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Validation of Burr XII inverse Rayleigh model via a modified chi-squared goodness-of-fit test

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ABSTRACT
In this work, we propose a new three parameter distribution called the Burr XII inverse Rayleigh model, this model is a generalization of the inverse Rayleigh distribution using the Burr XII family introduced by Cordeiro *et al.* [*The burr XII system of densities: properties, regression model and applications*. J. Stat. Comput. Simul. 88 (2018), pp. 432–456]. After studying the statistical characterization of this model, we construct a modified chi-squared goodness-of-fit test based on the Nikulin–Rao–Robson statistic in the presence of two cases: censored and complete data. We describe the theory and the mechanism of the Y_n^2 statistic test which can be used in survival and reliability data analysis. We use the maximum likelihood estimators based on initial non grouped data. Then, we conduct numerical simulations to reinforce the results. For showing the applicability of our model in various fields, we illustrate it and the proposed test by applications to two real data sets for complete data case and two other data sets in the presence of right censored.

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1. Introduction

An appropriate parametric model is often useful in the analysis of survival data because it provides insight into the characteristics of failure times and risk functions. In this work, we first introduce the first generalization of the inverse Rayleigh (IR) distribution using the Burr XII (XII-G) family originally introduced by Cordeiro *et al.* (2018). After studying the statistical characteristics of new model, we construct a modified chi-square goodness-of-fit tests for this model in the cases where the parameters are unknown, complete and right-censored data.

First, it is proposed to construct a modified chi-square type test for the BXII-IR model, in the case where the data are complete and the parameters unknown. This test is based on the Nikulin–Rao–Robson (N.R.R) statistic separately proposed by Nikulin (1973) and Rao and Robson [36]. Based on the maximum likelihood estimator (MLE) on the initial data, this Y^2 statistic is a natural modification of the classic chi-square χ^2 test. Then, we develop another goodness-of-fit test for our new model in the case where the parameters are unknown using a right-censored data. We use the approach proposed by Bagdonavicius

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and Nikulin [13], based on the MLEs on ungrouped data. This modified chi-square type test Y_n^2 is a modification of the N.R.R statistic that takes into account both unknown parameters and censorship. We calculate the MLEs in case of right censoring and all the elements constituting the test criteria. We are conducting a very important numerical simulation study. We end with applications on real data that confirm the results obtained.

The probability density function (PDF) and cumulative distribution function (CDF) of the IR distribution are given by (for $x \geq 0$)

$$\boldsymbol{\pi}_a(x) = 2a^2x^{-3}\exp\left[-(a/x)^2\right] \quad \text{and} \quad \boldsymbol{\Pi}_a(x) = \exp\left[-(a/x)^2\right], \tag{1}$$

respectively, where $a > 0$ is a scale parameter. Let Z be a random variable (rv) having the IR distribution (1) with parameter a. For $r < 2$, the r^{th} ordinary and incomplete moments of Z are given by $\mu_r' = a^r\Gamma(1 - \frac{1}{2}r)$ and $\varphi_r(t) = a^r\,\gamma(1 - \frac{1}{2}r,\ (\frac{a}{t})^2)$, respectively. Consider the BXII-G family of distributions is defined by

$$F_{\alpha,\beta,\boldsymbol{\psi}}(x) = 1 - \left\{1 + \left[\frac{\boldsymbol{\Pi}_{\boldsymbol{\psi}}(x)}{1 - \boldsymbol{\Pi}_{\boldsymbol{\psi}}(x)}\right]^{\alpha}\right\}^{-\beta}. \tag{2}$$

The PDF corresponding to (2) is given by

$$f_{\alpha,\beta,\boldsymbol{\psi}}(x) = \alpha\beta\frac{\boldsymbol{\pi}_{\boldsymbol{\psi}}(x)\,\boldsymbol{\Pi}_{\boldsymbol{\psi}}(x)^{\alpha-1}}{\left[1 - \boldsymbol{\Pi}_{\boldsymbol{\psi}}(x)\right]^{\alpha+1}}\left\{1 + \left[\frac{\boldsymbol{\Pi}_{\boldsymbol{\psi}}(x)}{1 - \boldsymbol{\Pi}_{\boldsymbol{\psi}}(x)}\right]^{\alpha}\right\}^{-\beta-1}, \tag{3}$$

where $\boldsymbol{\pi}_{\boldsymbol{\psi}}(x)$ is the baseline density. The hazard rate function (HRF) of X reduces to

$$\tau_{\alpha,\beta,\boldsymbol{\psi}}(x) = \alpha\beta\frac{\boldsymbol{\pi}_{\boldsymbol{\psi}}(x)\,\boldsymbol{\Pi}_{\boldsymbol{\psi}}(x)^{\alpha-1}}{\left[1 - \boldsymbol{\Pi}_{\boldsymbol{\psi}}(x)\right]^{\alpha+1}}\left\{1 + \left[\frac{\boldsymbol{\Pi}_{\boldsymbol{\psi}}(x)}{1 - \boldsymbol{\Pi}_{\boldsymbol{\psi}}(x)}\right]^{\alpha}\right\}^{-1}.$$

Inserting (1) in to (2) we have

$$F(x) = F_{\alpha,\beta,a}(x) = 1 - \left(1 + \left\{\frac{\exp\left[-(a/x)^2\right]}{1 - \exp\left[-(a/x)^2\right]}\right\}^{\alpha}\right)^{-\beta}, \tag{4}$$

Equation (4) represents the CDF of the proposed model (BXII-IR). The PDF corresponding to (4) is given by

$$\begin{aligned} f(x) = f_{\alpha,\beta,a}(x) &= 2\alpha\beta a^2x^{-3}\frac{\exp\left[-\alpha\,(a/x)^2\right]}{\left\{1 - \exp\left[-(a/x)^2\right]\right\}^{\alpha+1}} \\ &\times\left(1 + \left\{\frac{\exp\left[-(a/x)^2\right]}{1 - \exp\left[-(a/x)^2\right]}\right\}^{\alpha}\right)^{-\beta-1}, \end{aligned} \tag{5}$$

when $\alpha = 1$, the BXII-IR reduces to the Lomax-IR (Lx-IR), when $\beta = 1$, the BXII-IR reduces to the Log-logistic-IR (LL-IR). The HRF corresponding to (5) is given by

$$\tau_{\alpha,\beta,a}(x) = \alpha\beta 2a^2x^{-3}\frac{\exp\left[-\alpha\,(a/x)^2\right]}{\left\{1 - \exp\left[-(a/x)^2\right]\right\}^{\alpha+1}}\left(1 + \left\{\frac{\exp\left[-(a/x)^2\right]}{1 - \exp\left[-(a/x)^2\right]}\right\}^{\alpha}\right)^{-1}.$$

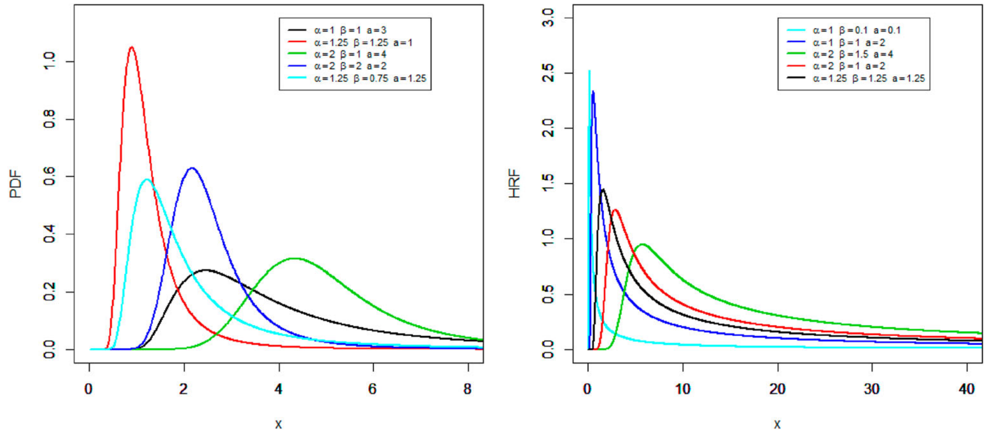

Figure 1. Plots of the BXII-IR PDF and HRF for selected parameter values.

The BXII-IR density can be symmetric and right-skewed, whereas the BXII-IR HRF can be upside down (see Figure 1). Hereafter, we denote by $X \sim \text{BXII} - \text{IR}(x, \alpha, \beta, a)$ a rv having density function (5).

The CDF (4) of X can be expressed as

$$F(x) = 1 - \underbrace{\left(1 + \left\{\frac{\exp\left[-(a/x)^2\right]}{1 - \exp\left[-(a/x)^2\right]}\right\}^{\alpha}\right)^{-\beta}}_{A}. \tag{6}$$

First, we consider two power series

$$(1+\tau)^{-c} = \sum_{h=0}^{\infty} 2^{-c-h} \binom{-c}{h} (\tau - 1)^h \tag{7}$$

and

$$(1-\tau)^{-c} = \sum_{h=0}^{\infty} \frac{\Gamma(c+h)}{h!\Gamma(c)} \tau^h \Big|_{(|\tau|<1,\, c>0)}. \tag{8}$$

Applying (7) for A in Equation (6) gives

$$F(x) = 1 - \sum_{k=0}^{\infty} 2^{-\beta-k} \binom{-\beta}{k} \left(\left\{\frac{\exp\left[-(a/x)^2\right]}{1 - \exp\left[-(a/x)^2\right]}\right\}^{\alpha} - 1\right)^k.$$

Second, using the binomial expansion, the last equation can be expressed as

$$F(x) = 1 - \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{(-1)^i \binom{k}{i} \binom{-\beta}{k}}{2^{\beta+k} \left\{\exp\left[-(a/x)^2\right]\right\}^{-(k-i)\alpha}} \underbrace{\left\{1 - \exp\left[-(a/x)^2\right]\right\}^{-(k-i)\alpha}}_{B}.$$

Third, applying (8) for B in the last equation gives

$$F(x) = 1 - \sum_{j,k=0}^{\infty} \sum_{i=0}^{k} c_{i,j,k} \boldsymbol{\Pi}_{(k-i)\alpha+j}(x, a), \tag{9}$$

where

$$c_{i,j,k} = \frac{(-1)^i \Gamma\left([k-i]\alpha + j\right)}{2^{\beta+k} j! \Gamma\left([k-i]\alpha\right)} \binom{k}{i} \binom{-\beta}{k}$$

and $\boldsymbol{\Pi}_{(k-i)\alpha+j}(x, a)$ is the CDF of the IR model with scale parameter $a[(k-i)\alpha + j]^{\frac{1}{2}}$. By differentiating (8), we obtain

$$f(x) = \sum_{\substack{j,k=0 \\ j+k\geq 1}}^{\infty} \sum_{i=0}^{k} \zeta_{i,j,k} \boldsymbol{\pi}_{(k-i)\alpha+j}(x, a), \tag{10}$$

where $\boldsymbol{\pi}_{(k-i)\alpha+j}(x, a)$ is the IR density with scale parameter $a[(k-i)\alpha + j]^{\frac{1}{2}}$ and $\zeta_{i,j,k} = -c_{i,j,k}$.

In the statistical literature, there are many useful studies based on the Burr type XII distribution can be cited, for example: group acceptance sampling plans for resubmitted lots under Burr Type XII distributions by Aslam *et al.* [7], double acceptance sampling plans for Burr type XII distribution percentiles under the truncated life test by Aslam *et al.* [9], two-stage group acceptance sampling plan for Burr type X percentiles by Aslam *et al.* [4], repetitive acceptance sampling plans for burr type XII percentiles by Aslam *et al.* [8], optimal designing of Skip lot sampling plan of type SkSP-2 with group acceptance sampling plan as a reference plan under Burr type XII distribution by Aslam *et al.* [6], multiple dependent state repetitive group sampling plan for Burr XII distribution by Aslam *et al.* [3] and time-truncated attribute sampling plans using EWMA for Weibull and Burr type X distributions by Aslam *et al.* [5], amonge others. On the other hand, man authors studied the inverse Rayleigh distribution, see for example: Yousof *et al.* [44], Korkmaz *et al.* [23], Aryal and Yousof [2], Merovci *et al.* [28], Brito *et al.* [14], Korkmaz *et al.* [24], Yousof *et al.* [45], Chakraborty *et al.* [15], Yousof *et al.* [46], among others.

