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A new two-parameter exponentiated discrete Lindley distribution: properties, estimation and applications

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ABSTRACT
This paper introduces a new two-parameter exponentiated discrete Lindley distribution. A wide range of its structural properties are investigated. This includes the shape of the probability mass function, hazard rate function, moments, skewness, kurtosis, stress–strength reliability, mean residual lifetime, mean past lifetime, order statistics and L-moment statistics. The hazard rate function can be increasing, decreasing, decreasing–increasing–decreasing, increasing–decreasing–increasing, unimodal, bathtub, and *J*-shaped depending on its parameters values. Two methods are used herein to estimate the model parameters, namely, the maximum likelihood, and the proportion. A detailed simulation study is carried out to examine the bias and mean square error of maximum likelihood and proportion estimators. The flexibility of the proposed model is explained by using four distinctive data sets. It can serve as an alternative model to other lifetime distributions in the existing statistical literature for modeling positive real data in many areas.

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Discrete lindley distribution; hazard rate function; L-moment statistics; mean residual lifetime; estimation methods

1. Introduction

Statistical distributions are commonly applied to describe and predict the probabilistic behavioral patterns of real-world phenomena. Several classical distributions have been extensively used for modeling data in several fields, especially, in medical, ecology, renewable energy and survival analysis fields. See for example, El-Gohary *et al.* [12], El-Bassiouny *et al.* [10,11], El-Morshedy *et al.* [14], El-Morshedy, Eliwa [13] and Alizadeh *et al.* [1], among others. The Lindley (Li) distribution is one of those distributions, since it has some favorable properties to be used in lifetime data analysis, and especially in applications modeling stress-strength model (see [29]). This distribution can be expressed as a mixture of exponential and gamma distributions. The cumulative distribution function (CDF), and the probability density function, of the Li distribution are respectively given by

$$\prod(x;\theta) = 1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta + 1}\right); \quad x > 0, \tag{1}$$

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$$\pi(x;\theta) = \frac{\theta^2}{1+\theta}(x+1)e^{-\theta x}; \quad x > 0, \tag{2}$$

where $\theta > 0$ is a scale parameter. The crucial importance of the Li distribution in solving lifetime modeling problems urges developing flexible flavors and generalizations of the Li distribution. See for example, Mahmoudi and Zakerzadeh [31], Nadarajah *et al.* [35], Bakouch *et al.* [4], Merovci and Elbatal [32], Merovci and Sharma [33], Liyanage and Pararai [30], Zeghdoudi and Nedjar [47], Özel *et al.* [39], Altun *et al.* [2], Jehhan *et al.* [23] and references cited therein.

On the other hand, in several cases, lifetimes need to be recorded on a discrete scale rather than on a continuous analogue. Therefore, discretizing continuous distributions has received much attention in the statistical literature. See for example, Roy [43], Inusah and Kozubowski [22], Krishna and Pundir [25], Ghitany and Al-Mutairi [16], Gó mez-Déniz [19], Gómez-Déniz and Calderín-Ojeda [20], Bebbington *et al.* [5], Calderín-Ojeda and Gómez-Déniz [6], Nekoukhou *et al.* [36], Bakouch *et al.* [3], Tanka and Srivastava [44], Munindra *et al.* [34], Nekoukhou and Bidram [37,38], Chandrakant *et al.* [8], Para and Jan [41], Kundu and Nekoukhou [27], Kus *et al.* [26], and references cited therein.

Although there are a number of discrete distributions in the literature, there is still a lot of space left to develop new discretized distributions that are suitable under different conditions like discrete Lindley (DLi) distribution for example. In this paper, we introduce a new discrete distribution with two parameters, referred to as the exponentiated discrete Lindley (EDLi) distribution.

