + r_2 + d_2) \dot{I}_{snd} \} \\
 & + \left(1 - \frac{I_{sd}^*}{I_{sd}}\right) \{ \tau_2 \dot{T} - (j_3 + \lambda_3 + r_3 + d_3) \dot{I}_{sd} \} \\
 & + \left(1 - \frac{Q^*}{Q}\right) \{ q \dot{I} - (\eta + j_4 + \lambda_4 + r_4 + d_4) \dot{Q} \} \\
 & + \left(1 - \frac{J^*}{J}\right) \{ j_1 \dot{A}_d + j_2 \dot{I}_{snd} + j_3 \dot{I}_{sd} + j_4 \dot{Q} - (\lambda_5 + r_5 + d_5) J \} \\
 & + \left(1 - \frac{L_t^*}{L_t}\right) \{ \lambda_1 \dot{A}_d + \lambda_2 \dot{I}_{snd} + \lambda_3 \dot{I}_{sd} + \lambda_4 \dot{Q} + \lambda_5 J - (r_6 + d_6) \dot{L}_t \} \\
 & + \left(1 - \frac{R^*}{R}\right) \{ r_0 \dot{I} + r_1 \dot{A}_d + r_2 \dot{I}_{snd} + r_3 \dot{I}_{sd} + r_4 \dot{Q} + r_5 J + r_6 \dot{L}_t \} \\
 & + \left(1 - \frac{D^*}{D}\right) \{ d_1 \dot{A}_d + d_2 \dot{I}_{snd} + d_3 \dot{I}_{sd} + d_4 \dot{Q} + d_5 J + d_6 \dot{L}_t \} + \left(1 - \frac{P^*}{P}\right) \alpha \dot{S}
 \end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
 \frac{d^2 L_f}{dt^2} = & \dot{\Pi}(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P) \\
 & + \left(1 - \frac{S^*}{S}\right) \{ -p_d \beta \dot{S} (\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J) \\
 & - p_d \beta S (\omega \dot{I} + \varepsilon \dot{A}_d + \tau \dot{I}_{snd} + \dot{I}_{sd} + \phi \dot{Q} + \sigma J) - l_{sr} \dot{S} + l_d \dot{L} - t_r \dot{S} T \\
 & - t_r S \dot{T} + h \dot{T} + \eta \dot{Q} - \alpha \dot{S} \} + \left(1 - \frac{L^*}{L}\right) (l_{sr} \dot{S} - l_d \dot{L}) \\
 & + \left(1 - \frac{T^*}{T}\right) (t_r S \dot{T} + t_r \dot{S} T - (h + \tau_1 + \tau_2) \dot{T}) \\
 & + \left(1 - \frac{I^*}{I}\right) \{ p_d \beta \dot{S} (\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J) \\
 & + p_d \beta S (\omega \dot{I} + \varepsilon \dot{A}_d + \tau \dot{I}_{snd} + \dot{I}_{sd} + \phi \dot{Q} + \sigma J) - (\gamma + q + r_0) \dot{I} \} \\
 & + \left(1 - \frac{A_d^*}{A_d}\right) \{ \tau_1 \dot{T} - (j_1 + \lambda_1 + r_1 + d_1) \dot{A}_d \} \\
 & + \left(1 - \frac{I_{snd}^*}{I_{snd}}\right) \{ \gamma \dot{I} - (j_2 + \lambda_2 + r_2 + d_2) \dot{I}_{snd} \} \\
 & + \left(1 - \frac{I_{sd}^*}{I_{sd}}\right) \{ \tau_2 \dot{T} - (j_3 + \lambda_3 + r_3 + d_3) \dot{I}_{sd} \} \\
 & + \left(1 - \frac{Q^*}{Q}\right) \{ q \dot{I} - (\eta + j_4 + \lambda_4 + r_4 + d_4) \dot{Q} \} \\
 & + \left(1 - \frac{J^*}{J}\right) \{ j_1 \dot{A}_d + j_2 \dot{I}_{snd} + j_3 \dot{I}_{sd} + j_4 \dot{Q} - (\lambda_5 + r_5 + d_5) J \} \\
 & + \left(1 - \frac{L_t^*}{L_t}\right) \{ \lambda_1 \dot{A}_d + \lambda_2 \dot{I}_{snd} + \lambda_3 \dot{I}_{sd} + \lambda_4 \dot{Q} + \lambda_5 J - (r_6 + d_6) \dot{L}_t \} \\
 & + \left(1 - \frac{R^*}{R}\right) \{ r_0 \dot{I} + r_1 \dot{A}_d + r_2 \dot{I}_{snd} + r_3 \dot{I}_{sd} + r_4 \dot{Q} + r_5 J + r_6 \dot{L}_t \} \\
 & + \left(1 - \frac{D^*}{D}\right) \{ d_1 \dot{A}_d + d_2 \dot{I}_{snd} + d_3 \dot{I}_{sd} + d_4 \dot{Q} + d_5 J + d_6 \dot{L}_t \} + \left(1 - \frac{P^*}{P}\right) \alpha \dot{S}
 \end{aligned} \tag{2.25}$$

Now substitute $\dot{S}, \dot{L}, \dot{T}, \dot{I}, \dot{A}_d, \dot{I}_{snd}, \dot{I}_{sd}, \dot{Q}, \dot{J}, \dot{L}_t, \dot{R}, \dot{D}, \dot{P}$ with their respective formulas and combine positive and negative components, we have

$$\frac{d^2 L_f}{dt^2} = \Sigma_1 - \Sigma_2 \tag{2.26}$$

As a result, it is clear that, if $\Sigma_1 > \Sigma_2$ then $\frac{d^2 L_f}{dt^2} > 0$

$$\text{if } \sum_1 < \sum_2 \text{ then } \frac{d^2 L_f}{dt^2} < 0 \tag{2.27}$$

if $\Sigma_1 = \Sigma_2$ then $\frac{d^2 L_f}{dt^2} = 0$.

After that, the interpretation associated with the second-order sign is as follows.

2.5. Existence and uniqueness

This section presents a detailed analysis of the existence and uniqueness of the system of equations that describes classical calculus's survival. The following theorem must be proved to do this.

Theorem 2. Assume that there exists a positive constant $\vartheta_i, \bar{\vartheta}_i$ such that

(i) $\forall i \in \{1, 2, 3, \dots, 13\}$

$$|f_i(x_i, t) - f_i(x'_i, t)|^2 \leq \vartheta_i |x_i - x'_i|^2. \tag{2.28}$$

(ii) $\forall (x, t) \in \mathbb{R}^{13} \times (0, T)$

$$|f_i(x_i, t)|^2 \leq \bar{\vartheta}_i (1 + |x_i|^2). \tag{2.29}$$

The proposed model can be written as

$$\frac{dS(t)}{dt} = -p_d \beta S(t) \{ \omega I(t) + \varepsilon A_d(t) + \tau I_{snd}(t) + I_{sd}(t) + \phi Q(t) + \sigma J(t) \} - l_{sr} S(t) + l_d L(t)$$

$$-t_r S(t) T(t) + h T(t) + \eta Q(t) - \alpha S(t) = f_1(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$$\frac{dL(t)}{dt} = l_{sr} S(t) - l_d L(t) = f_2(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$$\frac{dT(t)}{dt} = t_r S(t) T(t) - (h + \tau_1 + \tau_2) T(t) = f_3(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$$\frac{dI(t)}{dt} = p_d \beta S(t) \{ \omega I(t) + \varepsilon A_d(t) + \tau I_{snd}(t) + I_{sd}(t) + \phi Q(t) + \sigma J(t) \} - (\gamma + q + r_0) I(t) = f_4(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$$\frac{dA_d(t)}{dt} = \tau_1 T(t) - (j_1 + \lambda_1 + r_1 + d_1) A_d(t) = f_5(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$$\frac{dI_{snd}(t)}{dt} = \gamma I(t) - (j_2 + \lambda_2 + r_2 + d_2) I_{snd}(t) = f_6(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$$\frac{dI_{sd}(t)}{dt} = \tau_2 T(t) - (j_3 + \lambda_3 + r_3 + d_3) I_{sd}(t) = f_7(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$$\frac{dQ(t)}{dt} = q I(t) - (\eta + j_4 + \lambda_4 + r_4 + d_4) Q(t) = f_8(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$$\frac{dJ(t)}{dt} = j_1 A_d(t) + j_2 I_{snd}(t) + j_3 I_{sd}(t) + j_4 Q(t) - (\lambda_5 + r_5 + d_5) J(t) = f_9(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$$\frac{dL_t(t)}{dt} = \lambda_1 A_d(t) + \lambda_2 I_{snd}(t) + \lambda_3 I_{sd}(t) + \lambda_4 Q(t) + \lambda_5 J(t) - (r_6 + d_6) L_t(t) = f_{10}(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$$\frac{dR(t)}{dt} = r_0 I(t) + r_1 A_d(t) + r_2 I_{snd}(t) + r_3 I_{sd}(t) + r_4 Q(t) + r_5 J(t) + r_6 L_t(t) = f_{11}(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$$\frac{dD(t)}{dt} = d_1 A_d(t) + d_2 I_{snd}(t) + d_3 I_{sd}(t) + d_4 Q(t) + d_5 J(t) + d_6 L_t(t) = f_{12}(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$$\frac{dP(t)}{dt} = \alpha S(t) = f_{13}(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