In this work, we are interested in the Burr XII inverse Rayleigh distributions which is a new generalization of the IR distribution and whose mathematical form of its probability density is manageable thus allowing to calculate its various characteristics. This flexible model can describe different lifetimes from reliability, survival analysis, and other areas (see Section 3.2). Since the results of any statistical analysis depend on the chosen model, then we have constructed modified chi-square type fit tests to allow users to verify the adequacy of their observations to these types of distributions (see Section 5). The tests used take into account the unknown parameters of the models, right censorship generally present in the reliability and survival analysis studies, and use all the information provided by the sample. We have shown the applicability of this new model (BXII-IR) by a study of two real complete data and two others for the case of censored data (see Sections 6 and 7).

2. Properties

2.1. Moments and generating function

The rth ordinary moment of X say $\mu'_r = \mathbf{E}(X^r)$, is determined from (10) as

$$\mu'_r = \sum_{\substack{j,k=0 \\ j+k\geq 1}}^{\infty} \sum_{i=0}^{k} \varsigma_{i,j,k} a^r \left[(k-i)\,\alpha + j\right]^{\frac{1}{2}r} \Gamma\left(1 - \frac{1}{2}r\right)\Big|_{(r<2)},$$

where

$$\Gamma\left(1+\zeta\right)|_{(\zeta\in\mathbb{R}^+)} = \zeta! = \zeta \times (\zeta-1) \times (\zeta-2) \times \cdots \times 1 = \prod_{w=0}^{\zeta-1} (\zeta - w)$$

and

$$\int_0^\infty x^{\zeta-1}\,\mathrm{e}^{-t}\,\mathrm{d}x = \Gamma\left(\zeta\right).$$

The rth incomplete moment of X, say $\varphi_r(t)$, can be determined from (10) as

$$\varphi_r(t) = \int_{-\infty}^{t} x^r f(x)\,\mathrm{d}x = \sum_{\substack{j,k=0 \\ j+k\geq 1}}^{\infty} \sum_{i=0}^{k} \varsigma_{i,j,k} a^r \left[(k-i)\,\alpha + j\right]^{\frac{1}{2}r} \gamma\left(1 - \frac{1}{2}r, \left(\frac{a}{t}\right)^2\right)\Big|_{(r<2)}, \tag{11}$$

where

$$\gamma\left(\zeta, q\right)|_{(\zeta\neq 0,-1,-2,\ldots)} = \int_0^q t^{\zeta-1} \exp\left(-t\right)\mathrm{d}t = \frac{q^\zeta}{\zeta}\left\{{}_1\mathbf{F}_1\left[\zeta; \zeta a + 1; -q\right]\right\}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(\zeta+k)} q^{\zeta+k}$$

and ${}_1\mathbf{F}_1[\cdot,\cdot,\cdot]$ is a confluent hypergeometric function. The moment generating function $M(t) = \mathbf{E}(\mathrm{e}^{tX})$ of X follows from (10) as

$$M(t) = \sum_{\substack{j,k,r=0 \\ j+k\geq 1}}^{\infty} \sum_{i=0}^{k} \varsigma_{i,j,k} \left(t^r/r!\right) a^r \left[(k-i)\,\alpha + j\right]^{\frac{1}{2}r} \Gamma\left(1 - \frac{1}{2}r\right)\Big|_{(r<2)}.$$

2.2. Probability weighted moments (PWMs)

The PWMs are generally used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The (s,r)th PWM of X denoted by $\rho_{s,r}$ is formally defined by

$$\rho_{s,r} = \mathbf{E}\left\{X^s\,F(X)^r\right\} = \int_{-\infty}^{\infty} x^s F(x)^r f(x)\,\mathrm{d}x,$$

using (4), we have

$$F(x)^r = \left[1-\left(1+\left\{\frac{\exp\left[-(a/x)^2\right]}{1-\exp\left[-(a/x)^2\right]}\right\}^{\alpha}\right)^{-\beta}\right]^r.$$

Expanding z^η in Taylor series, we can write

$$z^\eta = \sum_{h=0}^{\infty} \frac{(\eta)_h}{h!}(z-1)^h = \sum_{i=0}^{\infty} \zeta_i(\eta) z^i, \tag{12}$$

where $(\eta)_h = \eta(\eta-1)\ldots(\eta-h+1)$ is the descending factorial and

$$\zeta_i(\eta) = \sum_{h=i}^{\infty} \frac{(-1)^{h-i}(\eta)_h}{h!}\binom{h}{i}.$$

Applying the Taylor series in z^η for $F(x)^r$, we obtain

$$F(x)^r = \sum_{i=0}^{\infty} (-1)^i \zeta_i(r)\left(1+\left\{\frac{\exp\left[-(a/x)^2\right]}{1-\exp\left[-(a/x)^2\right]}\right\}^{\alpha}\right)^{-i\beta}.$$

Second, using (5) and the last equation, we have

$$f(x)\,F(x)^r = \alpha\beta 2a^2 x^{-(2+1)} \exp\left[-(a/x)^2\right] \frac{\left\{\exp\left[-(a/x)^2\right]\right\}^{\alpha-1}}{\left\{1-\exp\left[-(a/x)^2\right]\right\}^{\alpha+1}}$$
$$\times \sum_{i=0}^{\infty}(-1)^i \zeta_i(r) \underbrace{\left(1+\left\{\frac{\exp\left[-(a/x)^2\right]}{1-\exp\left[-(a/x)^2\right]}\right\}^{\alpha}\right)^{-(1+i)\beta-1}}_{C},$$

applying (7) for C in the last equation, we obtain

$$f(x)\,F(x)^r = \alpha\beta \sum_{i,k=0}^{\infty} (-1)^i 2^{-(1+i)\beta-k-1} \zeta_i(r)\, 2a^2 x^{-(2+1)} \frac{\exp\left[-\alpha\,(a/x)^2\right]}{\left\{1-\exp\left[-(a/x)^2\right]\right\}^{\alpha+1}}$$
$$\times \underbrace{\left(-1+\left\{\frac{\exp\left[-(a/x)^2\right]}{1-\exp\left[-(a/x)^2\right]}\right\}^{\alpha}\right)^{k}}_{D}\binom{-(1+i)\beta-1}{k}.$$

Third, using the binomial expansion for D, the last equation be rewritten as

$$f(x)\,F(x)^r = \alpha\beta g(x) \sum_{i,k=0}^{\infty}\sum_{j=0}^{k} (-1)^{i+j} 2^{-(1+i)\beta-k-1}\binom{k}{j}\binom{-(1+i)\beta-1}{k}$$
$$\times \zeta_i(r)\left\{\exp\left[-(a/x)^2\right]\right\}^{(k-j+1)\alpha-1} \underbrace{\left\{1-\exp\left[-(a/x)^2\right]\right\}^{-[(k-j+1)\alpha+1]}}_{E}.$$

Applying (8) for E in the last equation gives

$$f(x)\,F(x)^r = \sum_{k,m=0}^{\infty}\sum_{j=0}^{k} c_{j,k,m}^{(r)}\boldsymbol{\pi}_{(k-j+1)\alpha+m}(x;a)\,|_{(j\leq k)},$$

where $c_{j,k,m}^{(r)} = \alpha\beta b_{j,k,m}\zeta_i(r)$, $\zeta_i(r)$ is defined in (12) and

$$b_{j,k,m} = \sum_{i=0}^{\infty}\frac{(-1)^{i+j}\left([k-j+1]\alpha+1\right)^{(m)}\binom{k}{j}\binom{-(1+i)\beta-1}{k}}{2^{(1+i)\beta+k+1}[(k-j+1)\alpha+m]m!},$$

where $m^{(\tau)} = \Gamma(m+\tau)/\Gamma(m)$ denotes the rising factorial. Finally, the (s,r)th PWM of X can be determined as

$$\rho_{s,r} = \sum_{k,m=0}^{\infty}\sum_{j=0}^{k} c_{j,k,m}^{(r)}a^s\left[(k-j+1)\alpha+m\right]^{\frac{s}{2}}\Gamma\left(1-\frac{s}{2}\right)\Big|_{(s<2\text{and}j\leq k)}.$$

2.3. Residual life and reversed residual life functions

The nth moment of the residual life, say

$$m_n(t)|\binom{n=1,2,\ldots}{X>t} = \mathbf{E}[(X-t)],$$

uniquely determines $F(x)$. The nth moment of the residual life of X is given by

$$m_n(t)|\binom{n=1,2,\ldots}{X>t} = \frac{\int_t^{\infty}(x-t)^n\,\mathrm{d}F(x)}{1-F(t)}.$$

Therefore

$$m_n(t)|\binom{n=1,2,\ldots}{X>t} = \frac{1}{1-F(t)}\sum_{\substack{j,k=0\\ j+k\geq 1}}^{\infty}\sum_{i=0}^{k}\zeta_{i,j,k}^{\star}a^n\left[(k-i)\,\alpha+j\right]^{\frac{n}{2}}\Gamma\left(1-\frac{n}{2},\left(\frac{a}{t}\right)^2\right)\Big|_{(n<2)},$$

where $\zeta_{i,j,k}^{\star} = \zeta_{i,j,k}(1-t)^n, \Gamma(\zeta,q)|_{(x>0)} = \int_q^{\infty} t^{\zeta-1}\,\mathrm{e}^{-t}\,\mathrm{d}t$ and $\Gamma(\zeta,q)+\gamma(\zeta,q) = \Gamma(\zeta)$. The mean residual life function or the life expectation at age t defined by $m_1(t)|\binom{n=1}{X>t} = \mathbf{E}[(X-t)]$ which represents the expected additional life length for a unit which is alive at age t. The mean residual life of X can be obtained by setting $n=1$ in the last equation. The nth moment of the reversed residual life, say

$$M_n(t)|\binom{n=1,2,\ldots}{X\leq t,\ t>0} = \mathbf{E}\left[(t-X)^n\right],$$

uniquely determines $F(x)$. We obtain

$$M_n(t)|\binom{n=1,2,\ldots}{X\leq t,\ t>0} = \frac{\int_0^t(t-x)^n\,\mathrm{d}F(x)}{F(t)}.$$

Then, the nth moment of the reversed residual life of X comes from

$$M_n(t)|\left(\substack{n=1,2,\dots\\ X\le t,\ t>0}\right) = \frac{1}{F(t)}\sum_{\substack{j,k=0\\ j+k\ge 1}}^{\infty}\sum_{i=0 i,j,k}^{k}\zeta^{\star\star}_{i,j,k}a^n\left[(k-i)\,\alpha+j\right]^{\frac{n}{2}}\gamma\left(1-\frac{n}{2},\left(\frac{a}{t}\right)^2\right)\Big|_{(n<2)}$$

where $\zeta^{\star\star}_{i,j,k}=\zeta_{i,j,k}\sum_{r=0}^{n}(-1)^r\binom{n}{r}t^{n-r}$. The mean inactivity time also called the mean reversed residual life function, is given by $M_1(t)|(\substack{n=1\\ X\le t,\ t>0})=\mathbf{E}[(t-X)]$ and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$. The mean inactivity time of the BXII-IR model is obtained easily by setting $n=1$ in the above equation.