Some characteristics of the EDLi distribution can be summarized as follows: it has closed forms for both reliability function (RF) and hazard rate function (HRF). Moreover, its HRF may assume different shapes, and consequently, the parameters of the underlying distribution can be adjusted to suit most data sets. Secondly, it provides more flexibility than the DLi distribution to model time and count data sets. It has more flexibility than the Poisson distribution, to model actuarial data that commonly suffers from the over-dispersion phenomenon. Lastly, the proposed EDLi distribution provides the best fit for both times and counts data in spite of having only two parameters. So, it can be used for modeling data in survival analysis, reliability and failure times. We believe that the EDLi distribution is well-suited to attract a wider set of applications and fields, including problems in medicine, engineering, amongst others.

2. The EDLi distribution

Recently, Gómez-Déniz and Calderín-Ojeda [20] introduced the DLi distribution. The CDF of the DLi distribution and its corresponding probability mass function (PMF) can be expressed as follows

$$W(x;a) = P(X \le x) = \frac{1 - a^{x+1} + \left[(2+x)\,a^{x+1} - 1\right]\log a}{1 - \log a}; \quad x \in \mathbb{N}_0 \tag{3}$$

and

$$w(x,a) = P(X = x) = \frac{a^x}{1 - \log a}[a\log a + (1-a)(1 - \log a^{x+1})]; \quad x \in \mathbb{N}_0, \tag{4}$$

respectively, where $0 < a = e^{-\theta} < 1$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. In the context of lifetime distributions with CDF W, the most widely used generalization technique is the exponentiated-W. Using this method, for $b > 0$, the CDF of the exponentiated-W class can be defined as follows

$$F(x; a, b) = [W(x; a)]^b, \tag{5}$$

(see [28]). Therefore, the random variable (RV) X is said to have the EDLi distribution with scale parameter $0 < a < 1$ and shape parameter $b > 0$ if its CDF is given by

$$F(x; a, b) = \frac{\Lambda(x+1; a, b)}{(1-\log a)^b}; \quad x \in \mathbb{N}_0, \tag{6}$$

where $\Lambda(x; a, b) = (1 - a^x + [(1+x)a^x - 1] \log a)^b$. The PMF of the EDLi distribution can be expressed as follows

$$f(x; a, b) = \frac{1}{(1-\log a)^b} [\Lambda(x+1; a, b) - \Lambda(x; a, b)]; \quad x \in \mathbb{N}_0. \tag{7}$$

Figure 1 shows the PMF plots for various values of the model parameters.

From Figure 1, it can be inferred that the EDLi distribution is always unimodal which is the case for log-concave PMFs in general. The HRF of the EDLi distribution can be expressed as follows

$$h(x; a, b) = \frac{f(x; a, b)}{R(x; a, b)} = \frac{\Lambda(x+1; a, b) - \Lambda(x; a, b)}{(1-\log a)^b - \Lambda(x; a, b)}; \quad x \in \mathbb{N}_0. \tag{8}$$

where $R(x; a, b) = P(X \geq x) = ((1-\log a)^b - \Lambda(x; a, b))/((1-\log a)^b)$. Figure 2 shows the HRF plots for various values of the model parameters.

As we see from Figure 2, a characteristic of the EDLi distribution is that its HRF can be increasing, decreasing, decreasing–increasing–decreasing, increasing–decreasing–increasing, unimodal, bathtub and J-shaped, which makes the proposed distribution more flexible to fit different data sets. Hence, the EDLi distribution is more flexible than other discrete distributions such as geometric (Geo), discrete generalized exponential second type (DGE II) and DLi distributions. Also, the reversed hazard rate function (RHRF) of the EDLi distribution can be expressed as follows

$$r(x; a, b) = 1 - \frac{\Lambda(x; a, b)}{\Lambda(x+1; a, b)}; \quad x \in \mathbb{N}_0. \tag{9}$$

Figure 3 shows the RHRF plots for various values of the model parameters.