Firstly, for the function $f_1(t, S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$ we will prove that

$$|f_1(S_1, t) - f_1(S_2, t)|^2 \leq \vartheta_1 |S_1 - S_2|^2. \tag{2.30}$$

Now

$$|f_1(S_1, t) - f_1(S_2, t)|^2 = | -p_d \beta (\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J)(S_1 - S_2) - l_{sr}(S_1 - S_2) - t_r T(S_1 - S_2) - \alpha(S_1 - S_2) |^2$$

$$= | \{ -p_d \beta (\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J) - l_{sr} - t_r T - \alpha \} (S_1 - S_2) |^2$$

$$\leq \{ 2p_d^2 \beta^2 (\omega^2 |I|^2 + \varepsilon^2 |A_d|^2 + \tau^2 |I_{snd}|^2 + |I_{sd}|^2 + \phi^2 |Q|^2 + \sigma^2 |J|^2) + 2l_{sr}^2 + 2t_r^2 |T|^2 + 2\alpha^2 \} |S_1 - S_2|^2$$

$$\begin{aligned} &\leq \left\{ 2p_d^2\beta^2 \left(\omega^2 \sup_{0 \leq t \leq T} |I|^2 + \varepsilon^2 \sup_{0 \leq t \leq T} |A_d|^2 + \tau^2 \sup_{0 \leq t \leq T} |I_{snd}|^2 + \sup_{0 \leq t \leq T} |I_{sd}|^2 + \phi^2 \sup_{0 \leq t \leq T} |Q|^2 + \sigma^2 \sup_{0 \leq t \leq T} |J|^2 \right) \right. \\ &\quad \left. + 2l_{sr}^2 + 2t_r^2 \sup_{0 \leq t \leq T} |T(t)|^2 + 2\alpha^2 \right\} |S_1 - S_2|^2 \\ &\leq \left\{ 2p_d^2\beta^2 (\omega^2 \|I(t)\|_\infty^2 + \varepsilon^2 \|A_d(t)\|_\infty^2 + \tau^2 \|I_{snd}(t)\|_\infty^2 + \|I_{sd}(t)\|_\infty^2 + \phi^2 \|Q(t)\|_\infty^2 + \sigma^2 \|J(t)\|_\infty^2) + 2l_{sr}^2 + 2t_r^2 \|T(t)\|_\infty^2 + 2\alpha^2 \right\} |S_1 - S_2|^2 \\ &\leq \vartheta_1 |S_1 - S_2|^2. \end{aligned}$$

where, $\vartheta_1 = 2p_d^2\beta^2 (\omega^2 \|I(t)\|_\infty^2 + \varepsilon^2 \|A_d(t)\|_\infty^2 + \tau^2 \|I_{snd}(t)\|_\infty^2 + \|I_{sd}(t)\|_\infty^2 + \phi^2 \|Q(t)\|_\infty^2 + \sigma^2 \|J(t)\|_\infty^2) + 2l_{sr}^2 + 2t_r^2 \|T(t)\|_\infty^2 + 2\alpha^2$.

Analogously we can prove that the remaining compartments hold the above inequality.

Secondly, we will prove that

$$|f_1(S, t)|^2 \leq \bar{\vartheta}_1 (1 + |S|^2). \tag{2.31}$$

Then

$$\begin{aligned} |f_1(S, t)|^2 &= | -p_d\beta(\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J)S - l_{sr}S - t_r TS - \alpha S + l_d L + hT + \eta Q|^2 \\ &= | \{-p_d\beta(\omega I + \varepsilon A_d + \tau I_{snd} + I_{sd} + \phi Q + \sigma J) - l_{sr} - t_r T - \alpha\}S + l_d L + hT + \eta Q|^2 \\ &\leq \left\{ 2p_d^2\beta^2 (\omega^2 |I|^2 + \varepsilon^2 |A_d|^2 + \tau^2 |I_{snd}|^2 + |I_{sd}|^2 + \phi^2 |Q|^2 + \sigma^2 |J|^2) + 2l_{sr}^2 + 2t_r^2 |T|^2 + 2\alpha^2 \right\} |S|^2 + 2l_d^2 |L|^2 + 2h^2 |T|^2 + 2\eta^2 |Q|^2 \\ &\leq \left\{ 2p_d^2\beta^2 \left(\omega^2 \sup_{0 \leq t \leq T} |I|^2 + \varepsilon^2 \sup_{0 \leq t \leq T} |A_d|^2 + \tau^2 \sup_{0 \leq t \leq T} |I_{snd}|^2 + \sup_{0 \leq t \leq T} |I_{sd}|^2 + \phi^2 \sup_{0 \leq t \leq T} |Q|^2 + \sigma^2 \sup_{0 \leq t \leq T} |J|^2 \right) + 2l_{sr}^2 + 2t_r^2 \sup_{0 \leq t \leq T} |T(t)|^2 + 2\alpha^2 \right\} |S|^2 \\ &\quad + 2l_d^2 \sup_{0 \leq t \leq T} |L|^2 + 2h^2 \sup_{0 \leq t \leq T} |T|^2 + 2\eta^2 \sup_{0 \leq t \leq T} |Q|^2 \\ &\leq \left\{ 2p_d^2\beta^2 (\omega^2 \|I(t)\|_\infty^2 + \varepsilon^2 \|A_d(t)\|_\infty^2 + \tau^2 \|I_{snd}(t)\|_\infty^2 + \|I_{sd}(t)\|_\infty^2 + \phi^2 \|Q(t)\|_\infty^2 + \sigma^2 \|J(t)\|_\infty^2) + 2l_{sr}^2 + 2t_r^2 \|T(t)\|_\infty^2 + 2\alpha^2 \right\} |S|^2 \\ &\quad + 2l_d^2 \|L(t)\|_\infty^2 + 2h^2 \|T(t)\|_\infty^2 + 2\eta^2 \|Q(t)\|_\infty^2 \\ &\leq \bar{\vartheta}_1 (1 + |S|^2). \end{aligned}$$

such that

$$\begin{aligned} &\left\{ 2p_d^2\beta^2 (\omega^2 \|I(t)\|_\infty^2 + \varepsilon^2 \|A_d(t)\|_\infty^2 + \tau^2 \|I_{snd}(t)\|_\infty^2 + \|I_{sd}(t)\|_\infty^2 + \phi^2 \|Q(t)\|_\infty^2 + \sigma^2 \|J(t)\|_\infty^2) + 2l_{sr}^2 + 2t_r^2 \|T(t)\|_\infty^2 + 2\alpha^2 \right\} / \\ &\left\{ l_d^2 \|L(t)\|_\infty^2 + h^2 \|T(t)\|_\infty^2 + \eta^2 \|Q(t)\|_\infty^2 \right\} < 1. \end{aligned}$$

where, $\bar{\vartheta}_1 = l_d^2 \|L(t)\|_\infty^2 + h^2 \|T(t)\|_\infty^2 + \eta^2 \|Q(t)\|_\infty^2$.

Similarly, we can prove that the remaining compartments hold the above inequality.

In conclusion, the solution of our system exists and is unique under the maximality condition, detailed in [48].