2.4. Order statistics

Let $X_1,\dots,X_n$ be a random sample from the BXII-IR model and let $X_{1:n},\dots,X_{n:n}$ be the corresponding order statistics. The PDF of the ith order statistic, say $X_{i:n}$, is given by

$$f_{i:n}(x)=\frac{f(x)}{\mathrm{B}(i,n-i+1)}\sum_{r=0}^{n-i}(-1)^r\binom{n-i}{r}F^{r+i-1}(x),$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function, then we can write

$$f(x)F(x)^{r+i-1}=\sum_{k,m=0}^{\infty}\sum_{j=0}^{k}c^{(r+i-1)}_{j,k,m}\boldsymbol{\pi}_{(k-j+1)\alpha+m}(x;a)\,|_{(j\le k)},$$

where $c^{(r+i-1)}_{j,k,m}$ is defined before. So, the PDF of $X_{i:n}$ follows using the last expression as

$$f_{i:n}(x)=\frac{1}{\mathrm{B}(i,n-i+1)}\sum_{k,m=0}^{\infty}\sum_{r=0}^{n-i}\sum_{j=0}^{k}(-1)^r\binom{n-i}{r}c^{(r+i-1)}_{j,k,m}\boldsymbol{\pi}_{(k-j+1)\alpha+m}(x;a)\,|_{(j\le k)}. \tag{13}$$

Then, the density function of the BXII-IR order statistics is a fourth linear combination of the IR density. Then The τth ordinary moment of $X_{i:n}$ say $\boldsymbol{E}(X^{\tau}_{i:n})$, is determined from (13) as

$$\mathbf{E}(X^{\tau}_{i:n})=\sum_{k,m=0}^{\infty}\sum_{r=0}^{n-i}\sum_{j=0}^{k}\frac{(-1)^r\binom{n-i}{r}c^{(r+i-1)}_{j,k,m}}{\mathrm{B}(i,n-i+1)}a^{\tau}\left[(k-j+1)\alpha+m\right]^{\frac{\tau}{2}}\Gamma\left(1-\frac{\tau}{2}\right)\Big|_{(\tau<2,\ j\le k)}.$$

3. Maximum likelihood estimation in case of complete data

3.1. Maximum likelihood estimation

Let $x_1,\dots,x_n$ be a RS from the BXII-IR model with parameters α,β and a. Let $\theta=(\alpha,\beta,a)^{\intercal}$ be the 3×1 parameter vector. For determining the MLE of θ, we have the log-likelihood function

$$\ell = \ell(\theta) = n\log 2 + n\log\alpha + n\log\beta + 2n\log a - 3\sum_{i=1}^{n} x_i - \alpha\sum_{i=1}^{n}\left(\frac{a}{x_i}\right)^2$$
$$-(\alpha+1)\sum_{i=1}^{n}\log(1-s_i) - (\beta+1)\sum_{i=1}^{n}\log z_i,$$

where

$$s_i = \exp\left[-\left(\frac{a}{x_i}\right)^2\right] \quad \text{and} \quad z_i = \left[1 + \left(\frac{s_i}{1-s_i}\right)^2\right],$$

the score vector $\mathbf{I}_{(\theta)} = \frac{\partial\ell}{\partial\theta} = (\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial a})^{\mathsf{T}}$ is given as

$$\mathbf{I}_{(\alpha)} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(\frac{a}{x_i}\right)^2 - \sum_{i=1}^{n}\log(1-s_i) - (\beta+1)\sum_{i=1}^{n}\frac{p_i}{z_i},$$

$$\mathbf{I}_{(\beta)} = \frac{n}{\beta} - \sum_{i=1}^{n}\log z_i,$$

$$\mathbf{I}_{(a)} = \frac{2n}{a} - 2\alpha\sum_{i=1}^{n}\frac{1}{x_i}\left(\frac{a}{x_i}\right) - 2(\alpha+1)\sum_{i=1}^{n}\frac{\left(\frac{a}{x_i}\right)\frac{s_i}{x_i}}{1-s_i} - (\beta+1)\sum_{i=1}^{n}\frac{m_i}{z_i},$$

where

$$p_i = \left(\frac{s_i}{1-s_i}\right)^{\alpha}\log\left(\frac{s_i}{1-s_i}\right) \quad \text{and} \quad m_i = -2\alpha\left(\frac{s_i}{1-s_i}\right)^{\alpha-1}\frac{\frac{s_i}{x_i}\left(\frac{a}{x_i}\right)}{[1-s_i]^2}.$$

Setting the nonlinear system of equations $\mathbf{I}_{(\alpha)} = 0, \mathbf{I}_{(\beta)} = 0$ and $\mathbf{I}_{(a)} = 0$ and solving them simultaneously yields the MLE $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{a})^{\mathsf{T}}$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. Since, we cannot find the explicit formulas of the maximum likelihood estimators of the parameters, we then use numerical methods such as the Newton Raphson method, the Monte Carlo method, the BB algorithm or others.

3.2. Data analysis: case of complete data

This section presents two applications of the BXII-IR distribution using real data sets. We shall compare the fit of the new distribution with the Weibull-inverse Weibull (W-IW) [1], exponentiated IW (E-IW) [31], Kumaraswamy IW (Kum-IW) [27], beta IW (B-IW) [30], transmuted IW (T-IW) [26], gamma extended IW (GaE-IW) [38], Marshall-Olkin IW (MO-IW) [25] and IW distributions with corresponding densities (for $x > 0$):

$$\mathrm{W-IW}: ab\beta\alpha^{\beta}x^{-(\beta+1)}\exp[-b(\alpha/x)^{\beta}]\{1-\exp[-(\alpha/x)^{\beta}]\}^{-(b+1)}$$
$$\times\exp\left\{-a\left[\frac{\exp[-(\alpha/x)^{\beta}]}{1-\exp[-(\alpha/x)^{\beta}]}\right]^{b}\right\};$$

$$\mathrm{K-IW}: f(x;\alpha,\beta,a,b) = ab\beta\alpha^{\beta}x^{-(\beta+1)}\exp[-a(\alpha/x)^{\beta}]\{1-\exp[-a(\alpha/x)^{\beta}]\}^{b-1};$$

$$\mathrm{E-IW}: f(x;\alpha,\beta,a) = a\beta\alpha^{\beta}x^{-(\beta+1)}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\{1-\exp[-(\alpha/x)^{\beta}]\}^{a-1};$$

$$\mathrm{B-IW}: f(x;\alpha,\beta,a,b) = \frac{\beta\alpha^{\beta}}{B(a,b)}x^{-(\beta+1)}\exp[-a(\alpha/x)^{\beta}]\{1-\exp[-(\alpha/x)^{\beta}]\}^{b-1};$$

$$\mathrm{GE-IW}: f(x;\alpha,\beta,a,b) = \frac{a\beta\alpha^{\beta}}{\Gamma(b)}x^{-(\beta+1)}\exp[-(\alpha/x)^{\beta}]\{1-\exp[-(\alpha/x)^{\beta}]\}^{a-1}$$

$$\times\{-\log\{1-\exp[-(\alpha/x)^{\beta}]\}^{a}\}^{b-1};$$

$$\mathrm{T-IW}: f(x;\alpha,\beta,a) = \beta\alpha^{\beta}x^{-(\beta+1)}\exp[-(\alpha/x)^{\beta}]\{(a+1)-2a\exp[-(\alpha/x)^{\beta}]\};$$

$$\mathrm{MO-IW}: f(x;\alpha,\beta,a) = a\beta\alpha^{\beta}x^{-(\beta+1)}\exp[-(\alpha/x)^{\beta}]$$

$$\times\{a+(1-a)\exp[-(\alpha/x)^{\beta}]\}^{-2}.$$

The unknown parameters of the above PDFs are all positive real numbers except for the T-IW distribution for which $|a| \leq 1$. The first dataset consists of 100 observations of breaking stress of carbon fbres (in Gba) given by Nichols and Padgett [32]. The data are: 0.92, 0.928, 0.997, 0.9971, 1.061, 1.117, 1.162, 1.183, 1.187, 1.192, 1.196, 1.213, 1.215, 1.2199, 1.22, 1.224, 1.225, 1.228, 1.237, 1.24, 1.244, 1.259, 1.261, 1.263, 1.276, 1.31, 1.321, 1.329, 1.331, 1.337, 1.351, 1.359, 1.388, 1.408, 1.449, 1.4497, 1.45, 1.459, 1.471, 1.475, 1.477, 1.48, 1.489, 1.501, 1.507, 1.515, 1.53, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.562, 1.566, 1.585, 1.586, 1.599, 1.602, 1.614, 1.616, 1.617, 1.628, 1.684, 1.711, 1.718, 1.733, 1.738, 1.743, 1.759, 1.777, 1.794, 1.799, 1.806, 1.814, 1.816, 1.828, 1.83, 1.884, 1.892, 1.944, 1.972, 1.984, 1.987, 2.02, 2.0304, 2.029, 2.035, 2.037, 2.043, 2.046, 2.059, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.306. The second dataset [39] consists of 63 observations of the strengths of 1.5 cm glass fibers, originally obtained by workers at the UK National Physical Laboratory. Unfortunately, the units of measurement are not given in the paper. The data are: 1.014, 1.081, 1.082, 1.185, 1.223, 1.248, 1.267, 1.271, 1.272, 1.275, 1.276, 1.278, 1.286, 1.288, 1.292, 1.304, 1.306, 1.355, 1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 1.46, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535, 1.541, 1.568, 1.579, 1.581, 1.591, 1.593, 1.602, 1.666, 1.67, 1.684, 1.691, 1.704, 1.731, 1.735, 1.747, 1.748, 1.757, 1.800, 1.806, 1.867, 1.876, 1.878, 1.91, 1.916, 1.972, 2.012, 2.456, 2.592, 3.197, 4.121. In order to compare the distributions, we consider the following criteria: the $-2\hat{\ell}$ (Maximized Log-Likelihood), AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian information criterion) and HQIC (Hannan-Quinn information Criterion). These statistics are given by

$$AIC = -2\hat{\ell} + 2k, \quad BIC = -2\hat{\ell} + k\log(n), \quad HQIC = -2\hat{\ell} + 2k\log[\log(n)]$$

and

$$CAIC = -2\hat{\ell} + 2kn/(n-k-1),$$

where $\hat{\ell}$ denotes the log-likelihood function evaluated at the MLEs, k is the number of model parameters and n is the sample size. The model with minimum values for these

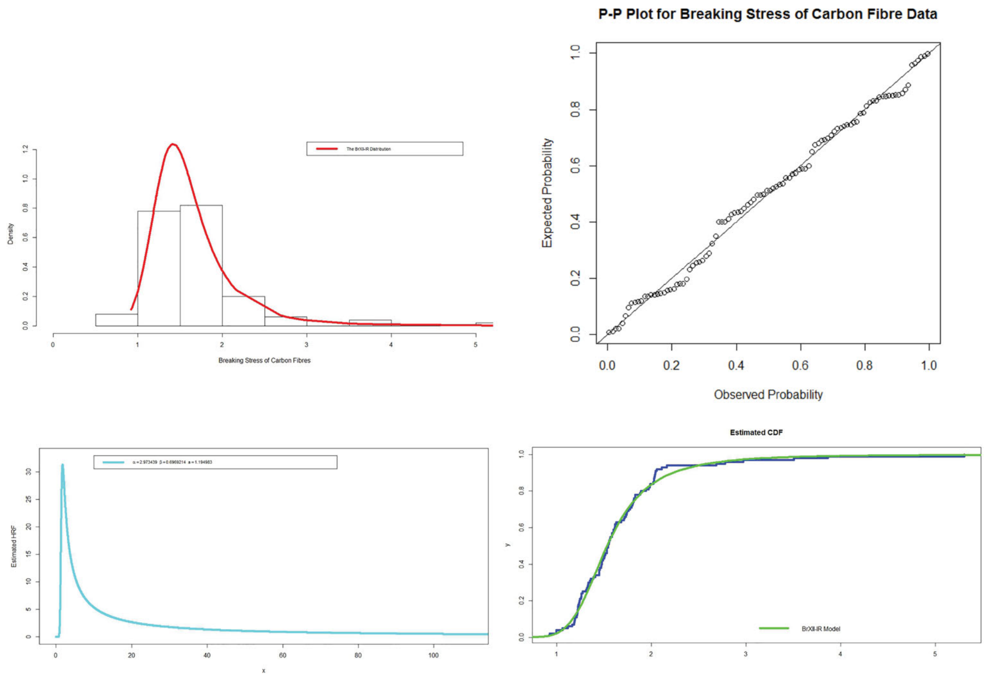

Figure 2. Estimated PDF, P-P plot, estimated HRF and estimated CDF for the first data.

Table 1. $-2\hat{\ell}$, *AIC*, *BIC*, *HQIC* and *CAIC* for the first data.

	Goodness of fit criteria				
Model	$-2\hat{\ell}$	*AIC*	*BIC*	*HQIC*	*CAIC*
BXII-IR	**103.8**	**109.9**	**117.7**	**113.0**	**110.1**
LL-IR	111.3	115.06	120.27	117.17	115.19
W-IW	286.5	294.5	304.9	298.7	294.9
E-IW	289.7	295.7	303.5	298.9	296.0
Kum-IW	289.1	297.1	307.5	301.3	297.5
B-IW	303.1	311.1	321.6	315.4	311.6
GaE-IW	304	312	332.4	316.2	312.4
IW	344.3	348.3	353.5	350.4	348.4
T-IW	344.5	350.5	358.3	353.6	350.7
MO-IW	345.3	351.3	359.1	354.5	351.6

statistics could be chosen as the best model to fit the data. All results are obtained using the R PROGRAM (see Appendix 1). Figure 2 gives the estimated PDF, P-P plot, estimated HRF and estimated CDF for the first data. Figure 3 gives the estimated PDF, P-P plot, estimated HRF and estimated CDF for the second data.