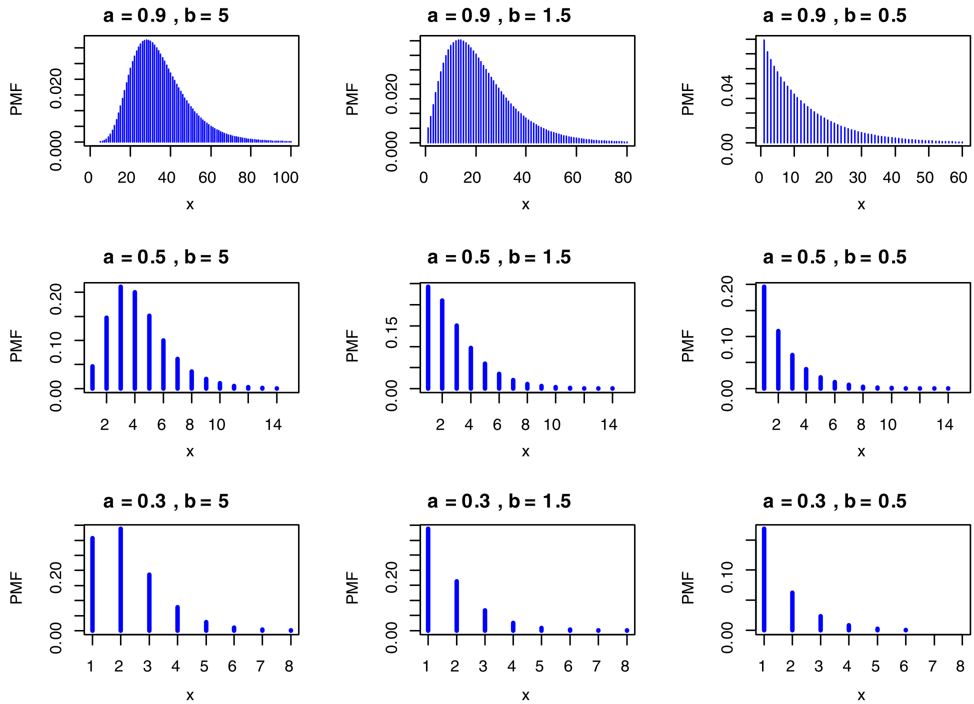

Figure 1. The PMF of the EDLi distribution.

3. Different properties

3.1. Moments

Assume non-negative RV $X \sim \text{EDLi}(a, b)$. Then, the rth moment, say $\varpi_r^{'}$, can be expressed as follows

$$\varpi_r^{'} = \sum_{x=0}^{\infty} x^r f(x; a, b)$$
$$= \frac{1}{(1 - \log a)^b} \sum_{x=0}^{\infty} x^r \left[\Lambda(x+1; a, b) - \Lambda(x; a, b)\right]. \qquad (10)$$

It is not possible to get a closed form of the rth moment, and consequently, Maple software is required to discuss this property numerically. Equation (10) is convergence for $0 < a < 1$ and $b > 0$. The mean and variance of the EDLi distribution for different values of its parameters are listed in Tables 1 and 2, respectively, based on a unique random sample.

It is evident that the mean and variance increase with $a \longrightarrow 1$ for fixed value of b or with $b \longrightarrow \infty$ for fixed value of a. In addition, the EDLi distribution is appropriate for modeling both over- and under-dispersed data since, in this model, the variance can be larger or smaller than the mean which is not the case with some standard classical discrete distributions. Hence, the parameters of the underlying distribution can be adjusted to suit most data sets. The skewness and kurtosis are reported in Tables 3 and 4, respectively.