3. Epidemic model based on Caputo fractional derivative

By implementing the well-known Caputo fractional-order derivative [45], we intend to modify our proposed epidemic dynamics as follows,

$${}^C_0 D_t^p f(t) = \frac{1}{\Gamma(n-p)} \int_0^t \frac{f^{(n)}(\eta)}{(t-\eta)^{p-n-1}} d\eta, t > 0, p > 0, n-1 < p \leq n, n \in \mathbb{N} \tag{3}$$

where, Γ is the well-known Gamma function, p is the order of the Caputo fractional derivative operator ${}^C D^p$.

Although integer derivative-based mathematical models have been implemented in the modern decades with tremendous progress, sometimes such models cannot perfectly replicate the real-world phenomenon due to the scarcity of information or exactness in transforming reality into a mathematical formula. Therefore, their use is essential to humanity for prediction, which helps humans understand what could happen soon, such that to avoid worst-case situations, they can take some control measures. Thus, in the current section, a Caputo fractional derivative-based mathematical model is devolved, predicting the outbreak of covid-19 for the Italian populations. In this regard, Caputo fractional derivative [50–55] has been applied in the conventional proposed mathematical model (Eqs. (1.1)–(1.14)). Then the system of the nonlinear fractional-order differential equation is as follows:

$$\begin{aligned} {}^C_0 D_t^p S(t) &= -p_d\beta S(t) \{ \omega I(t) + \varepsilon A_d(t) + \tau I_{snd}(t) + I_{sd}(t) + \phi Q(t) + \sigma J(t) \} \\ &\quad - l_{sr}S(t) + l_d L(t) - t_r S(t)T(t) + hT(t) + \eta Q(t) - \alpha S(t) \end{aligned} \tag{4.1}$$

$${}^C_0 D_t^p L(t) = l_{sr}S(t) - l_d L(t) \tag{4.2}$$

$${}^C_0 D_t^p T(t) = t_r S(t)T(t) - (h + \tau_1 + \tau_2)T(t) \tag{4.3}$$

$${}^C_0 D_t^p I(t) = p_d\beta S(t) \{ \omega I(t) + \varepsilon A_d(t) + \tau I_{snd}(t) + I_{sd}(t) + \phi Q(t) + \sigma J(t) \} - (\gamma + q + r_0)I(t)$$

$${}^C_0D_t^p A_d(t) = \tau_1 T(t) - (j_1 + \lambda_1 + r_1 + d_1)A_d(t) \tag{4.5}$$

$${}^C_0D_t^p I_{snd}(t) = \gamma I(t) - (j_2 + \lambda_2 + r_2 + d_2)I_{snd}(t) \tag{4.6}$$

$${}^C_0D_t^p I_{sd}(t) = \tau_2 T(t) - (j_3 + \lambda_3 + r_3 + d_3)I_{sd}(t) \tag{4.7}$$

$${}^C_0D_t^p Q(t) = qI(t) - (\eta + j_4 + \lambda_4 + r_4 + d_4)Q(t) \tag{4.8}$$

$${}^C_0D_t^p J(t) = j_1 A_d(t) + j_2 I_{snd}(t) + j_3 I_{sd}(t) + j_4 Q(t) - (\lambda_5 + r_5 + d_5)J(t) \tag{4.9}$$

$${}^C_0D_t^p L_t(t) = \lambda_1 A_d(t) + \lambda_2 I_{snd}(t) + \lambda_3 I_{sd}(t) + \lambda_4 Q(t) + \lambda_5 J(t) - (r_6 + d_6)L_t(t) \tag{4.10}$$

$${}^C_0D_t^p R(t) = r_0 I(t) + r_1 A_d(t) + r_2 I_{snd}(t) + r_3 I_{sd}(t) + r_4 Q(t) + r_5 J(t) + r_6 L_t(t) \tag{4.11}$$

$${}^C_0D_t^p D(t) = d_1 A_d(t) + d_2 I_{snd}(t) + d_3 I_{sd}(t) + d_4 Q(t) + d_5 J(t) + d_6 L_t(t) \tag{4.12}$$

$${}^C_0D_t^p P(t) = \alpha S(t) \tag{4.13}$$

As the above mathematical model (Eqs. (4.1)–(4.13)) of covid-19 outbreak predicts a real-world problem’s characteristic, [38] helps analyze the model’s positivity. Then

$$R_+^{13} = \{ \zeta \in R^{13} : \zeta \geq 0 \},$$

$$\zeta(t) = [S(t), L(t), T(t), I(t), A_d(t), I_{snd}(t), I_{sd}(t), Q(t), J(t), L_t(t), R(t), D(t), P(t)]^T.$$

Since summation of all Eq. of the system (4.1–4.13) gives zero, the system is classified and exhibits the preservation characteristic of mass. Directly,

$${}^C_0D_t^p \{S(t) + L(t) + T(t) + I(t) + A_d(t) + I_{snd}(t) + I_{sd}(t) + Q(t) + J(t) + L_t(t) + R(t) + D(t) + P(t)\} = 0$$

which signifies that the total population is constant.

As the all-state variables imply the whole population portions, we can suppose that $\sum_{\forall i} \zeta(i) = 1$, where 1 denotes the total population $N(t)$.

Lemma. The proposed model (Eqs. (4.1)–(4.13)) solution $\zeta(t)$ is positive, unique, and lies in R_+^{13} .

Proof. Since the proposed model deals with the population model, all the ingredients are confined in the positive quadrant to analyze its positivity. Therefore, the vector field tends to R_+^{13} , then

$${}^C_0D_t^p S(t) = l_d L(t) + hT(t) + \eta Q(t) \geq 0 \tag{5.1}$$

$${}^C_0D_t^p L(t) = l_{sr} S(t) \geq 0 \tag{5.2}$$

$${}^C_0D_t^p T(t) = 0 \tag{5.3}$$

$${}^C_0D_t^p I(t) = p_d \beta S(t) \{ \varepsilon A_d(t) + \tau I_{snd}(t) + I_{sd}(t) + \phi Q(t) + \sigma J(t) \} \geq 0 \tag{5.4}$$

$${}^C_0D_t^p A_d(t) = \tau_1 T(t) \geq 0 \tag{5.5}$$

$${}^C_0D_t^p I_{snd}(t) = \gamma I(t) \geq 0 \tag{5.6}$$

$${}^C_0D_t^p I_{sd}(t) = \tau_2 T(t) \geq 0 \tag{5.7}$$

$${}^C_0D_t^p Q(t) = qI(t) \geq 0 \tag{5.8}$$

$${}^C_0D_t^p J(t) = j_1 A_d(t) + j_2 I_{snd}(t) + j_3 I_{sd}(t) + j_4 Q(t) \geq 0 \tag{5.9}$$

$${}_0^C D_t^p L_t(t) = \lambda_1 A_d(t) + \lambda_2 I_{snd}(t) + \lambda_3 I_{sd}(t) + \lambda_4 Q(t) + \lambda_5 J(t) \geq 0 \tag{5.10}$$

$${}_0^C D_t^p R(t) = r_0 I(t) + r_1 A_d(t) + r_2 I_{snd}(t) + r_3 I_{sd}(t) + r_4 Q(t) + r_5 J(t) + r_6 L_t(t) \geq 0 \tag{5.11}$$

$${}_0^C D_t^p D(t) = d_1 A_d(t) + d_2 I_{snd}(t) + d_3 I_{sd}(t) + d_4 Q(t) + d_5 J(t) + d_6 L_t(t) \geq 0 \tag{5.12}$$

$${}_0^C D_t^p P(t) = \alpha S(t) \geq 0. \tag{5.13}$$

3.1. Existence of uniformly stable solution

Assume that

$${}_0^C D_t^p S(t) = -p_d \beta S(t) \{ \omega I(t) + \varepsilon A_d(t) + \tau I_{snd}(t) + I_{sd}(t) + \phi Q(t) + \sigma J(t) \} - l_{sr} S(t) + l_d L(t)$$