Tables 1 and 3 compare the BXII-IR model model with the W-IW, TMO-IW, K-IW, B-IW, E-IW, GE-IW, T-IW, MO-IW and -IW distributions. The new model gives the lowest values for the AIC, BIC, HQIC and CAIC statistics (in bold values) among all fitted models to these data. So, it could be chosen as the best model among them. Figure 3 displays the plots of estimated density for the proposed model models and estimated CDF of the new model for the first data. Figure 4 displays the plots of estimated density for the proposed model models and estimated CDF of the proposed model for the second data. These plots

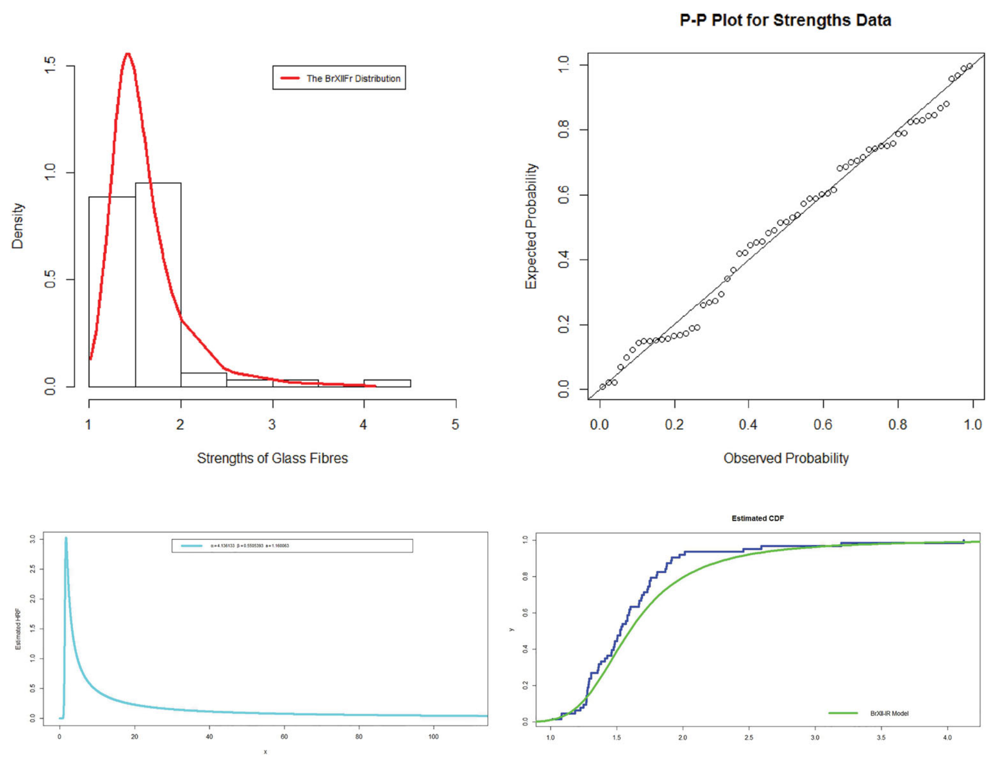

Figure 3. Estimated PDF, P-P plot, estimated HRF and estimated CDF for the second data.

reveal that the proposed distribution yields a better ft than other nested and non-nested models for both data sets (Tables 2 and 4).

3.3. Simulations: case of complete data

We consider the BXII-IR model. The data were simulated $N = 10{,}000$ times; with sample sizes $n = 30,\ n = 100,\ n = 250,\ n = 500$, and parameter values $\alpha = 0.6,\ \beta = 3.7$, $a = 1.5$. Using the R software and the Barzilai-Borwein (BB) algorithm [37] for calculating the averages of the simulated values of the maximum likelihood estimators $\hat{\alpha}, \hat{\beta}, \hat{a}$ parameters and their mean squared errors (noted MSE), we obtain the results presented in Table 5 (see the R code in Appendix 2). From Table 5, we can notice that the maximum likelihood estimators are convergent.

4. Maximum likelihood estimation for censored data

4.1. Maximum likelihood estimation

Consider this time the case of right-censored data. Let T a random variable distributed according to a BXII-IR distribution with $\boldsymbol{\theta} = (\alpha, \beta, a)^{\mathrm{T}}$. For i (individual); T_i is the lifetime and C_i is the censorship time, where T_i and C_i are independent random variables. Suppose

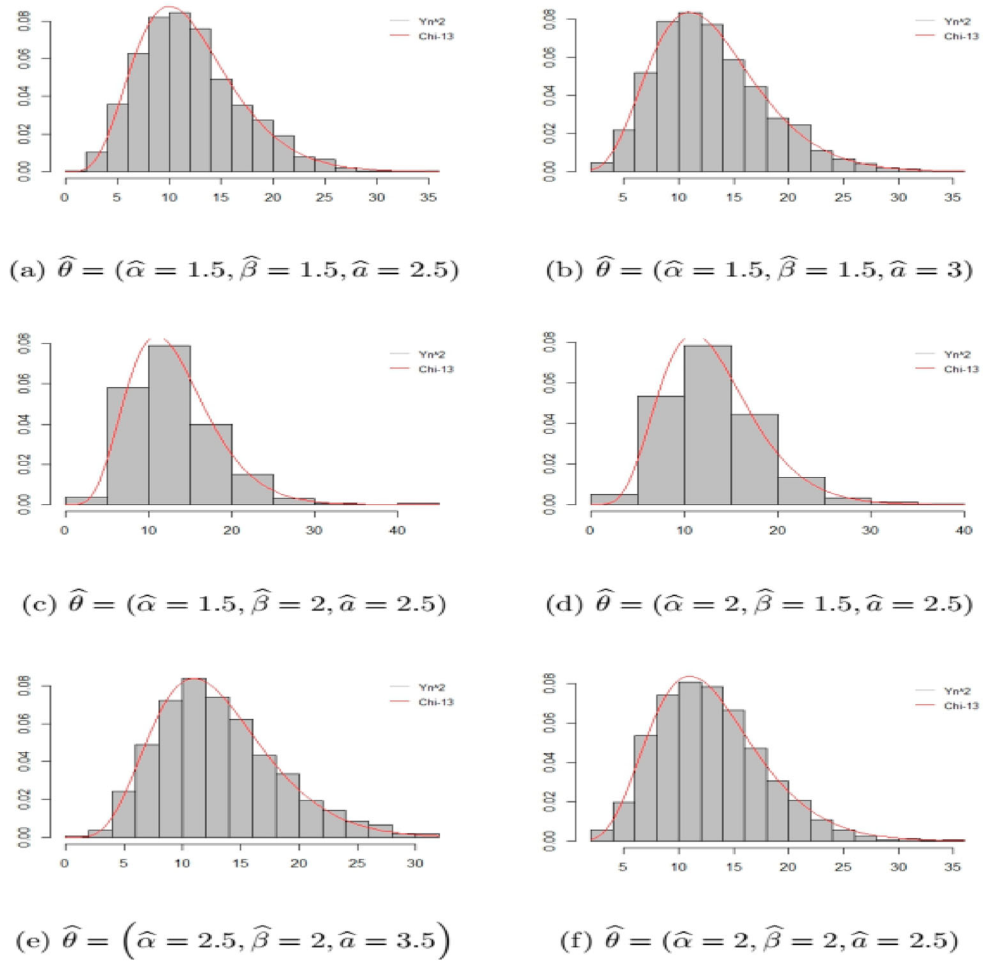

Figure 4. Simulated distribution of the Y^2 statistic under the null hypothesis H_0, with different parameters of $\hat{\theta}$ versus the chi-squared distribution with 13 degrees of freedom, with $n = 200,\ N = 10,000$.

the data consists of n independent observations

$$t_i = \min(T_i, C_i) \quad \text{for } i = 1, \dots, n.$$

Censorship is assumed to be non-informative (the distribution of C_i does not depend on the unknown parameters of T_i). The likelihood function in the case of censored data can be given by:

$$\mathbf{L}(t, \boldsymbol{\theta}) = \prod_{i=1}^{n} \lambda^{\delta_i}(t_i, \boldsymbol{\theta}) S(t_i, \boldsymbol{\theta}); \quad \boldsymbol{\theta} = (\alpha, \beta, a)^{\mathrm{T}} \text{ is the vector of parameters, } \delta_i = 1_{\{T_i \leq C_i\}}.$$

Table 2. MLEs and their standard errors for the first data.

	Estimates			
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
BXII-IR	2.973	0.697	1.1949	
	(0.6069)	(0.2488)	(0.0831)	
LL-IR		4.641	2.197	
		(0.802)	(0.102)	
W-IW	2.2231	0.355	6.9721	4.9179
	(11.409)	(0.411)	(113.811)	(3.756)
EF	69.1489	0.5019	145.3275	
	(57.349)	(0.08)	(122.924)	
Kum-IW	2.0556	0.4654	6.2815	224.18
	(0.071)	(0.00701)	(0.063)	(0.164)
B-IW	1.6097	0.4046	22.0143	29.7617
	(2.498)	(0.108)	(21.432)	(17.479)
GaE-IW	1.3692	0.4776	27.6452	17.4581
	(2.017)	(0.133)	(14.136)	(14.818)
IW	1.8705	1.7766		
	(0.112)	(0.113)		
T-IW	1.9315	1.7435	0.0819	
	(0.097)	(0.076)	(0.198)	
MO-IW	2.3066	1.5796	0.5988	
	(0.498)	(0.16)	(0.3091)	

Table 3. $-2\hat{\ell}$, *AIC*, *BIC*, *HQIC* and *CAIC* for second data.

	Measures				
Model	$-2\hat{\ell}$	*AIC*	*BIC*	*HQIC*	*CAIC*
BXII-IR	**38.5**	**44.5**	**50.9**	**47.04**	**44.9**
LL-IR	48.6	52.1	56.3	53.7	52.3
Kum-IW	39.6	47.6	56.2	51	48.3
E-IW	44.3	50.5	56.7	52.8	50.7
B-IW	60.6	68.6	77.2	72.0	69.3
GaE-IW	61.6	69.6	78.1	72.9	70.3
IW	93.7	97.7	102	99.4	97.9
T-IW	94.1	100.1	106.5	102.6	100.5
MO-IW	95.7	101.7	108.2	104.2	102.1

In our case, let T_i be a random variable distributed with the vector of parameters $\boldsymbol{\theta} = (b, \alpha, \gamma, \beta_0, \beta_1)^{\mathrm{T}}$, so the likelihood function reduces to

$$\mathbf{L}(t, \boldsymbol{\theta}) = \prod_{i=1}^{n} \left[\begin{array}{c} \alpha\beta 2a^2 t_i^{-3} \dfrac{\exp\left[-\alpha\,(a/t_i)^2\right]}{\left\{1-\exp\left[-(a/t_i)^2\right]\right\}^{\alpha+1}} \\ \times \left(1+\left\{\dfrac{\exp\left[-(a/t_i)^2\right]}{1-\exp\left[-(a/t_i)^2\right]}\right\}^{\alpha}\right)^{-1} \end{array} \right]^{\delta_i} \times \left[\left(1+\left\{\dfrac{\exp\left[-(a/t_i)^2\right]}{1-\exp\left[-(a/t_i)^2\right]}\right\}^{\alpha}\right)^{-\beta}\right]$$

Table 4. MLEs and their standard errors for second data.

	Estimates			
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
BXII-IR	4.1361	00.5505	1.1601	
	1.04398	0.23159	0.0702	
LL-IR		6.752	2.369	
		(1.553)	(0.128)	
Kum-IW	2.116	0.740	5.504	857.343
	(4.555)	(0.071)	(7.982)	(153.948)
E-IW	7.816	0.999	132.827	
	(2.945)	(0.136)	(116.63)	
B-IW	2.0518	0.6466	15.0756	36.9397
	(0.986)	(0.163)	(12.057)	(22.649)
GaE-IW	1.6625	0.7421	32.112	13.2688
	(0.952)	(0.197)	(17.397)	(9.967)
IW	1.264	2.888		
	(0.059)	(0.234)		
T-IW	1.3068	2.7898	0.1298	
	(0.034)	(0.165)	(0.208)	
MO-IW	1.5441	2.3876	0.4816	
	(0.226)	(0.253)	(0.252)	

Table 5. Maximum likelihood estimators $(\hat{\alpha}, \hat{\beta}, \hat{a})$ of the parameters and their mean squared errors.