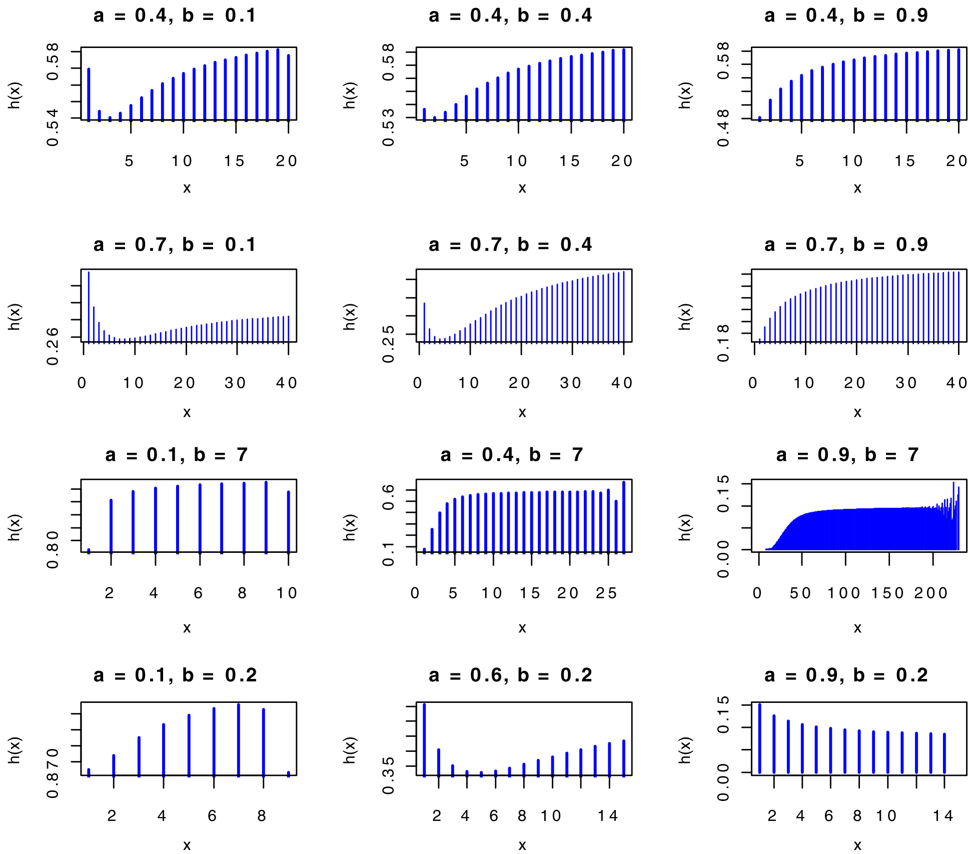

Figure 2. The HRF of the EDLi distribution.

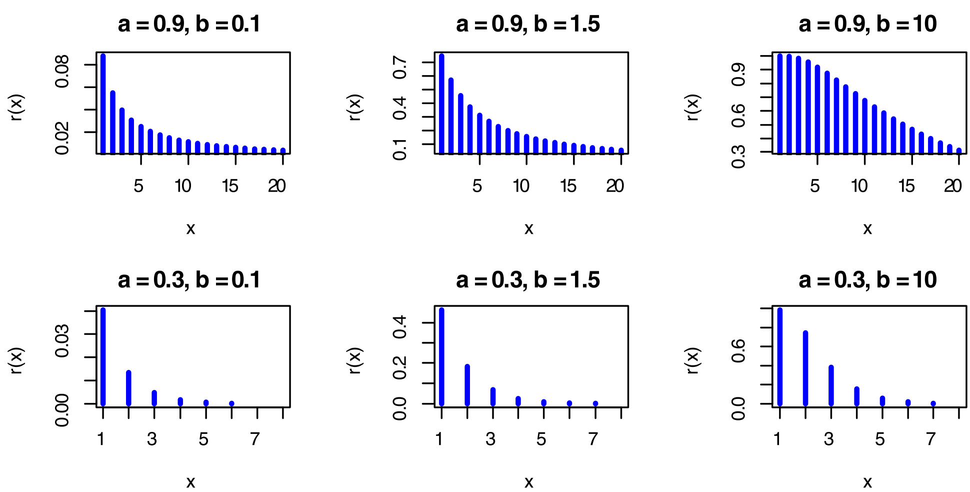

Figure 3. The RHRF of the EDLi distribution.

Table 1. The mean of the EDLi distribution.