$$-t_r S(t) T(t) + h T(t) + \eta Q(t) - \alpha S(t) = f_1(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$${}_0^C D_t^p L(t) = l_{sr} S(t) - l_d L(t) = f_2(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$${}_0^C D_t^p T(t) = t_r S(t) T(t) - (h + \tau_1 + \tau_2) T(t) = f_3(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$${}_0^C D_t^p I(t) = p_d \beta S(t) \{ \omega I(t) + \varepsilon A_d(t) + \tau I_{snd}(t) + I_{sd}(t) + \phi Q(t) + \sigma J(t) \} - (\gamma + q + r_0) I(t) = f_4(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$${}_0^C D_t^p A_d(t) = \tau_1 T(t) - (j_1 + \lambda_1 + r_1 + d_1) A_d(t) = f_5(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$${}_0^C D_t^p I_{snd}(t) = \gamma I(t) - (j_2 + \lambda_2 + r_2 + d_2) I_{snd}(t) = f_6(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$${}_0^C D_t^p I_{sd}(t) = \tau_2 T(t) - (j_3 + \lambda_3 + r_3 + d_3) I_{sd}(t) = f_7(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$${}_0^C D_t^p Q(t) = q I(t) - (\eta + j_4 + \lambda_4 + r_4 + d_4) Q(t) = f_8(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$${}_0^C D_t^p J(t) = j_1 A_d(t) + j_2 I_{snd}(t) + j_3 I_{sd}(t) + j_4 Q(t) - (\lambda_5 + r_5 + d_5) J(t) = f_9(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$${}_0^C D_t^p L_t(t) = \lambda_1 A_d(t) + \lambda_2 I_{snd}(t) + \lambda_3 I_{sd}(t) + \lambda_4 Q(t) + \lambda_5 J(t) - (r_6 + d_6) L_t(t) = f_{10}(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$${}_0^C D_t^p R(t) = r_0 I(t) + r_1 A_d(t) + r_2 I_{snd}(t) + r_3 I_{sd}(t) + r_4 Q(t) + r_5 J(t) + r_6 L_t(t) = f_{11}(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$${}_0^C D_t^p D(t) = d_1 A_d(t) + d_2 I_{snd}(t) + d_3 I_{sd}(t) + d_4 Q(t) + d_5 J(t) + d_6 L_t(t) = f_{12}(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

$${}_0^C D_t^p P(t) = \alpha S(t) = f_{13}(S, L, T, I, A_d, I_{snd}, I_{sd}, Q, J, L_t, R, D, P)$$

Assume for the total population $N(t)$ that

$$\Phi = \{ (S(t) + L(t) + T(t) + I(t) + A_d(t) + I_{snd}(t) + I_{sd}(t) + Q(t) + J(t) + L_t(t) + R(t) + D(t) + P(t)) \in R^{13} : |\zeta(i)| \leq N(t) \text{ and } t \in [0, T].$$

Then over Φ , we have

$$\frac{\partial f_1}{\partial S} = -p_d \beta \{ \omega I(t) + \varepsilon A_d(t) + \tau I_{snd}(t) + I_{sd}(t) + \phi Q(t) + \sigma J(t) \} - l_{sr} - t_r T(t) - \alpha \Rightarrow \left| \frac{\partial f_1}{\partial S} \right| \leq a_{11};$$

$$\frac{\partial f_1}{\partial L} = l_d \Rightarrow \left| \frac{\partial f_1}{\partial L} \right| \leq a_{12}; \quad \frac{\partial f_1}{\partial T} = h - t_r S(t) \Rightarrow \left| \frac{\partial f_1}{\partial T} \right| \leq a_{13}; \quad \frac{\partial f_1}{\partial I} = -p_d \beta \omega S(t) \Rightarrow \left| \frac{\partial f_1}{\partial I} \right| \leq a_{14};$$

$$\frac{\partial f_1}{\partial A_d} = -p_d \beta \varepsilon S(t) \Rightarrow \left| \frac{\partial f_1}{\partial A_d} \right| \leq a_{15}; \quad \frac{\partial f_1}{\partial I_{snd}} = -p_d \beta \tau S(t) \Rightarrow \left| \frac{\partial f_1}{\partial I_{snd}} \right| \leq a_{16};$$

$$\frac{\partial f_1}{\partial I_{sd}} = -p_d \beta S(t) \Rightarrow \left| \frac{\partial f_1}{\partial I_{sd}} \right| \leq a_{17}; \quad \frac{\partial f_1}{\partial Q} = \eta - p_d \beta \phi S(t) \Rightarrow \left| \frac{\partial f_1}{\partial Q} \right| \leq a_{18};$$

$$\frac{\partial f_1}{\partial J} = -p_d \beta \sigma S(t) \Rightarrow \left| \frac{\partial f_1}{\partial J} \right| \leq a_{19}; \frac{\partial f_1}{\partial L_t} = 0 \Rightarrow f_1(L_t) = a_{110}; \frac{\partial f_1}{\partial R} = 0 \Rightarrow f_1(R) = a_{111};$$

$$\frac{\partial f_1}{\partial D} = 0 \Rightarrow f_1(D) = a_{112}; \frac{\partial f_1}{\partial P} = 0 \Rightarrow f_1(P) = a_{113}.$$

$$\frac{\partial f_2}{\partial S} = I_{sr} \Rightarrow \left| \frac{\partial f_2}{\partial S} \right| \leq a_{21}; \frac{\partial f_2}{\partial L} = -I_d \Rightarrow \left| \frac{\partial f_2}{\partial L} \right| \leq a_{22}; \frac{\partial f_2}{\partial T} = 0 \Rightarrow f_2(T) = a_{23};$$

$$\frac{\partial f_2}{\partial I} = 0 \Rightarrow f_2(I) = a_{24}; \frac{\partial f_2}{\partial A_d} = 0 \Rightarrow f_2(A_d) = a_{25}; \frac{\partial f_2}{\partial I_{snd}} = 0 \Rightarrow f_2(I_{snd}) = a_{26}; \frac{\partial f_2}{\partial I_{sd}} = 0 \Rightarrow f_2(I_{sd}) = a_{27};$$

$$\frac{\partial f_2}{\partial Q} = 0 \Rightarrow f_2(Q) = a_{28}; \frac{\partial f_2}{\partial J} = 0 \Rightarrow f_2(J) = a_{29}; \frac{\partial f_2}{\partial L_t} = 0 \Rightarrow f_2(L_t) = a_{210};$$

$$\frac{\partial f_2}{\partial R} = 0 \Rightarrow f_2(R) = a_{211}; \frac{\partial f_2}{\partial D} = 0 \Rightarrow f_2(D) = a_{212}; \frac{\partial f_2}{\partial P} = 0 \Rightarrow f_2(P) = a_{213}.$$

$$\frac{\partial f_3}{\partial S} = t_r T(t) \Rightarrow \left| \frac{\partial f_3}{\partial S} \right| \leq a_{31}; \frac{\partial f_3}{\partial L} = 0 \Rightarrow f_3(L) = a_{32}; \frac{\partial f_3}{\partial T} = -h - \tau_1 - \tau_2 \Rightarrow \left| \frac{\partial f_3}{\partial T} \right| \leq a_{33};$$

$$\frac{\partial f_3}{\partial I} = 0 \Rightarrow f_3(I) = a_{34}; \frac{\partial f_3}{\partial A_d} = 0 \Rightarrow f_3(A_d) = a_{35}; \frac{\partial f_3}{\partial I_{snd}} = 0 \Rightarrow f_3(I_{snd}) = a_{36}; \frac{\partial f_3}{\partial I_{sd}} = 0 \Rightarrow f_3(I_{sd}) = a_{37};$$

$$\frac{\partial f_3}{\partial Q} = 0 \Rightarrow f_3(Q) = a_{38}; \frac{\partial f_3}{\partial J} = 0 \Rightarrow f_3(J) = a_{39}; \frac{\partial f_3}{\partial L_t} = 0 \Rightarrow f_3(L_t) = a_{310};$$

$$\frac{\partial f_3}{\partial R} = 0 \Rightarrow f_3(R) = a_{311}; \frac{\partial f_3}{\partial D} = 0 \Rightarrow f_3(D) = a_{312}; \frac{\partial f_3}{\partial P} = 0 \Rightarrow f_3(P) = a_{313}.$$

$$\frac{\partial f_4}{\partial S} = p_d \beta \{\omega I(t) + \varepsilon A_d(t) + \tau I_{snd}(t) + I_{sd}(t) + \phi Q(t) + \sigma J(t)\} \Rightarrow \left| \frac{\partial f_4}{\partial S} \right| \leq a_{41}; \frac{\partial f_4}{\partial L} = 0 \Rightarrow f_4(L) = a_{42};$$

$$\frac{\partial f_4}{\partial T} = 0 \Rightarrow f_4(T) = a_{43}; \frac{\partial f_4}{\partial I} = p_d \beta \omega S(t) - \gamma - q - r_0 \Rightarrow \left| \frac{\partial f_4}{\partial I} \right| \leq a_{44}; \frac{\partial f_4}{\partial A_d} = p_d \beta \varepsilon S(t) \Rightarrow \left| \frac{\partial f_4}{\partial A_d} \right| \leq a_{45};$$