$N = 10{,}000$	$n = 30$	$n = 100$	$n = 250$	$n = 500$
$\hat{\alpha}$	0.6308	0.6247	0.6201	0.5985
MSE	$4.51.10^{-03}$	$4.11.10^{-03}$	$2.76.10^{-03}$	$1.83.10^{-03}$
$\hat{\beta}$	3.7501	3.4822	3.6689	3.6978
MSE	0.0519	0.0308	0.0024	0.0011
$\hat{a}$	1.4922	1.4957	1.4971	1.5094
MSE	0.04215	0.0167	0.0021	0.0013

and the loglikelihood function is given by

$$l(t,\boldsymbol{\theta}) = -\beta \ln\left(1 + \left\{\frac{\exp\left[-(a/t_i)^2\right]}{1-\exp\left[-(a/t_i)^2\right]}\right\}^{\alpha}\right) \times \sum_{i=1}^{n}\left[\delta_i\begin{pmatrix}\ln(\alpha\beta 2a^2) - 3\ln(t_i) - \alpha\,(a/t_i)^2 \\ -(\alpha+1)\ln\left\{1-\exp\left[-(a/t_i)^2\right]\right\}\end{pmatrix} - \ln\left(1 + \left\{\frac{\exp\left[-(a/t_i)^2\right]}{1-\exp\left[-(a/t_i)^2\right]}\right\}^{\alpha}\right)\right],$$

then

$$l(t,\boldsymbol{\theta}) = r\left[\log(\alpha) + \log(\beta) + 4\log(a)\right] - (\alpha+1)\sum_{i\in F}\ln\left\{1-\exp\left[-(a/t_i)^2\right]\right\} - 3\sum_{i\in F}\log(t_i) - \sum_{i\in F}\ln\left(1 + \left\{\frac{\exp\left[-(a/t_i)^2\right]}{1-\exp\left[-(a/t_i)^2\right]}\right\}^{\alpha}\right)$$

$$-\alpha\sum_{i\in F}(a/t_i)^2-\beta\sum_{i\in C}\ln\left(1+\left\{\frac{\exp\left[-(a/t_i)^2\right]}{1-\exp\left[-(a/t_i)^2\right]}\right\}^{\alpha}\right),$$

where r is the number of failures, F and C denote the sets of uncensored and censored observations, respectively. The maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}$ can be find by solving the system formed by equalizing the following score functions to zero

$$\begin{aligned}\frac{\partial l(t,\boldsymbol{\theta})}{\partial\alpha}&=\frac{r}{\alpha}-\sum_{i\in F}(a/t_i)^2-\sum_{i\in F}\ln\left\{1-\exp\left[-(a/t_i)^2\right]\right\}\\&\quad-\sum_{i\in F}\frac{\ln\left[m(t_i,\boldsymbol{\theta})\right]m^{\alpha}(t_i,\boldsymbol{\theta})}{1+m^{\alpha}(t_i,\boldsymbol{\theta})}-\beta\sum_{i\in C}\frac{\ln\left[m(t_i,\boldsymbol{\theta})\right]m^{\alpha}(t_i,\boldsymbol{\theta})}{1+m^{\alpha}(t_i,\boldsymbol{\theta})},\\\frac{\partial l(t,\boldsymbol{\theta})}{\partial\beta}&=\frac{r}{\beta}-\sum_{i\in C}\ln\left[1+m^{\alpha}(t_i,\boldsymbol{\theta})\right],\\\frac{\partial l(t,\boldsymbol{\theta})}{\partial a}&=\frac{4r}{a}-\alpha\sum_{i\in F}\frac{2a}{t_i^2}-(\alpha+1)\sum_{i\in F}\frac{2a}{t_i^2}m(t_i,\boldsymbol{\theta})\\&\quad-\sum_{i\in F}\frac{2a\alpha m^{\alpha}(t_i,\boldsymbol{\theta})}{t_i^2\left\{1-\exp\left[-(a/t_i)^2\right]\right\}^2\left[1+m^{\alpha}(t_i,\boldsymbol{\theta})\right]}\\&\quad-\beta\sum_{i\in C}\frac{2a\alpha m^{\alpha}(t_i,\boldsymbol{\theta})}{t_i^2\left\{1-\exp\left[-(a/t_i)^2\right]\right\}^2\left[1+m^{\alpha}(t_i,\boldsymbol{\theta})\right]},\end{aligned}$$

where

$$m(t_i,\boldsymbol{\theta})=\frac{\exp\left[-(a/t_i)^2\right]}{1-\exp\left[-(a/t_i)^2\right]}.$$

To solve the system of score functions, quite complicated, we use numerical methods, such as the Monte Carlo method, the Barzilai-Borwein (BB) algorithm or others.

4.2. Simulations: case of censored data

We consider the BXII-IR model. The data were simulated $N=10{,}000$ times (with sample sizes $n=30$, $n=100$, $n=250$, $n=500$) and parameter values $\alpha=0.58$, $\beta=2.64$ and $a=1.5$. The averages of the simulated values of the maximum likelihood estimators $\hat{\alpha},\hat{\beta},\hat{a}$ Parameters, and their mean squared errors (MSE) are calculated and presented in Table 6 (see the R code in Appendix 2). From Table 6, we can notice that the mean squared errors are very small, which confirms the convergence of the maximum likelihood estimators.

5. Goodness-of-fit test

In case of complete data, various techniques are used to verify the adequacy of mathematical models to data from observation. The most common tests are those based on Pearson's Chi-square statistics. Nevertheless, these cannot be applied in all situations, especially when the parameters of the model are unknown or when the data is censored. Since

Table 6. Maximum likelihood estimators $(\hat{\alpha}, \hat{\beta}, \hat{a})$ of the parameters and their mean squared errors (censored data).

$N = 10{,}000$	$n = 30$	$n = 100$	$n = 250$	$n = 500$
$\hat{\alpha}$	0.59007	0.58511	0.58054	0.58010
MSE	$8.33.10^{-03}$	$6.91.10^{-03}$	$5.79.10^{-03}$	$2.11.10^{-04}$
$\hat{\beta}$	2.6257	2.6251	2.6227	2.6514
MSE	0.01974	0.00198	$7.12.10^{-06}$	$3.09.10^{-06}$
$\hat{a}$	1.4842	1.4867	1.4895	1.5161
MSE	0.0018	0.0010	0.0005	0.0003

the middle of the last century, researchers have begun to propose modifications of existing statistics to take into account unknown parameters on the one hand and censorship on the other. For the complete data, Nikulin (1973) and Rao and Robson [36] separately proposed a statistic known today as the N.R.R statistic (Nikulin-Rao-Robson). This statistical test, which follows a chi-square distribution, is a natural modification of the Pearson statistic.

If, in addition to the unknown parameters, the data are censored, the classical tests are inadequate to verify a hypothesis H_0 according to which a series of observations comes from a parametric family $F(t)$. Habib and Thomas [22] considered the natural modifications of the N.R.R statistic. These tests are based on the differences between two probability estimators, one based on the Kaplan–Meier estimator, the other based on the maximum likelihood estimators of the unknown parameters of the cumulative distribution function of the Kaplan-Meier estimator. model tested. When to Bagdonavicius and Nikulin [13]; Bagdonavicius *et al.* [11], they proposed a modification of the N.R.R statistic that takes into account random right censorship. This statistic, based on the maximum likelihood estimators on the initial data, also follows a chi-square distribution at the limit. For more details on the construction of these statistics, we can see Voinov *et al.* [42]. These techniques were used to adjust observations to the generalized inverse Weibull model [19], the distribution of Birbaurm Saunders [34], the kumaraswamy generalized inverse Weibull distribution [20], Bertholon model [16].

In this work, we construct a modified chi-square type tests for the BXII-IR model case of complete and censored data. The N.R.R statistic is used on case of complete data. In the presence of censorship, we work with the modification of the N.R.R statistic proposed by Bagdonavicius and Nikulin [13].

5.1. Nikulin–Rao–Robson (N.R.R) statistic test

To test the hypothesis H_0 according to which $T_1, T_2, \ldots, T_n$, an $n-$sample comes from a parametric family $F(t;\boldsymbol{\theta})$

$$H_0 : P\{T_i \le t\} = F(t, \boldsymbol{\theta}), \quad t \in \mathbb{R}, \;\; \boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_s)^{\mathrm{T}},$$

where $\boldsymbol{\theta}$ represents the vector of unknown parameters, Nikulin (1973) and Rao and Robson [36] proposed Y^2 the N.R.R statistic defined as following:

Observations $T_1, T_2, \ldots, T_n$ are grouped in r subintervals $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_r$ mutually disjoint $\mathbf{I}_j =]a_j - 1; a_j]$; where $j = \overline{1; r}$.

The limits aj of the intervals $\mathbf{I}_j$ are obtained such that

$$p_j(\boldsymbol{\theta}) = \int_{a_{j-1}}^{a_j} f(t,\boldsymbol{\theta})\,\mathrm{d}t\Big|_{(j=1,2,\ldots,r)},$$

so

$$a_j = F^{-1}\left(\frac{j}{r}\right)\Big|_{(j=1,\ldots,r-1)}.$$

If $\nu_j = (\nu_1, \nu_2, \ldots, \nu_r)^{\mathrm{T}}$ is the vector of frequencies obtained by the grouping of data in these $\mathbf{I}_j$ intervals

$$\nu_j = \sum_{i=1}^{n} 1_{\{t_i \in \mathbf{I}_j\}}\,|_{(j=1,\ldots,r)}.$$

The N.R.R statistic is given by

$$Y^2(\hat{\boldsymbol{\theta}}_n) = X_n^2(\hat{\boldsymbol{\theta}}_n) + \frac{1}{n}\mathbf{L}^{\mathrm{T}}(\hat{\boldsymbol{\theta}}_n)(\mathbf{I}(\hat{\boldsymbol{\theta}}_n) - \mathbf{J}(\hat{\boldsymbol{\theta}}_n))^{-1}\mathbf{L}(\hat{\boldsymbol{\theta}}_n),$$

where

$$X_n^2(\boldsymbol{\theta}) = \left(\frac{\nu_1 - np_1(\boldsymbol{\theta})}{\sqrt{np_1(\boldsymbol{\theta})}}, \frac{\nu_2 - np_2(\boldsymbol{\theta})}{\sqrt{np_2(\boldsymbol{\theta})}}, \ldots, \frac{\nu_r - np_r(\boldsymbol{\theta})}{\sqrt{np_r(\boldsymbol{\theta})}}\right)^{\mathrm{T}}$$

and $\mathbf{J}(\boldsymbol{\theta})$ is the information matrix for the grouped data defined by

$$\mathbf{J}(\boldsymbol{\theta}) = B(\boldsymbol{\theta})^{\mathrm{T}} B(\boldsymbol{\theta}),$$

with

$$B(\boldsymbol{\theta}) = \left[\frac{1}{\sqrt{p_i}}\frac{\partial p_i(\boldsymbol{\theta})}{\partial \mu}\right]_{r\times s}\Bigg|_{(i=1,2,\ldots,r \text{ and } k=1,\ldots,s)},$$

then

$$\mathbf{L}(\boldsymbol{\theta}) = (\mathbf{L}_1(\boldsymbol{\theta}), \ldots, \mathbf{L}_s(\boldsymbol{\theta}))^{\mathrm{T}} \quad \text{with } \mathbf{L}_k(\boldsymbol{\theta}) = \sum_{i=1}^{r} \frac{\nu_i}{p_i}\frac{\partial}{\partial \boldsymbol{\theta}_k} p_i(\boldsymbol{\theta}),$$

where $\mathbf{I}_n(\hat{\boldsymbol{\theta}}_n)$ represents the estimated Fisher information matrix and $\hat{\boldsymbol{\theta}}_n$ is the maximum likelihood estimator of the parameter vector. The Y^2 statistic follows a distribution of chi-square χ^2_{r-1} with $(r-1)$ degrees of freedom.