$b \downarrow a \rightarrow$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
2	0.364	0.772	1.269	1.916	2.816	4.165	6.424	10.969
3	0.508	1.023	1.626	2.398	3.463	5.053	7.708	13.033
4	0.631	1.219	1.893	2.752	3.934	5.055	8.637	14.526
5	0.737	1.376	2.102	3.029	4.305	6.204	9.365	15.693

Table 2. The variance of the EDLi distribution.

$b \downarrow a \rightarrow$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
2	0.356	0.793	1.443	2.514	4.430	8.225	17.001	43.702
3	0.435	0.875	1.530	2.623	4.588	8.479	17.463	44.766
4	0.477	0.899	1.553	2.624	4.645	8.568	17.616	45.100
5	0.496	0.901	1.560	2.675	4.668	8.599	17.661	45.181

Table 3. The skewness of the EDLi distribution.

$b \downarrow a \rightarrow$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
2	1.667	1.335	1.264	1.248	1.240	1.232	1.223	1.215
3	1.222	1.176	1.111	1.139	1.148	1.148	1.145	1.142
4	0.966	0.980	1.066	1.098	1.108	1.110	1.109	1.107
5	0.809	0.957	1.051	1.075	1.084	1.088	1.088	1.088

Table 4. The kurtosis of the EDLi distribution.

$b \downarrow a \rightarrow$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
2	6.100	5.491	5.466	5.468	5.452	5.423	5.393	5.368
3	4.679	4.923	5.135	5.200	5.210	5.201	5.189	5.178
4	4.181	4.860	5.051	5.097	5.108	5.108	5.104	5.099
5	4.051	4.845	5.002	5.038	5.055	5.061	5.059	5.059

Tables 3 and 4 show that the EDLi distribution is positively skewed for some values of a and b. Also, the skewness and kurtosis decrease with $b \longrightarrow \infty$ for fixed value of a.

3.2. *Stress–strength* (*S–S* *) *analysis*

S–S* analysis has been used in mechanical component design. The probability of failure is based on the probability of S exceeding S*. Assume that both S and S* are in the positive domain. The expected reliability (R^*) can be calculated by

$$R^* = P\left[X_S \leq X_{S^*}\right] = \sum_{x=0}^{\infty} f_{X_S}(x) R_{X_{S^*}}. \tag{11}$$

If $X_S \sim \text{EDLi}(a_1, b_1)$ and $X_{S^*} \sim \text{EDLi}(a_2, b_2)$, then R^* can be expressed as follows

$$R^* = \frac{1}{\left(1 - \log a_1\right)^{b_1}\left(1 - \log a_2\right)^{b_2}} \sum_{x=0}^{\infty} \times \left(\left[\Lambda(x+1; a_1, b_1) - \Lambda(x; a_1, b_1)\right]\left[\left(1 - \log a_2\right)^{b_2} - \Lambda(x+1; a_2, b_2)\right]\right). \tag{12}$$

Table 5. The numerical values of R^* at $a_1 = 0.2$ and $a_2 = 0.3$.

Parameter ↓ →			b_2								
b_1	a_1	a_2	0.1	0.4	0.7	1.0	1.3	1.6	1.9	2.2	2.5
0.1			0.058	0.214	0.344	0.452	0.542	0.616	0.678	0.729	0.772
0.4			0.054	0.198	0.319	0.419	0.503	0.574	0.633	0.682	0.724
0.7	0.2	0.3	0.049	0.183	0.296	0.389	0.469	0.536	0.592	0.640	0.681
1.0			0.046	0.170	0.275	0.363	0.438	0.502	0.556	0.602	0.642
1.3			0.042	0.158	0.256	0.339	0.410	0.471	0.523	0.568	0.606

Table 6. The numerical values of R^*at $a_1 = 0.6$ and $a_2 = 0.9$.