$$\frac{\partial f_4}{\partial Q} = p_d \beta \phi S(t) \Rightarrow \left| \frac{\partial f_4}{\partial Q} \right| \leq a_{48}; \frac{\partial f_4}{\partial J} = p_d \beta \sigma S(t) \Rightarrow \left| \frac{\partial f_4}{\partial J} \right| \leq a_{49}; \frac{\partial f_4}{\partial L_t} = 0 \Rightarrow f_4(L_t) = a_{410};$$

$$\frac{\partial f_4}{\partial R} = 0 \Rightarrow f_4(R) = a_{411}; \frac{\partial f_4}{\partial D} = 0 \Rightarrow f_4(D) = a_{412}; \frac{\partial f_4}{\partial P} = 0 \Rightarrow f_4(P) = a_{413}.$$

$$\frac{\partial f_5}{\partial S} = 0 \Rightarrow f_5(S) = a_{51}; \frac{\partial f_5}{\partial L} = 0 \Rightarrow f_5(L) = a_{52}; \frac{\partial f_5}{\partial T} = \inf A_d \Rightarrow \left| \frac{\partial f_5}{\partial T} \right| \leq a_{53}; \frac{\partial f_5}{\partial I} = \gamma \Rightarrow \left| \frac{\partial f_5}{\partial I} \right| \leq a_{54};$$

$$\frac{\partial f_5}{\partial A_d} = -j_1 - \lambda_1 - r_1 - d_1 \Rightarrow \left| \frac{\partial f_5}{\partial A_d} \right| \leq a_{55}; \frac{\partial f_5}{\partial I_{snd}} = 0 \Rightarrow f_5(I_{snd}) = a_{56}; \frac{\partial f_5}{\partial I_{sd}} = 0 \Rightarrow f_5(I_{sd}) = a_{57};$$

$$\frac{\partial f_5}{\partial Q} = 0 \Rightarrow f_5(Q) = a_{58}; \frac{\partial f_5}{\partial J} = 0 \Rightarrow f_5(J) = a_{59}; \frac{\partial f_5}{\partial L_t} = 0 \Rightarrow f_5(L_t) = a_{510};$$

$$\frac{\partial f_5}{\partial R} = 0 \Rightarrow f_5(R) = a_{511}; \frac{\partial f_5}{\partial D} = 0 \Rightarrow f_5(D) = a_{512}; \frac{\partial f_5}{\partial P} = 0 \Rightarrow f_5(P) = a_{513}.$$

$$\frac{\partial f_6}{\partial S} = 0 \Rightarrow f_6(S) = a_{61}; \frac{\partial f_6}{\partial L} = 0 \Rightarrow f_6(L) = a_{62}; \frac{\partial f_6}{\partial T} = 0 \Rightarrow f_6(T) = a_{63}; \frac{\partial f_6}{\partial I} = \gamma \Rightarrow \left| \frac{\partial f_6}{\partial I} \right| \leq a_{64};$$

$$\frac{\partial f_6}{\partial A_d} = 0 \Rightarrow f_6(A_d) = a_{65}; \frac{\partial f_6}{\partial I_{snd}} = -j_2 - \lambda_2 - r_2 - d_2 \Rightarrow \left| \frac{\partial f_6}{\partial I_{snd}} \right| \leq a_{66}; \frac{\partial f_6}{\partial I_{sd}} = 0 \Rightarrow f_6(I_{sd}) = a_{67};$$

$$\frac{\partial f_6}{\partial Q} = 0 \Rightarrow f_6(Q) = a_{68}; \frac{\partial f_6}{\partial J} = 0 \Rightarrow f_6(J) = a_{69}; \frac{\partial f_6}{\partial L_t} = 0 \Rightarrow f_6(L_t) = a_{610};$$

$$\frac{\partial f_6}{\partial R} = 0 \Rightarrow f_6(R) = a_{611}; \frac{\partial f_6}{\partial D} = 0 \Rightarrow f_6(D) = a_{612}; \frac{\partial f_6}{\partial P} = 0 \Rightarrow f_6(P) = a_{613}.$$

$$\frac{\partial f_7}{\partial S} = 0 \Rightarrow f_7(S) = a_{71}; \frac{\partial f_7}{\partial L} = 0 \Rightarrow f_7(L) = a_{72}; \frac{\partial f_7}{\partial T} = \inf I_{sd} \Rightarrow \left| \frac{\partial f_7}{\partial T} \right| \leq a_{73}; \frac{\partial f_7}{\partial I} = 0 \Rightarrow f_7(I) = a_{74};$$

$$\frac{\partial f_7}{\partial A_d} = 0 \Rightarrow f_7(A_d) = a_{75}; \frac{\partial f_7}{\partial I_{snd}} = 0 \Rightarrow f_7(I_{snd}) = a_{76}; \frac{\partial f_7}{\partial I_{sd}} = -j_3 - \lambda_3 - r_3 - d_3 \Rightarrow \left| \frac{\partial f_7}{\partial I_{sd}} \right| \leq a_{77};$$

$$\frac{\partial f_7}{\partial Q} = 0 \Rightarrow f_7(Q) = a_{78}; \frac{\partial f_7}{\partial J} = 0 \Rightarrow f_7(J) = a_{79}; \frac{\partial f_7}{\partial L_t} = 0 \Rightarrow f_7(L_t) = a_{710};$$

$$\frac{\partial f_7}{\partial R} = 0 \Rightarrow f_7(R) = a_{711}; \frac{\partial f_7}{\partial D} = 0 \Rightarrow f_7(D) = a_{712}; \frac{\partial f_7}{\partial P} = 0 \Rightarrow f_7(P) = a_{713}.$$

$$\frac{\partial f_8}{\partial S} = 0 \Rightarrow f_8(S) = a_{81}; \frac{\partial f_8}{\partial L} = 0 \Rightarrow f_8(L) = a_{82}; \frac{\partial f_8}{\partial T} = 0 \Rightarrow f_8(T) = a_{83}; \frac{\partial f_8}{\partial I} = q \Rightarrow \left| \frac{\partial f_8}{\partial I} \right| \leq a_{84};$$

$$\frac{\partial f_8}{\partial A_d} = 0 \Rightarrow f_8(A_d) = a_{85}; \frac{\partial f_8}{\partial I_{snd}} = 0 \Rightarrow f_8(I_{snd}) = a_{86}; \frac{\partial f_8}{\partial I_{sd}} = 0 \Rightarrow f_8(I_{sd}) = a_{87};$$

$$\frac{\partial f_8}{\partial Q} = -\eta - j_4 - \lambda_4 - r_4 - d_4 \Rightarrow \left| \frac{\partial f_8}{\partial Q} \right| \leq a_{88}; \frac{\partial f_8}{\partial J} = 0 \Rightarrow f_8(J) = a_{89}; \frac{\partial f_8}{\partial L_t} = 0 \Rightarrow f_8(L_t) = a_{810};$$

$$\frac{\partial f_8}{\partial R} = 0 \Rightarrow f_8(R) = a_{811}; \frac{\partial f_8}{\partial D} = 0 \Rightarrow f_8(D) = a_{812}; \frac{\partial f_8}{\partial P} = 0 \Rightarrow f_8(P) = a_{813}.$$

$$\frac{\partial f_9}{\partial S} = 0 \Rightarrow f_9(S) = a_{91}; \frac{\partial f_9}{\partial L} = 0 \Rightarrow f_9(L) = a_{92}; \frac{\partial f_9}{\partial T} = 0 \Rightarrow f_9(T) = a_{93}; \frac{\partial f_9}{\partial I} = 0 \Rightarrow f_9(I) = a_{94};$$

$$\frac{\partial f_9}{\partial A_d} = j_1 \Rightarrow \left| \frac{\partial f_9}{\partial A_d} \right| \leq a_{95}; \frac{\partial f_9}{\partial I_{snd}} = j_2 \Rightarrow \left| \frac{\partial f_9}{\partial I_{snd}} \right| \leq a_{96}; \frac{\partial f_9}{\partial I_{sd}} = j_3 \Rightarrow \left| \frac{\partial f_9}{\partial I_{sd}} \right| \leq a_{97};$$