5.2. N.R.R statistic for the BXII-IR model

Consider a sample $T = (T_1, T_2, \ldots, T_n)^{\mathrm{T}}$. To verify if these data are distributed according to the BXII-IR model, $P\{T_i \le t\} = F_{B\mathrm{XII}-IR}(t, \boldsymbol{\theta})$; with unknown parameters $\boldsymbol{\theta} = (\alpha, \beta, a)^{\mathrm{T}}$, a chi-square goodness-of-fit test is constructed by fitting the N.R.R statistic developed in the previous section. The maximum likelihood estimators $\hat{\boldsymbol{\theta}}_n$ of the unknown parameters of the BXII-IR distribution are computed on the initial data. The statistic Y^2 does not depend on the parameters, we can therefore use the Fisher information matrix estimated $\mathbf{I}_n(\hat{\boldsymbol{\theta}}_n)$. All the components of the statistic Y^2, for the distribution BXII-IR are provided, therefore Y^2 can be deduced easily.

Table 7. Empirical levels and corresponding theoretical levels ($\epsilon = 0.02, 0.05, 0.01, 0.1$).

$N = 10{,}000$	$\epsilon = 0.02$	$\epsilon = 0.05$	$\epsilon = 0.01$	$\epsilon = 0.1$
$n = 30$	0.9830	0.9524	0.9952	0.9050
$n = 50$	0.9823	0.9552	0.9930	0.9032
$n = 100$	0.9820	0.9519	0.9927	0.9028
$n = 250$	0.9807	0.9508	0.9913	0.9010
$n = 500$	0.9802	0.9501	0.9904	0.8999

5.3. Simulation studies (N.R.R statistics Y^2)

To support the results obtained in this work, we conduct an intensive study by numerical simulations. Thus, to test the null hypothesis H_0 that a sample belongs to the BXII-IR model, we calculate Y^2 the N.R.R statistic of 10,000 simulated samples with sizes $n = 30$, $n = 50$, $n = 100$, $n = 250$ and $n = 500$, respectively. For different theoretical levels ($\epsilon = 0.02, 0.05, 0.01, 0.1$); we calculate the average of the non-rejection numbers of the null hypothesis, when $Y^2 \leq \chi^2_\epsilon(r-1)$ then, we present the results of the corresponding empirical and theoretical levels in Table 7 (see the R code in Appendix 2). As can be seen, the values of the empirical levels calculated are very close to those of their corresponding theoretical levels. Thus, we conclude that the proposed test is well suited to the BXII-IR distribution.

6. Simulated distribution of Y^2 statistic for BXII-IR model

For demonstrating that the Y^2 statistic follows in the limit; a chi-squared distribution with $k = r - 1$ degrees of freedom; we compute $N = 10{,}000$ times, the simulated distribution of $Y^2(\hat{\theta})$ under the null hypothesis H_0 with different values of parameters BXII-IR (α, β, a), and $r = 14$ intervals, versus the chi-squared distribution with $k = 13$ degree of freedom. Their histograms are represented in Figure 4 versus the chi-squared distribution with k degree of freedom.

From Figure 4, we can observe that the statistical distribution of Y^2 with different values of parameters and different numbers k of grouping cells; in the limit follows a chi-squared with k degrees of freedom within the statistical errors of simulation. The same results is obtained for different number of equiprobable grouping intervals and different value of parameters. It is means that the limiting distribution of the generalized chi-squared Y^2 statistic is distribution free.

6.1. Applications to real data

6.1.1. Breaking stress of carbon fibers (in Gba) data

To test the null hypothesis H_0 that these data are adjusted by a BXII-IR distribution, we use the N.R.R statistic obtained previously. Using the R software and the BB algorithm [37], we compute the maximum likelihood estimators (MLE) $\hat{\alpha} = 23.1482$, $\hat{\beta} = 0.9578$ and $\hat{a} = 0.095541$. The estimated Fisher information matrix is then

$$\mathbf{I}(\hat{\boldsymbol{\theta}}) = \begin{pmatrix} 0.001547892 & 0.12487592 & 8.01975548 \\ 0.12487592 & 78.0218845 & 298.845512 \\ 8.01975548 & 98.8455122 & 187.001587 \end{pmatrix}.$$

We then deduce the value of $Y^2 = 20.128745$. For significance level $\epsilon = 0.01$, the critical value is $\chi^2_{0.01}(10-1) = 21.66599$, then, the N.R.R Y^2 statistic is less than the critical value, this allows us to say that these data correspond appropriately to the BXII-IR model.

6.1.2. Strengths of 1.5 cm glass fibers

Assuming that the Strengths of 1.5 cm glass fibers data can be fitted by our BXII-IR model, we can find (using the BB algorithm) the MLE's of the $\boldsymbol{\theta}$ vector of parameters as:

$$\hat{\boldsymbol{\theta}} = \left(\hat{\alpha}, \hat{\beta}, \hat{a}\right)^{\mathrm{T}} = (9.12547, 1.0548, 0.85472)^{\mathrm{T}}.$$

Using the $\hat{\boldsymbol{\theta}}$ value, the estimated Fisher information matrix is can be writing as:

$$\mathbf{I}(\hat{\boldsymbol{\theta}}) = \begin{pmatrix} 0.01024879 & 0.032541225 & 8.12459001 \\ 0.032541225 & 38.9517789 & 88.1299547 \\ 8.12459001 & 88.1299547 & 345.9001756 \end{pmatrix}.$$

After calculate, we give the N.R.R statistic test and the critical value as: $Y^2 = 10.96241$ and $\chi^2_{0.05}(7-1) = 12.59159$, respectively. We can affirm that data of 1.5 cm glass fibers can be modeled by our BXII-IR model with a satisfactory manner.

6.1.3. Gene expression Breast cancer data

We illustrate the use of our BXII-IR model by applying it to gene expression Breast cancer data, namely gene expression from breast tumors. We can find this data in R as a Package 'breastCancerNKI'. Genexpression dataset from a breast cancer study published by van't Veer *et al.* [41] and van de Vijver *et al.* [40], provided as an eSet. The source of this data is available at: http://www.rii.com/publications/2002/vantveer.html.

Using the BB algorithm and 'breastCancerNKI' package; the MLE's of the $\boldsymbol{\theta}$ vector of parameters are giving as follow:

$$\hat{\boldsymbol{\theta}} = \left(\hat{\alpha}, \hat{\beta}, \hat{a}\right)^{\mathrm{T}} = (3.2154, 1.18049, 1.5417)^{\mathrm{T}}.$$

The estimated Fisher information matrix is can be composing as:

$$\mathbf{I}(\hat{\boldsymbol{\theta}}) = \begin{pmatrix} 1.095475 & 0.0431855 & 432.07822 \\ 0.0431855 & 21.99547 & 53.4446 \\ 432.07822 & 53.4446 & 108.6587 \end{pmatrix}.$$

After calculate, the N.R.R statistic test Y^2 is 15.7210 and the critical value $\chi^2_{0.05}(10-1) = 16.91898$. We can aver that data of Gene expression Breast cancer can fit our BXII-IR model adequately.

7. Goodness-of-fit test for right censored data

To verify the adequacy of the BXII-IR model when the parameters are unknown and the data censored, we use the approach proposed by Bagdonavicius and Nikulin [13]; Bagdonavicius *et al.* [12] that we develop in this paragraph. It is a chi-square type test based on a modification of the N.R.R statistic. We adapt this test for a BXII-IR model. Let us consider the composite hypothesis

$$H_0 : F(t) \in F_0 = \left\{F_0(t, \boldsymbol{\theta}),\ t \in R^1,\ \boldsymbol{\theta} \in \Theta \subset R^s\right\},$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_s)^{\mathrm{T}} \in \Theta \subset R^s$ is an unknown m-dimensional parameter and F_0 is a differentiated completely specified cdf with the support $(0, \infty)$. Let us consider a finite time interval only say $[0, \tau]$, where τ is the maximum time of the study, and divide it into $k > s$ smaller intervals $\mathbf{I}_j = (a_{j-1}, a_j]$, where

$$0 =< a_0 < a_1 \cdots < a_{k-1} < a_k = +\infty.$$

In this case the estimated $\hat{a}_j$ is given by

$$\hat{a}_j = \boldsymbol{\Lambda}^{-1}\left((E_j - \sum_{l=1}^{i-1} \boldsymbol{\Lambda}(T_{(l)}, \hat{\boldsymbol{\theta}}))/(n-i+1), \hat{\theta}\right), \quad \widehat{a_k} = T_{(n)}|_{(j=1,\ldots,k)},$$

where $\hat{\boldsymbol{\theta}}$ is the maximum likelihood estimator of the parameter $\boldsymbol{\theta}$, $\boldsymbol{\Lambda}^{-1}$ is the inverse of cumulative hazard function $\boldsymbol{\Lambda}$, $T_{(i)}$ is the i^{th} element in the ordered statistics $(T_{(1)},,, T_{(n)})$ and

$$E_j = (n-i+1)\boldsymbol{\Lambda}(\widehat{a_j}, \hat{\boldsymbol{\theta}}) + \sum_{l=1}^{i-1} \boldsymbol{\Lambda}(T_{(l)}, \hat{\boldsymbol{\theta}}),$$

and a_j are random data functions such as the k intervals chosen have equal expected numbers of failures e_j. Usually in real application, we fix k. Bagdonavicius *et al.* [10] and Greenwood and Nikulin [21] give some recommendations for the choice of intervals. The test is based on the vector

$$\mathbf{Z} = (Z_1, \ldots, Z_k)^{\mathrm{T}},\ Z_j = \frac{1}{\sqrt{n}}(\mathbf{U}_j - e_j)\bigg|_{(j=1,2,\ldots,k)},$$

where $\mathbf{U}_j$ represent the numbers of observed failures in these intervals. The test for hypothesis H_0 can be based on the statistic

$$Y_n^2 = \mathbf{Z}^{\mathrm{T}}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{Z},$$

where

$$\hat{\boldsymbol{\Sigma}}^{-1} = \hat{\mathbf{A}}^{-1} + \hat{\mathbf{C}}^{-1}\hat{\mathbf{A}}^{\mathrm{T}}\hat{\mathbf{G}}^{-1}\hat{\mathbf{C}}\hat{\mathbf{A}}^{-1}$$

and

$$\hat{\mathbf{G}} = \hat{i} - \hat{\mathbf{C}}\hat{\mathbf{A}}^{-1}\hat{\mathbf{C}}^{\mathrm{T}}.$$

The test statistic can be written in the following form

$$Y_n^2 = \sum_{j=1}^{k} \frac{(\mathbf{U}_j - e_j)^2}{\mathbf{U}_j} + \mathbf{Q},$$

where

$$\hat{\mathbf{A}}_j = n^{-1}\mathbf{U}_j,$$

$$\hat{\mathbf{G}} = [\hat{g}_{ll'}]_{s\times s},$$
$$\mathbf{U}_j = \sum_{i:X_i\in \mathbf{I}_j} \delta_i,$$
$$\mathbf{Q} = \hat{\mathbf{W}}^{\mathrm{T}}\hat{\mathbf{G}}^{-1}\hat{\mathbf{W}},$$
$$\hat{\mathbf{C}}_{lj} = \frac{1}{n}\sum_{i:X_i\in \mathbf{I}_j} \delta_i \frac{\partial}{\partial \boldsymbol{\theta}} \ln\left[\lambda_i(t_i,\hat{\boldsymbol{\theta}})\right],$$
$$\hat{\mathbf{W}}_l = \sum_{j=1}^{k} \hat{\mathbf{C}}_{lj}\hat{\mathbf{A}}_j^{-1}\mathbf{Z}_j, \quad l,l' = 1,\ldots,s,$$
$$\hat{\mathbf{W}} = (\hat{\mathbf{W}}_1,\ldots,\hat{\mathbf{W}}_s)^{\mathrm{T}},$$
$$\hat{i}_{ll'} = n^{-1}\sum_{i=1}^{n} \delta_i \frac{\partial}{\partial \boldsymbol{\theta}_l} \ln\left[\lambda_i(t_i,\hat{\boldsymbol{\theta}})\right] \frac{\partial}{\partial \boldsymbol{\theta}_{l'}} \ln\left[\lambda_i(t_i,\hat{\boldsymbol{\theta}})\right]$$

and

$$\hat{g}_{ll'} = \hat{i}_{ll'} - \sum_{j=1}^{k} \hat{\mathbf{C}}_{lj}\hat{\mathbf{C}}_{l'j}\hat{A}_j^{-1}, \quad \hat{\mathbf{C}}_{lj} = \frac{1}{n}\sum_{i:X_i\in \mathbf{I}_j} \delta_i \frac{\partial}{\partial \boldsymbol{\theta}} \ln \lambda_i(t_i,\hat{\boldsymbol{\theta}}),$$

calculation of the matrices $\hat{\mathbf{W}}$ and $\hat{\mathbf{I}}$ are given in the Appendix 3. The limit distribution of the statistic Y_n^2 is chi-square with $r = \mathrm{rank}(\boldsymbol{\Sigma}) = tr(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma})$ degrees of freedom. If $\mathbf{G}$ is non-degenerate then $r = k$. The hypothesis is rejected with approximate significance level ϵ if $Y_n^2 > \chi_\epsilon^2(r)$ where $\chi_\epsilon^2(r)$ is the quantile of chi-square with r degrees of freedom. For more details, see Bagdonavicius and Nikulin [13] and Bagdonavicius *et al.* [12].