Parameter ↓ →			b_2								
b_1	a_1	a_2	0.1	0.4	0.7	1.0	1.3	1.6	1.9	2.2	2.5
0.1			0.327	0.786	0.921	0.963	0.977	0.981	0.983	0.984	0.9846
0.4			0.282	0.705	0.850	0.904	0.924	0.933	0.937	0.939	0.9399
0.7	0.6	0.9	0.248	0.641	0.790	0.850	0.876	0.888	0.893	0.896	0.8973
1.0			0.222	0.589	0.738	0.803	0.832	0.845	0.852	0.855	0.8568
1.3			0.201	0.545	0.693	0.759	0.790	0.805	0.812	0.816	0.8181

Table 7. The numerical values of R^* at $a_1 = a_2 = 0.5$.

Parameter ↓ →			b_2								
b_1	a_1	a_2	0.1	0.4	0.7	1.0	1.3	1.6	1.9	2.2	2.5
0.1			0.106	0.359	0.536	0.660	0.748	0.810	0.855	0.886	0.909
0.4			0.085	0.290	0.439	0.549	0.629	0.689	0.736	0.771	0.798
0.7	0.5	0.5	0.069	0.239	0.367	0.464	0.538	0.596	0.642	0.679	0.709
1.0			0.057	0.201	0.312	0.398	0.467	0.523	0.568	0.605	0.637
1.3			0.048	0.171	0.269	0.347	0.411	0.464	0.508	0.545	0.577

Table 8. The numerical values of R^* at $b_1 = b_2 = 0.6$.

Parameter ↓ →			a_2				
a_1	b_1	b_2	0.1	0.3	0.5	0.7	0.9
0.1			0.096	0.290	0.494	0.706	0.915
0.3			0.076	0.241	0.433	0.653	0.893
0.5	0.6	0.6	0.054	0.182	0.349	0.566	0.837
0.7			0.031	0.113	0.232	0.412	0.669
0.9			0.009	0.035	0.077	0.152	0.272

We cannot get a closed form to Equation (12), and consequently, Maple software is required to discuss this property numerically. Tables 5–8 show the numerical values of R^* for various values of the model parameters.

From Tables 5–7, it is clear that the reliability increases with $b_2 \longrightarrow \infty$ for fixed values of a_1, a_2 and b_1. But, the reliability decreases with $b_1 \longrightarrow \infty$ for fixed values of a_1, a_2 and b_2. Table 8 shows the numerical values of R^* with $a_1 \longrightarrow 1$ and $a_2 \longrightarrow 1$ for fixed values of b_1 and b_2. From Table 8, it is clear that the reliability increases with $a_2 \longrightarrow 1$ for fixed values of a_1, b_1 and b_2. But, the reliability decreases with $a_1 \longrightarrow 1$ for fixed values of a_2, b_1 and b_2.

3.3. Mean residual lifetime (MRL) and mean past lifetime (MPL)

There are several measures in the reliability and survival analysis literature that are defined to study the aging behavior of components. One of those measures is the MRL tool, say $\varsigma(i)$, which is a helpful tool to model and analyze the burn-in and maintenance policies. In the discrete setting, the MRL is defined as follows

$$\varsigma(i) = E(X - i|X \geq i) = \frac{1}{R(i)} \sum_{j=i+1}^{l} R(j); \quad i \in \mathbb{N}_0, \tag{13}$$

where $0 < l < \infty$. If the RV $X \sim$ EDLi(a, b), then the MRL can be expressed as follows

$$\varsigma(i) = \frac{1}{\left(1 - \log a\right)^b - \Lambda(i; a, b)} \sum_{j=i+1}^{l} \left[\left(1 - \log a\right)^b - \Lambda(j; a, b)\right]. \tag{14}$$

Another measure of interest in survival analysis is the MPL, say $\varsigma^*(i)$, it measures the time elapsed since the failure of X given that the system has failed sometime before i. In the discrete setting, the MPL is defined as follows

$$\varsigma^*(i) = E(i - X|X < i) = \frac{1}{F(i-1)} \sum_{m=1}^{i} F(m-1); \quad i \in \mathbb{N}_0 - \{0\}, \tag{15}$$

where $\varsigma^*(i) = 0$ (see [18]).