$$\frac{\partial f_9}{\partial Q} = j_4 \Rightarrow \left| \frac{\partial f_9}{\partial Q} \right| \leq a_{98}; \frac{\partial f_9}{\partial J} = -\lambda_5 - r_5 - d_5 \Rightarrow \left| \frac{\partial f_9}{\partial J} \right| \leq a_{99}; \frac{\partial f_9}{\partial L_t} = 0 \Rightarrow f_9(L_t) = a_{910};$$

$$\frac{\partial f_9}{\partial R} = 0 \Rightarrow f_9(R) = a_{911}; \frac{\partial f_9}{\partial D} = 0 \Rightarrow f_9(D) = a_{912}; \frac{\partial f_9}{\partial P} = 0 \Rightarrow f_9(P) = a_{913}.$$

$$\frac{\partial f_{10}}{\partial S} = 0 \Rightarrow f_{10}(S) = a_{101}; \frac{\partial f_{10}}{\partial L} = 0 \Rightarrow f_{10}(L) = a_{102}; \frac{\partial f_{10}}{\partial T} = 0 \Rightarrow f_{10}(T) = a_{103}; \frac{\partial f_{10}}{\partial I} = r_0 \Rightarrow \left| \frac{\partial f_{10}}{\partial I} \right| \leq a_{104};$$

$$\frac{\partial f_{10}}{\partial A_d} = \lambda_1 \Rightarrow \left| \frac{\partial f_{10}}{\partial A_d} \right| \leq a_{105}; \frac{\partial f_{10}}{\partial I_{snd}} = \lambda_2 \Rightarrow \left| \frac{\partial f_{10}}{\partial I_{snd}} \right| \leq a_{106}; \frac{\partial f_{10}}{\partial I_{sd}} = \lambda_3 \Rightarrow \left| \frac{\partial f_{10}}{\partial I_{sd}} \right| \leq a_{107};$$

$$\frac{\partial f_{10}}{\partial Q} = \lambda_4 \Rightarrow \left| \frac{\partial f_{10}}{\partial Q} \right| \leq a_{108}; \frac{\partial f_{10}}{\partial J} = \lambda_5 \Rightarrow \left| \frac{\partial f_{10}}{\partial J} \right| \leq a_{109}; \frac{\partial f_{10}}{\partial L_t} = -r_6 - d_6 \Rightarrow \left| \frac{\partial f_{10}}{\partial L_t} \right| \leq a_{1010};$$

$$\frac{\partial f_{10}}{\partial R} = 0 \Rightarrow f_{10}(R) = a_{1011}; \frac{\partial f_{10}}{\partial D} = 0 \Rightarrow f_{10}(D) = a_{1012}; \frac{\partial f_{10}}{\partial P} = 0 \Rightarrow f_{10}(P) = a_{1013}.$$

$$\frac{\partial f_{11}}{\partial S} = 0 \Rightarrow f_{11}(S) = a_{111}; \frac{\partial f_{11}}{\partial L} = 0 \Rightarrow f_{11}(L) = a_{112}; \frac{\partial f_{11}}{\partial T} = 0 \Rightarrow f_{11}(T) = a_{113}; \frac{\partial f_{11}}{\partial I} = r_0 \Rightarrow f_{11}(I) = a_{114};$$

$$\frac{\partial f_{11}}{\partial A_d} = r_1 \Rightarrow \left| \frac{\partial f_{11}}{\partial A_d} \right| \leq a_{115}; \frac{\partial f_{11}}{\partial I_{snd}} = r_2 \Rightarrow \left| \frac{\partial f_{11}}{\partial I_{snd}} \right| \leq a_{116}; \frac{\partial f_{11}}{\partial I_{sd}} = r_3 \Rightarrow \left| \frac{\partial f_{11}}{\partial I_{sd}} \right| \leq a_{117};$$

$$\frac{\partial f_{11}}{\partial Q} = r_4 \Rightarrow \left| \frac{\partial f_{11}}{\partial Q} \right| \leq a_{118}; \frac{\partial f_{11}}{\partial J} = r_5 \Rightarrow \left| \frac{\partial f_{11}}{\partial J} \right| \leq a_{119}; \frac{\partial f_{11}}{\partial L_t} = r_6 \Rightarrow \left| \frac{\partial f_{11}}{\partial L_t} \right| \leq a_{1110};$$

$$\frac{\partial f_{11}}{\partial R} = 0 \Rightarrow f_{11}(R) = a_{1111}; \frac{\partial f_{11}}{\partial D} = 0 \Rightarrow f_{11}(D) = a_{1112}; \frac{\partial f_{11}}{\partial P} = 0 \Rightarrow f_{11}(P) = a_{1113}.$$

$$\frac{\partial f_{12}}{\partial S} = 0 \Rightarrow f_{12}(S) = a_{121}; \frac{\partial f_{12}}{\partial L} = 0 \Rightarrow f_{12}(L) = a_{122}; \frac{\partial f_{12}}{\partial T} = 0 \Rightarrow f_{12}(T) = a_{123}; \frac{\partial f_{12}}{\partial I} = 0 \Rightarrow f_{12}(I) = a_{124};$$

$$\frac{\partial f_{12}}{\partial A_d} = d_1 \Rightarrow \left| \frac{\partial f_{12}}{\partial A_d} \right| \leq a_{125}; \frac{\partial f_{12}}{\partial I_{snd}} = d_2 \Rightarrow \left| \frac{\partial f_{12}}{\partial I_{snd}} \right| \leq a_{126}; \frac{\partial f_{12}}{\partial I_{sd}} = d_3 \Rightarrow \left| \frac{\partial f_{12}}{\partial I_{sd}} \right| \leq a_{127};$$

$$\frac{\partial f_{12}}{\partial Q} = d_4 \Rightarrow \left| \frac{\partial f_{12}}{\partial Q} \right| \leq a_{138}; \frac{\partial f_{12}}{\partial J} = d_5 \Rightarrow \left| \frac{\partial f_{12}}{\partial J} \right| \leq a_{129}; \frac{\partial f_{12}}{\partial L_t} = d_6 \Rightarrow \left| \frac{\partial f_{12}}{\partial L_t} \right| \leq a_{1210};$$

$$\frac{\partial f_{12}}{\partial R} = 0 \Rightarrow f_{12}(R) = a_{1211}; \frac{\partial f_{12}}{\partial D} = 0 \Rightarrow f_{12}(D) = a_{1212}; \frac{\partial f_{12}}{\partial P} = 0 \Rightarrow f_{12}(P) = a_{1213}.$$

$$\frac{\partial f_{13}}{\partial S} = \alpha \Rightarrow \left| \frac{\partial f_{13}}{\partial S} \right| \leq a_{131}; \frac{\partial f_{13}}{\partial L} = 0 \Rightarrow f_{13}(L) = a_{132}; \frac{\partial f_{13}}{\partial T} = 0 \Rightarrow f_{13}(T) = a_{133}; \frac{\partial f_{13}}{\partial I} = 0 \Rightarrow f_{13}(I) = a_{134};$$

$$\frac{\partial f_{13}}{\partial A_d} = 0 \Rightarrow f_{13}(A_d) = a_{135}; \frac{\partial f_{13}}{\partial I_{snd}} = 0 \Rightarrow f_{13}(I_{snd}) = a_{136}; \frac{\partial f_{13}}{\partial I_{sd}} = 0 \Rightarrow f_{13}(I_{sd}) = a_{137};$$

$$\frac{\partial f_{13}}{\partial Q} = 0 \Rightarrow f_{13}(Q) = a_{138}; \frac{\partial f_{13}}{\partial J} = 0 \Rightarrow f_{13}(J) = a_{139}; \frac{\partial f_{13}}{\partial L_t} = 0 \Rightarrow f_{13}(L_t) = a_{1310};$$

$$\frac{\partial f_{13}}{\partial R} = 0 \Rightarrow f_{13}(R) = a_{1311}; \frac{\partial f_{13}}{\partial D} = 0 \Rightarrow f_{13}(D) = a_{1312}; \frac{\partial f_{13}}{\partial P} = 0 \Rightarrow f_{13}(P) = a_{1313}.$$

where $a_{ij}(i \geq 1$ and $j \leq 13)$ all are the positive constants. Then each of the thirteen functions f_1, f_2, \dots, f_{13} agreed with the Lipchitz condition [56,57]. Concerning the above thirteen arguments, it is clear that all functions are absolutely continuous.