7.1. Goodness-of-fit test for the BXII-IR model in case of censored data

In this section, we study the validity of the BXII-IR model, by a goodness-of-fit test based on Y_n^2, the modified N.R.R statistic presented in the previous section. Suppose H_0 is checked, that is, the failure rate T_i follows an BXII-IR distribution, the survival function is:

$$S(t,\boldsymbol{\theta}) = 1 - F(t;\alpha,\beta,a) = \left(1 + \left\{\frac{\exp\left[-(a/t)^2\right]}{1-\exp\left[-(a/t)^2\right]}\right\}^{\alpha}\right)^{-\beta}.$$

The choice of $\hat{a}_j$ when the baseline distribution is the BXII-IR model, is obtained as follows:

First, we have

$$\boldsymbol{\Lambda}_{BXII-IR}(t,\boldsymbol{\theta}) = -\ln S(t,\boldsymbol{\theta}) = \beta \ln\left(1 + \left\{\frac{\exp\left[-(a/t)^2\right]}{1-\exp\left[-(a/t)^2\right]}\right\}^{\alpha}\right)$$

and

$$t = a\left(\ln\left[1 + \left[\exp\left(y/\beta\right) - 1\right]^{\frac{-1}{\alpha}}\right]\right)^{\frac{-1}{2}},$$

Table 8. Power of Y^2 for BXII-IR against.

W-IW ($a = 0.2,\ b = 0.8,\ \alpha = 1.3,\ \beta = 0.5$).			
$N = 10{,}000$	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.05$
$n = 30$	0.486	0.511	0.566
$n = 100$	0.549	0.642	0.751
$n = 200$	0.693	0.860	0.895
$n = 500$	0.892	0.909	0.981
Power of Y^2 for BXII-IR against K-IW ($a = 1.3,\ b = 0.5,\ \alpha = 1.5,\ \beta = 0.9$).			
$N = 10{,}000$	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.05$
$n = 30$	0.497	0.538	0.551
$n = 100$	0.562	0.687	0.722
$n = 200$	0.699	0.860	0.889
$n = 500$	0.907	0.919	0.989

$$\widehat{a_j} = a\left[\ln\left(1+\left\{\exp\left[\frac{E_j - \sum_{l=1}^{i-1}\boldsymbol{\Lambda}(x_{(l)},\hat{\boldsymbol{\theta}})}{\beta\,(n-i+1)}\right]-1\right\}^{\frac{-1}{\alpha}}\right)\right]^{\frac{-1}{2}}\Bigg|_{(j=1,\ldots,k-1)},$$

$$\hat{a}_k = t_{(n)},$$

where

$$E_j = \sum_{i:X_i>a_j}(\boldsymbol{\Lambda}(a_j \wedge t_i,\hat{\boldsymbol{\theta}}) - \boldsymbol{\Lambda}(a_{j-1},\hat{\boldsymbol{\theta}}) \quad \text{and} \quad E_k = \sum_{i=1}^{n}\boldsymbol{\Lambda}(t_i,\hat{\boldsymbol{\theta}}).$$

Under such choice of intervals we have a constant value of $e_j = E_k/k$ for any j. There is no explicit form of the inverse hazard function of BXII-IR distribution, so we can estimate intervals by iterative method.

7.2. Simulation study

7.2.1. Power of Y^2

In order to evaluate the powerful of Y^2 statistic tests for BXII-IR model, we have considered two alternative hypotheses Weibull-inverse Weibull (W-IW) and Kumaraswamy IW (K-IW) distributions. The generated samples ($N = 10,000$) are assumed to be censored at 25% and $r = 6$ grouping intervals. As expected, the results given in Tables 8 show that our models BXII-IR model can be used instead of the W-IW and K-IW models.

To test the null hypothesis H_0 that a sample comes from a BXII-IR model, we calculate Y_n^2 the N.R.R statistic of 10,000 simulated samples with sizes $n = 30,\ n = 150,\ n = 250,\ n = 500$, respectively. For different levels of meaning ($\epsilon = 0.02, 0.05, 0.01, 0.1$); we calculate the mean of the number of no rejections of the null hypothesis when $Y_n^2 \leq \chi_\epsilon^2(r)$, then we present the results of the empirical values and the corresponding theoretical values in Table 9.

According to this results, we find that the empirical signification levels of the Y_n^2 statistic coincide with those corresponding to the theoretical levels of the chi-square distributions at r degrees of freedom. Therefore, we can say that the proposed test can properly fit censored data from the BXII-IR distribution.

Table 9. Empirical levels and corresponding theoretical levels ($\epsilon = 0.02; 0.05; 0.01; 0.1$).

$N = 10{,}000$	$\epsilon = 0.02$	$\epsilon = 0.05$	$\epsilon = 0.01$	$\epsilon = 0.1$
$n = 30$	0.9831	0.9521	0.9929	0.9022
$n = 150$	0.9815	0.9510	0.9918	0.9018
$n = 250$	0.9803	0.9505	0.9910	0.9011
$n = 500$	0.9798	0.9498	0.9903	0.9898

7.3. Application to real data

7.3.1. Aluminum reduction cells data

The data of Whitmore [43], who considered the times of failures for 20 aluminum reduction cells, and the numbers of failures in 1,000 days units are : 0.468, 0.725, 0.838, 0.853, 0.965-1.139, 1.142, 1.304, 1.317, 1.427, 1.554, 1.658, 1.764, 1.776, 1.990, 2.010, 2.224, 2.279*, 2.244*, 2.286*. (* censoring). Assuming that these data are distributed according to the BXII-IR distribution, the maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ of the parameter vector $\boldsymbol{\theta}$ is:

$$\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{a})^{\mathrm{T}} = (1.02587, 1.2549, 0.94127)^{\mathrm{T}}$$

We choose $r = 4$ a number of classes. The element of the statistic test Y_n^2 are presented as:

$\widehat{a_j}$	0.9432	1.2109	1.6681	2.2945
$\widehat{\mathbf{U}_j}$	4	3	5	8
e_j	2.8541	2.8541	2.8541	2.8541
$\widehat{\mathbf{C}_{1j}}$	−0.54876	−0.28891	0.31025	0.01872
$\widehat{\mathbf{C}_{2j}}$	0.21547	−0.23541	0.18759	0.21973
$\widehat{\mathbf{C}_{3j}}$	0.00458	0.01224	−0.81207	0.06301

The fisher's estimated matrix is given by:

$$\hat{\mathbf{I}} = \begin{pmatrix} -1.52403 & 3.25487 & -0.45178 \\ 3.25487 & -0.98547 & 0.84512 \\ -0.45178 & 0.84512 & 2.00548 \end{pmatrix}$$

Then, we can calculate the value of the statistic test $Y_n^2 = 9.1098$. The critical value is $\chi^2_{0.05}(4) = 9.4877 > Y_n^2$, we conclude that the data of Aluminum reduction cells is in concordance with the BXII-IR model.

7.3.2. Arm-A head and neck cancer data

The data considered below (was conducted by northern California oncology group) was used by Efron [18] for logistic distribution. Mudholkar *et al.* [29] and Nikulin and Haghighi [35] reanalysed the same data and give the acceptable fit (chi-square type test) to the exponentiated Weibull and generalized Weibull distribution families, respectively.

The survival times in days for the patients ($n = 51$) were as below ($\delta = 42$). 7, 34, 42, 63, 64, 74*, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 173, 176, 185*, 218, 225, 241, 248, 273, 277, 279*, 297, 319*, 405, 417, 420, 440, 523*, 523, 583, 594, 1101, 1116*, 1146, 1226*, 1349*, 1412*, 1417. * censoring We use the data after transforming the survival times in months (1month = 30.438 days). The maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ of the parameter vector $\boldsymbol{\theta}$ is, if we suppose that this data are distributed according to the BXII-IR distribution:

$$\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{a})^{\mathrm{T}} = (2.1451, 0.9845, 3.1477)^{\mathrm{T}}$$

We choose $r = 7$ as a number of classes. The elements of the test statistic Y_n^2 was presented as follows:

$\widehat{a_j}$	2.742	4.547	9.512	21.008	37.0094	44.479	46.903
$\widehat{\mathbf{U}_j}$	7	7	20	10	2	3	2
e_j	2.8105	2.8105	2.8105	2.8105	2.8105	2.8105	2.8105
$\widehat{\mathbf{C}_{1j}}$	−0.2014	−0.3225	0.3580	0.0589	−0.2185	0.4001	0.0457
$\widehat{\mathbf{C}_{2j}}$	−0.5119	0.4058	0.0984	0.2546	0.6521	−0.2142	0.6300
$\widehat{\mathbf{C}_{3j}}$	0.09447	0.1882	0.0921	0.1163	0.2481	0.0887	0.7102

The fisher's estimated matrix is:

$$\hat{\mathbf{I}} = \begin{pmatrix} -1.458702 & 1.569821 & -1.922301 \\ -1.922301 & 3.988510 & 0.719854 \\ 0.3987176 & 0.719854 & -0.249836 \end{pmatrix},$$

after calculate, we find $Y_n^2 = 13.67849$. The critical value $\chi^2_{0.05}(7) = 14.06714 > Y_n^2 = 13.67849$, we can say that this data can be well modelised by the our BXII-IR model.

8. Conclusion

In this work, we are interested in the Burr XII inverse Rayleigh distribution (BXII-IR) model which is a generalization of the IR distribution and whose mathematical form of its probability density is manageable thus allowing to calculate its various characteristics. This flexible model can describe different lifetimes from reliability, survival analysis, and other areas. Since the results of any statistical analysis depend on the chosen model, then we have constructed modified chi-square type fit tests to allow users to verify the adequacy of their observations to these types of distributions. The tests used take into account the unknown parameters of the models, right censorship generally present in the reliability and survival analysis studies, and use all the information provided by the sample. We have shown the applicability of this new model (BXII-IR) by a study of two real complete data and two others for the case of censored data.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix 1