3.4. Order statistics and L-moment statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from the EDLi distribution, and let $X_{1:n}, X_{2:n}, \ldots,$ $X_{n:n}$ be their corresponding order statistics (Os). Then, the CDF of ith Os for an integer value of x can be expressed as follows

$$\begin{aligned} F_{i:n}(x; a, b) &= \sum_{k=i}^{n} \binom{n}{k} \left[F_i(x; a, b)\right]^k \left[1 - F_i(x; a, b)\right]^{n-k} \\ &= \sum_{k=i}^{n} \sum_{j=0}^{n-k} \ominus_{(j)}^{(n,k)} \frac{\Lambda(x+1; a, b(k+j))}{\left(1 - \log a\right)^{b(k+j)}}, \end{aligned} \tag{16}$$

where $\ominus_{(j)}^{(n,k)} = (-1)^j \binom{n}{k} \binom{n-k}{j}$. Furthermore, the PMF of the ith Os can be expressed as follows

$$f_{i:n}(x; a, b) = \sum_{k=i}^{n} \sum_{j=0}^{n-k} \ominus_{(j)}^{(n,k)} \frac{\left[\Lambda(x+1; a, b(k+j)) - \Lambda(x; a, b(k+j))\right]}{\left(1 - \log a\right)^{b(k+j)}}. \tag{17}$$

So, the vth moments of $X_{i:n}$ can be written as follows

$$E(X_{i:n}^v) = \sum_{x=0}^{\infty} \sum_{k=i}^{n} \sum_{j=0}^{n-k} \ominus_{(j)}^{(n,k)} x^v \frac{\left[\Lambda(x+1; a, b(k+j)) - \Lambda(x; a, b(k+j))\right]}{\left(1 - \log a\right)^{b(k+j)}}. \tag{18}$$

Hosking [21] has defined the L-moments (Lms) to summaries theoretical distribution and observed samples. He has shown that the Lms have good properties as measure of distributional shape and are useful for fitting distribution to data. Lms are expectation of certain linear combinations of Os. The Lms of the RV X can be expressed as follows

$$\Delta_s = \frac{1}{s}\sum_{j=0}^{s-1}(-1)^j \binom{s-1}{j} E\left(X_{s-j:s}\right). \tag{19}$$

Since Hosking has defined the Lms of the RV X to be the quantities. Then, we can propose some statistical measures such as L-moment (Lm) of mean $= \Delta_1$, Lm coefficient of variation $= \Delta_2/\Delta_1$, Lm coefficient of skewness $= \Delta_3/\Delta_2$, and Lm coefficient of kurtosis $= \Delta_4/\Delta_2$.

4. Estimation methods

In this section, two estimation methods are used to estimate the unknown parameters of the EDLi distribution. Several authors in the literature prefer to use different estimation methods to study which is the best method for estimating the model parameters. See for example, Eliwa *et al.* [15], Cordeiro *et al.* [9], among others.

4.1. Maximum likelihood estimation (MLE)

In this section, we determine the MLEs of the model parameters from complete samples. Assume $X_1, X_2, \ldots, X_n$ be a random sample of size n from the EDLi(a,b). The log-likelihood function (L) can be expressed as follows

$$L(x; a, b) = -nb\log\left(1-\log a\right) + \sum_{i=1}^{n}\log\left[\Lambda(x+1; a, b) - \Lambda(x; a, b)\right]. \tag{20}$$

By differentiating Equation (20) with respect to the parameters a and b, we get the normal nonlinear likelihood equations as follows

$$\frac{n\widehat{b}}{\widehat{a}\left(1-\log\widehat{a}\right)} + \widehat{b}\sum_{i=1}^{n}\frac{[V_1(x_i+1;\widehat{a})]^{\widehat{b}-1}\, V_2(x_i+1;\widehat{a}) - [V_1(x_i;\widehat{a})]^{\widehat{b}-1}\, V_2(x_i;\widehat{a})}{\Lambda(x_i+1;\widehat{a},\widehat{b}) - \Lambda(x_i;\widehat{a},\widehat{b})} = 0 \tag{21}$$