4. Numerical simulation

In this section, the proposed model is generalized via applying the fractional Caputo derivative and numerically simulated based on parameter values presented in Table 1. Numerical simulations were carried out by the Adams-Bashforth-Moulton algorithm [58].

First, let us recall the primary method produced to solve initial value problems with Caputo derivatives (Eqs. (4.1)–(4.13)). The technique extends the familiar Adams-Bashforth-Moulton integrator that is well known for the numerical simulation of first-order differential equations [56]. This method relies on the feature that the initial value problem is equal to the Volterra integral equation. The fractional Adams-Bashforth-Moulton method is fully described by the following Equations (all other states can be found same as S). Let $[0, T]$ is the domain of the solution and, $n = 0, 1, 2, \dots, N, h = T/N, t_n = nh$:

$$S(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} S_0^{(k)} \frac{t_{n+1}^{k+1}}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} f_1(t_{n+1}, S^p(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^n A_{j,n+1} f_1(t_j, S(t_j)),$$

$$S^p(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} S_0^{(k)} \frac{t_{n+1}^{k+1}}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n B_{j,n+1} f_1(t_j, S(t_j)),$$

where

$$A_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & \text{if } 1 \leq j \leq n, \\ 1 & \text{if } j = n+1, \end{cases}$$

and

$$B_{j,n+1} = \frac{h^\alpha}{\alpha} ((n+1-j)^\alpha - (n-j)^\alpha).$$

For more details about the method, the reader can see [56].

The initial values [19] of the thirteen compartments are taken as follows:

$$L(0) = 0, T(0) = \frac{200}{N(0)}, I(0) = \frac{200}{N(0)}, A_{sd}(0) = \frac{20}{N(0)}, I_{snd}(0) = \frac{1}{N(0)}, I_{sd} = \frac{2}{N(0)},$$

$$Q(0) = 0, J(0) = 0, L_t(0) = 0, R(0) = 0, D(0) = 0, P(0) = 0$$

$$S(0) = N(0) - L(0) - T(0) - I(0) - A_{sd}(0) - I_{snd}(0) - I_{sd}(0) - Q(0) - J(0) - L_t(0) - R(0) - D(0) - P(0).$$

Table 1
Parameter values of the proposed epidemic dynamics and their references.

Parameter	Value/Range	Current Model	Refs.
$N(t)$	60,403,693	60,403,693	[19]
p_d	$(1/6 - 1/3)ft^{-1}$	1/3	[25]
β	$(0.2 - 1.5) \text{ days}^{-1}$	0.5	[59–61]
ω	0.57	0.57	[19]
ε	0.0114	0.0114	[19]
τ	0.456	0.456	[19]
ϕ	–	0.0114	Estimated
σ	–	0.0114	Estimated
η	$(1/14 - 1/3) \text{ days}^{-1}$	1/13	[[61],[62]]
l_{sr}	–	0.8	Estimated
l_d	16	16	[63]
t_r	0.0251	0.0251	[64]
h	0.9829	0.9829	[64]
τ_1	0.171	0.0171	[19]
τ_2	0.3705	0.03705	[19]
γ	0.264	0.264	[19]
q	–	0.5	Estimated
j_1	$(1/14 - 1/5) \text{ days}^{-1}$	0.1	[[65],[66]]
j_2	–	0.2	Estimated
j_3	0.2–0.9	0.3	[[65],[66]]
j_4	–	0.15	Estimated
λ_1	0.08–	0.08	Estimated
λ_2	0.0171	0.0171	[19]
λ_3	0.0274	0.0274	[19]
λ_4	–	0.05	Estimated
λ_5	–	0.08	Estimated
r_0	–	0.4	Estimated
r_1	0.0342	0.0342	[19]
r_2	0.0171	0.0171	[19]
r_3	0.02	0.02	[[61],[66],[67]]
r_4	0.0239	0.0239	[61]
r_5	$(1/30 - 1/3) \text{ days}^{-1}$	1/30	[[66],[67]]
r_6	0.0171	0.0171	[19]
d_1	–	0.05	Estimated
d_2	0.001–0.1	0.005	[65]
d_3	0.001–0.01	0.003	[[68],[69]]
d_4	0.001–0.01	0.001	[[68],[69]]
d_5	0.001–0.1	0.005	[65]
d_6	0.01	0.01	[19]
α	–	0.5	Estimated

5. Results and discussion

To study the sensitivity of fractional-order-based epidemic dynamics along time-elapsd around the steady-state situation called equilibrium, we present line graphs for $p \in \{0.9, 0.95, 1.0\}$, depicted in Fig. 2. It displays the fraction of susceptible, lockdown, quarantine, infected (asymptomatic, mild-symptomatic-infected, minor or moderate infection but not detected), symptomatic infected with not detected, symptomatic infected with detected, Tested, asymptomatic infected with detected, isolated or hospitalized, life-threatening condition, self-protected and recovered individuals from (i) to (xii), respectively. According to the simulated results, the fractional-order can significantly influence the changing pattern of different epidemic compartments. Thus, we can confer that the decreasing of the fractional-order p , lessened the portion of susceptible, quarantined, infected, and tested individuals, as expected.

Fig. 3 displays the effect of $S(t)$ and $L(t)$ to illustrate the susceptibility and lockdown state concerning the lockdown success rate more profoundly. It seems that the increase of lockdown success rate lessens the amount of suspected susceptible individuals. However, the rise in lockdown rate increased the fraction of individuals in lockdown compartments, reducing infection risk.

Next, we inspect the relation of test rate vs. time (i–iii) and lockdown success rate vs. time (iv–vi) for a fraction of $S(t)$, $T(t)$, and $I(t)$ individuals depicted in Fig. 4. As a general tendency, we can confirm that the fraction of infected and tested individuals

is lessened for increasing of both test rate and lockdown success rate.

6. Conclusion

Motivated by the current COVID-19 situation, we proposed the protecting measures-based epidemic models by incorporating the fractional-order approach to study the disease behavior. Model investigation and analysis are carried out by presuming the Caputo fractional-order derivative notion to generate the fractional-order mathematical epidemic model. Further, the numerical simulation of the suggested system is carried out by consuming the Adams-Bashforth-Moulton algorithm. It is observed that irrespective of introducing a vaccine policy, the combined effect of several self-protecting measures helps to reduce the disease risk.

Furthermore, because a first derivative analysis does not always offer an apparent indication of function change a priori, a second derivative analysis is necessary. The study of the second derivative discloses infection points, as well as local maximum and minimum values. These fundamental analyses may be employed in epidemiological modeling to understand dissemination patterns better. A new concept called Strength number was recently proposed. It is derived by taking the second derivative of the nonlinear section of a particular infectious disease model, then applying the next generation matrix approach to obtain the strength number. Such numbers, it was suggested, may aid in detecting waves or instability in