Application I (Tables 1–2)

```
# Application I ===== Breaking stress of carbon fibers
x = c(0.92, 0.928, 0.997, 0.9971, 1.061, 1.117, 1.162, 1.183, 1.187, 1.192, 1.196, 1.213,
1.215, 1.2199, 1.22, 1.224, 1.225, 1.228, 1.237, 1.24, 1.244, 1.259, 1.261, 1.263, 1.276,
1.31, 1.321, 1.329, 1.331, 1.337, 1.351, 1.359, 1.388, 1.408, 1.449, 1.4497, 1.45, 1.459,
1.471, 1.475, 1.477, 1.48, 1.489, 1.501, 1.507, 1.515, 1.53, 1.5304, 1.533, 1.544, 1.5443,
1.552, 1.556, 1.562, 1.566, 1.585, 1.586, 1.599, 1.602, 1.614, 1.616, 1.617, 1.628, 1.684,
1.711, 1.718, 1.733, 1.738, 1.743, 1.759, 1.777, 1.794, 1.799, 1.806, 1.814, 1.816, 1.828,
1.83, 1.884, 1.892, 1.944, 1.972, 1.984, 1.987, 2.02, 2.0304, 2.029, 2.035, 2.037, 2.043,
2.046, 2.059, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.306)
hist(x)
#===================================
cdf_BXIIIR <- function(par,x){
alpha=par[1]
beta=par[2]
a=par[3]
g=2*(a^2)*x^(-(2+1))*(exp(-((a/x)^(2))))
G=exp(-((a/x)^(2)))
g.BXII =alpha*beta*g*G^(alpha-1)*(1-G)^ (-alpha-1)*(1+(G/(1-G))^alpha)^(-beta-1)
G.BXII =1-(1+(G/(1-G))^(alpha))^(-beta)
return(G.BXII)
}
pdf_BXIIIR <- function(par,x){
alpha=par[1]
beta=par[2]
a=par[3]
g=b*(a^2)*x^(-(2+1))*(exp(-((a/x)^(2))))
G=exp(-((a/x)^(2)))
g.BXII =alpha*beta*g*G^(alpha-1)*(1-G)^ (-alpha-1)*(1+(G/(1-G))^alpha)^(-beta-1)
G.BXII =1-(1+(G/(1-G))^(alpha))^(-beta)
return(g.BXII)
}
goodness.fit(pdf=pdf_BXIIIR, cdf=cdf_BXIIFr, starts = c(1,1,1), data=x,
method="N", domain=c(0,Inf), mle=NULL)
#===================================
```

Application II (Tables 3–4)

```
# Application I ===== Strength of glass fibers
x = c(1.014, 1.081, 1.082, 1.185, 1.223,
1.248, 1.267, 1.271, 1.272, 1.275, 1.276, 1.278, 1.286, 1.288, 1.292, 1.304, 1.306, 1.355,
1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 1.46, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524,
1.526, 1.535, 1.541, 1.568, 1.579, 1.581, 1.591, 1.593, 1.602, 1.666, 1.67, 1.684, 1.691,
1.704, 1.731, 1.735, 1.747, 1.748, 1.757, 1.800, 1.806, 1.867, 1.876, 1.878, 1.91, 1.916,
1.972, 2.012, 2.456, 2.592, 3.197, 4.121)
hist(x)
#===================================
cdf_BXIIIR <- function(par,x){
alpha=par[1]
beta=par[2]
a=par[3]
```

```
g = 2*(a^2)*x^(-(2+1))*(exp(-((a/x)^(2))))
G = exp(-((a/x)^(2)))
g.BXII = alpha*beta*g*G^(alpha-1)*(1-G)^ (-alpha-1)*(1+(G/(1-G))^alpha)^(-beta-1)
G.BXII = 1-(1+(G/(1-G))^(alpha))^(-beta)
return(G.BXII)
}
pdf_BXIIIR < - function(par,x){
alpha = par[1]
beta = par[2]
a = par[3]
g = b*(a^2)*x^(-(2+1))*(exp(-((a/x)^(2))))
G = exp(-((a/x)^(2)))
g.BXII = alpha*beta*g*G^(alpha-1)*(1-G)^ (-alpha-1)*(1+(G/(1-G))^alpha)^(-beta-1)
G.BXII = 1-(1+(G/(1-G))^(alpha))^(-beta)
return(g.BXII)
}
goodness.fit(pdf = pdf_BXIIIR, cdf = cdf_BXIIFr, starts = c(1,1,1), data = x,
method = "N", domain = c(0,Inf), mle = NULL)
```

Appendix 2 (Table 5)

```
#p represents parameters
#dd represents score fonctions
library(BB)
library(nleqslv)
g < - function(p){n = 30;
si < - exp(-((p[3]/(t))^2)
zi < - (1+(((si)/(1-si)))^2)
pi < - (((si)/(1-si)))^(p[1])*log(((si)/(1-si)))
mi < - -2*p[1]*(((si)/(1-si)))^ (p[1]-1)*((((si)/(t))((p[3]/(t))))/((1-si)^2))
dd < - rep(NA, length(p))
dd[1] < -(n/p[1])-sum((p[3])/(t)^ 2)-sum(log(1-si))-(p[2]+1)*sum((pi/zi))
dd[2] < -(n/p[2])-sum(log(zi))
dd[3] < -((2*n)/p[3])-2*p[1]*sum((1/t)*(p[3]/(t)))-2*(p[1]+1)*sum((((p[3]/t))((si)/(t)))/(1-si))
-(p[2]+1)*sum((mi)/(zi))
dd
}
p0 < - rep(0.7, 0.5, 6, 5, 0.5) ##We can chage it##
BBsolve(par = p0, fn = g)
BBsolve(par = p0, fn = g)$par
nleqslv(x = p0,fn = g)
```

(Table 6)

```
library(BB)
library(nleqslv)
gg < - function(p){n = 100;
mii = ((exp[-(p[3]/t)^2])/(1-exp[-(p[3]/t)^2]))
dd < - rep(NA, length(p))
dd[1] < -(r/p[1])-sum(p[3]/t)^2-sum(log(1-exp(-(p[3]/t) ^2)))-sum((log(mii)*mii^(p[1]))/(1+
mii^ (p[1])*mii))
-p[2]*sum(log(mii)*mii^(p[1])/(1+mii^(p[1])))
dd[2] < - (r/p[2])-sum(log(1+mii^(p[1])))
dd[3] < - ((4*r)/p[3])-p[1]*sum((2*p[3])/(t^ 2))-(p[1]+1)*sum((2*p[3])/t^2*mii)
-sum((2*p[3]*p[1]*mii^(p[1]))/(t^2*(1-exp(-(p[3]/t ^2))^2*(1+mii^(p[1])))))
```

```
-p[2]*sum((2*p[3]*p[1]*mii^(p[1]))/(t^2(1-exp(-(p[3]/t) ^2))^2(1+mii^(p[1]))))
dd
}
p0 < - matrix(runif(300), 100, 3)
BBsolve(par = p0, fn = gg, method=c(2,3,1))
nleqslv(x=p0,fn=g)
```

(Table 7)

```
r < -round(1+2.303*log(n,10))
a < -1:(r-1); p < -1:r ; a0 < -+1e-50; ar < -10000; v < -1:r
for (j in 1:r) {v[j] < -0}
XX < -1:r;
for (i in 1:r) {XX[i] < -(v[i]-n*p[i])^2/(n*p[i])};
chi2 < -sum(XX)
WW < -t(informant)%*% solve(mat)%*%(informant)
Y2n < -chi2+(1/n)*WW
ca < -qchisq(0.95,r-1)
ca
if (Y2n < ca) {print(" H0 est acceptée ")} else print(" H0 est rejetée")
Y2n
```

Appendix 3

Calculation of the matrix $\hat{W}$

The elements of the estimated matrix $\hat{\mathbf{W}}$ defined by

$$\hat{\mathbf{W}}_l = \sum_{j=1}^{k} \hat{\mathbf{C}}_{lj} \hat{\mathbf{A}}_j^{-1} \mathbf{Z}_j|_{(l=1,2,3.\ j=1,\ldots,k)}$$

are obtained as follows

$$\widehat{\mathbf{C}_{lj}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \frac{\partial}{\partial \boldsymbol{\theta}} \ln \lambda(t_i, \hat{\boldsymbol{\theta}}),$$

$$\ln \lambda(t, \hat{\boldsymbol{\theta}}) = \ln(\alpha) + \ln(\beta) + 4\ln(a) - \alpha\,(a/t_i)^2 - (\alpha+1)\ln\left\{1 - \exp\left[-(a/t_i)^2\right]\right\}$$
$$- 3\ln(t_i) - \ln\left(1 + \left\{\frac{\exp\left[-(a/t_i)^2\right]}{1 - \exp\left[-(a/t_i)^2\right]}\right\}^{\alpha}\right).$$

The expressions of the elements of the matrix $\widehat{\mathbf{C}_{lj}}$ are given as follows

$$\widehat{\mathbf{C}_{1j}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \left[\frac{1}{\alpha} - (a/t_i)^2 - \ln\left\{1 - \exp\left[-(a/t_i)^2\right]\right\} - \frac{m^{\alpha}(t_i, \boldsymbol{\theta}) \ln\left(m(t_i, \boldsymbol{\theta})\right)}{1 + m^{\alpha}(t_i, \boldsymbol{\theta})}\right],$$

$$\widehat{\mathbf{C}_{2j}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \left[\frac{1}{\beta}\right],$$

$$\widehat{\mathbf{C}_{3j}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \left\{\begin{array}{c} \frac{4}{a} - 2a\alpha/t_i - 2a/t_i^2\,(\alpha+1)\,m(t_i, \boldsymbol{\theta}) \\ -\alpha\left[-2a/t_i^2 m(t_i, \boldsymbol{\theta})\right]^{\alpha-1}\left[1 + m^{\alpha}(t_i, \boldsymbol{\theta})\right]^{-1} \end{array}\right\},$$

where

$$m(t_i, \boldsymbol{\theta}) = \frac{\exp\left[-(a/t_i)^2\right]}{1 - \exp\left[-(a/t_i)^2\right]}.$$

Calculation of the matrix $\hat{I}$

The formulas of the elements of the Fisher's information matrix $\hat{\mathbf{I}} = (\hat{i}_{ll'})_{3\times 3}$ is

$$\widehat{i_{ll'}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \frac{\partial \ln \lambda(t_i, \hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}_l} \frac{\partial \ln \lambda(t_i, \hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}_{l'}}.$$

In our case we have:

$$\widehat{i_{11}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \left[\left(\frac{1}{\alpha} - (a/t_i)^2 - \ln\left\{1 - \exp\left[-(a/t_i)^2\right]\right\} - \frac{m^{\alpha}(t_i, \boldsymbol{\theta}) \ln\left[m(t_i, \boldsymbol{\theta})\right]}{1 + m^{\alpha}(t_i, \boldsymbol{\theta})} \right)^2 \right],$$

$$\widehat{i_{12}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \left[\frac{1}{\beta} \left(\frac{1}{\alpha} - (a/t_i)^2 - \ln\left\{1 - \exp\left[-(a/t_i)^2\right]\right\} - \frac{m^{\alpha}(t_i, \boldsymbol{\theta}) \ln\left[m(t_i, \boldsymbol{\theta})\right]}{1 + m^{\alpha}(t_i, \boldsymbol{\theta})} \right) \right],$$

$$\widehat{i_{13}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \left(\begin{array}{c} \left\{ \begin{array}{c} \frac{1}{\alpha} - (a/t_i)^2 - \ln\left\{1 - \exp\left[-(a/t_i)^2\right]\right\} \\ -\frac{m^{\alpha}(t_i, \boldsymbol{\theta}) \ln\left[m(t_i, \boldsymbol{\theta})\right]}{1 + m^{\alpha}(t_i, \boldsymbol{\theta})} \end{array} \right\} \\ \times \left\{ \begin{array}{c} \frac{4}{a} - 2a\alpha/t_i - 2a/t_i^2 (\alpha + 1)\, m(t_i, \boldsymbol{\theta}) \\ -\alpha\left[-2a/t_i^2 m(t_i, \boldsymbol{\theta})\right]^{\alpha - 1} \left[1 + m^{\alpha}(t_i, \boldsymbol{\theta})\right]^{-1} \end{array} \right\} \end{array} \right),$$

$$\widehat{i_{22}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \left(\frac{1}{\beta^2} \right),$$

$$\widehat{i_{23}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \left\{ \frac{1}{\beta^2} \left[\begin{array}{c} \frac{4}{a} - 2a\alpha/t_i - 2a/t_i^2 (\alpha + 1)\, m(t_i, \boldsymbol{\theta}) \\ -\alpha\left[-2a/t_i^2 m(t_i, \boldsymbol{\theta})\right]^{\alpha - 1} \left(1 + m^{\alpha}(t_i, \boldsymbol{\theta})\right)^{-1} \end{array} \right] \right\},$$

and

$$\widehat{i_{33}} = \frac{1}{n} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \left\{ \frac{4}{a} - 2a\alpha/t_i - 2a/t_i^2 (\alpha + 1)\, m(t_i, \boldsymbol{\theta}) - \alpha\left[-2a/t_i^2 m(t_i, \boldsymbol{\theta})\right]^{\alpha - 1} \left[1 + m^{\alpha}(t_i, \boldsymbol{\theta})\right]^{-1} \right\}^2.$$

Notice that, the components of the information matrix $\hat{\mathbf{I}}$ are required for computation of the statistic Y_n^2.