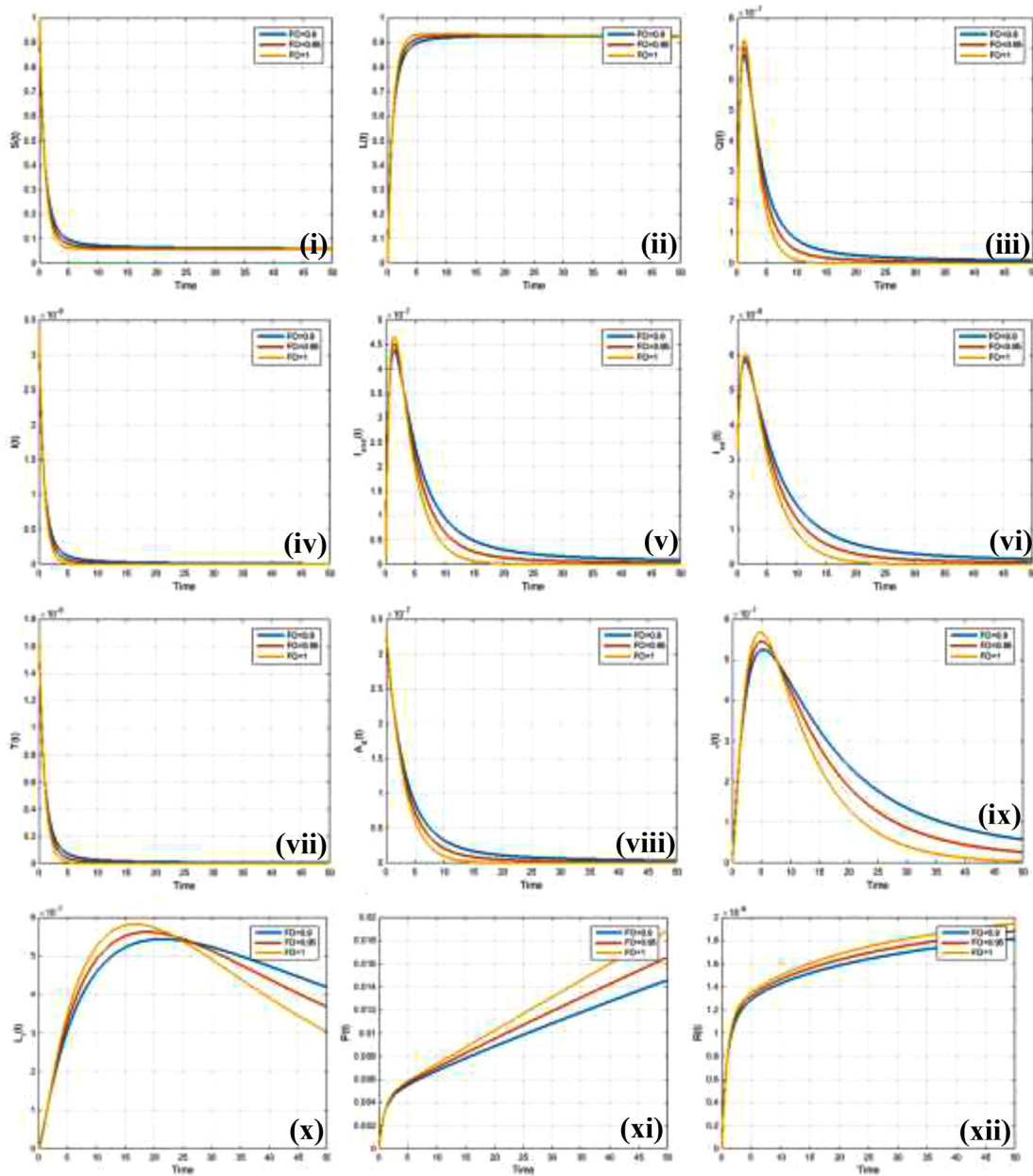


Fig. 2. Impact of different fractional order $p \in \{0.9, 0.95, 1.0\}$ on (i) susceptible $(S(t))$, (ii) lock-down $(L(t))$, (iii) quarantine, (iv) infected $I(t)$ (asymptomatic, mild-symptomatic-infected, minor or moderate infection but not detected), v) symptomatic infected with not detected $I_{snd}(t)$, (vi) symptomatic infected with detected $I_{sd}(t)$, (vii) Tested state $(T(t))$, (viii) asymptomatic infected with detected $A_d(t)$, (ix) isolated or hospitalized $(J(t))$ (x) life threatening condition $(L_t(t))$, (xi) self-protected $(P(t))$ (xii) recovered $(R(t))$.

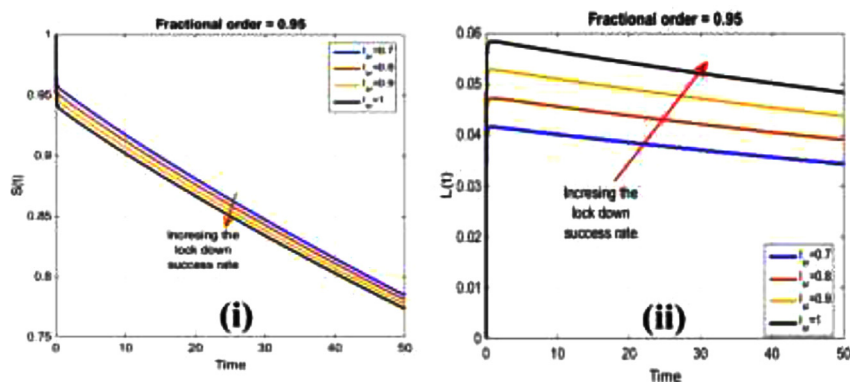


Fig. 3. The sensitivity of different lockdown success rate I_{sr} on (i) susceptible $(S(t))$, and (ii) lockdown $(L(t))$ for fractional-order $p = 0.95$.

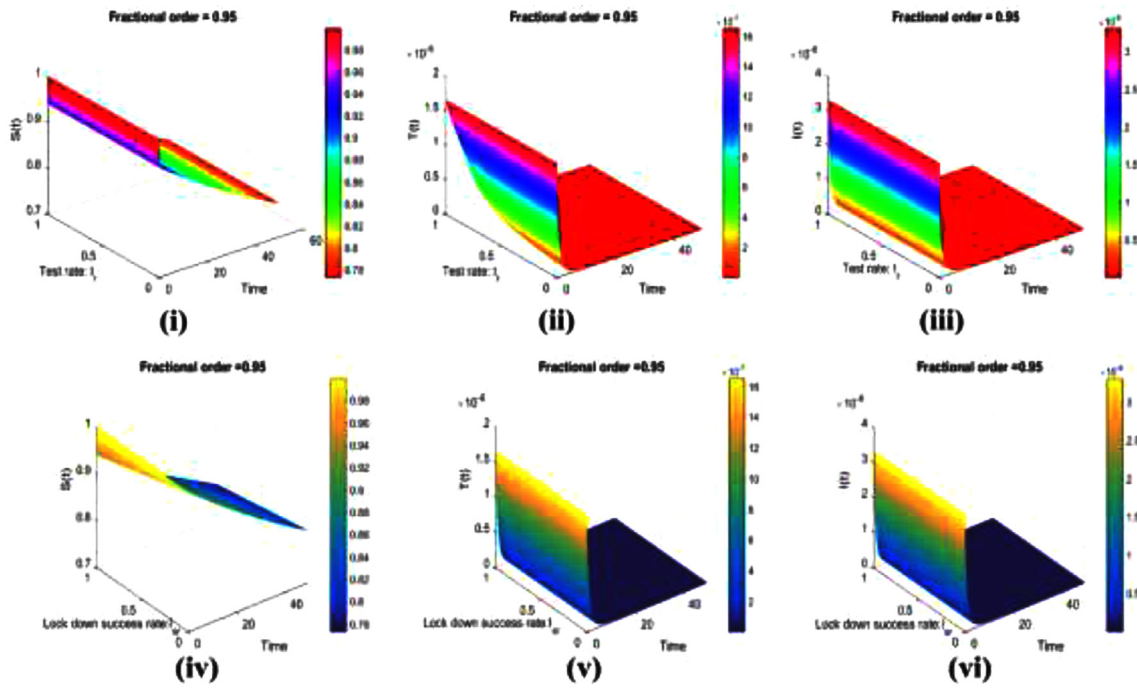


Fig. 4. The impact of changing the test rate t_r vs. time on susceptible ($S(t)$), Tested state ($T(t)$), infected $I(t)$ (asymptomatic, mild-symptomatic-infected, minor or moderate infection but not detected) are presented in (i), (ii) and (iii). Similarly, the impact of changing lockdown success rate l_{sr} vs. time are depicted in (iv) susceptible ($S(t)$), (v) Tested state ($T(t)$), and (vi) infected $I(t)$ (asymptomatic, mild-symptomatic-infected, minor or moderate infection but not detected). Throughout, we used the fractional order of $p = 0.95$.

a model. This paper uses a similar technique in conjunction with second derivative analysis in a complex $SLTIA_d I_{smd} I_{sd} QJL_t RDP$ problem. The computed strength number was negative, implying that the model with second derivatives could only produce one wave before dying out.

In the current work, we only developed and analyzed our proposed model theoretically. In practice and reality, the successful model should depend on actual data fitting and deliberate numerical analysis. Such a complex phenomenon and numerical analysis contain meaningful suggestions to develop health policy and public health measurement to explore future studies.

CRediT authorship contribution statement

Mohammad Sharif Ullah: Software, Formal analysis, Writing – original draft. **M. Higazy:** Methodology, numerical analysis. **K.M. Ariful Kabir:** Conceptualization, Supervision, Writing – original draft. All author’s critically revised the manuscript and gave final approval.

Declaration of Competing Interest

We declare we have no competing interests.

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