and

$$-n\log\left(1-\log\widehat{a}\right) + \sum_{i=1}^{n}\frac{\begin{array}{c}\Lambda(x_i+1;\widehat{a},\widehat{b})\log(V_1(x_i+1;\widehat{a}))\\ -\Lambda(x_i;\widehat{a},\widehat{b})\log(V_1(x_i;\widehat{a}))\end{array}}{\Lambda(x_i+1;\widehat{a},\widehat{b}) - \Lambda(x_i;\widehat{a},\widehat{b})} = 0, \tag{22}$$

respectively, where $V_1(x;\widehat{a}) = 1-\widehat{a}^x + [(1+x)\widehat{a}^x - 1]\log\widehat{a}$, and $V_2(x;\widehat{a}) = x(x+1)\widehat{a}^{x-1}\log\widehat{a} - x\widehat{a}^{x-1} + 1/\widehat{a}[(1+x)\widehat{a}^x - 1]$. These equations cannot be solved analytically, therefore an iterative procedure like Newton Raphson is required to solve them numerically.

4.2. Proportion estimation (PnE)

Assume $X_1, X_2, \ldots, X_n$ be a random sample of size n from the EDLi(a,b) distribution. Since we have two unknown parameters, we define two indicators as follows:

$$I_1(x_i) = \begin{cases} 1 & \text{if } x_i = 0 \\ 0 & \text{if otherwise} \end{cases} \tag{23}$$

and

$$I_2(x_i) = \begin{cases} 1 & \text{if } x_i = 1 \\ 0 & \text{if otherwise.} \end{cases} \tag{24}$$

Assume, $Q = \sum_{i=1}^{n} I_1(x_i)$ and $V = \sum_{i=1}^{n} I_2(x_i)$ denote respectively the number of 0's and 1's in the sample. By using Equations (6), (23) and (24), we get $P(X \leq 0) = Q/n$ and $P(X \leq 1) = (Q + V)/n$. Thus, the unknown parameters a and b are estimated by solving the following two equations

$$\left[1 - \frac{\widehat{a}\left(1 - 2\log \widehat{a}\right)}{1 - \log \widehat{a}}\right]^{\widehat{b}} - \frac{Q}{n} = 0 \tag{25}$$

and

$$\left[1 - \frac{(\widehat{a})^2\left(1 - 3\log \widehat{a}\right)}{1 - \log \widehat{a}}\right]^{\widehat{b}} - \frac{Q+V}{n} = 0. \tag{26}$$

Since Q/n and V/n are unbiased and consistent empirical estimators of probabilities $P(X \leq 0)$ and $P(X \leq 1)$, the $\widehat{a}$ of a and $\widehat{b}$ of b are also unbiased and consistent. For more detail, see Khan *et al.* [24].

5. Simulation results: MLE versus PnE

In this section, we assess the performance of the MLE and PnE with respect to sample size n. The assessment is based on a simulation study:

(1) Generate 10000 samples of size $n =$ 50, 150, 250, 350 from EDLi(0.5,0.3), EDLi(0.5,0.5), EDLi(0.7,0.7), EDLi(0.7,0.9), EDLi(0.9,1.5) and EDLi(0.9,2.5), respectively. A general form to generate a random variable X from the EDLi distribution is first to generate the value Y from the continuous exponentiated Li distribution and then to discretize this value to obtain X. The following formula can be used to generate a random variable Y ,

$$Q(u) = \left\{-1 - \frac{1}{\theta} - \frac{1}{\theta} W_0\left(-(1+\theta)e^{-(1+\theta)}(1 - u^{1/b})\right)\right\}; \quad 0 < u < 1,$$

where W_0 represents Lambert function.

(2) Compute the MLEs and PnEs for the 10,000 samples, say $\widehat{a}_j$ and $\widehat{b}_j$ for $j = 1, 2, \ldots, 10000$.

Table 9. The average bias and average MSE (with in parenthesis) for the MLEs and PnEs.

Parameter		Sample size	MLE		PnE	
a	*b*	*n*	*a*	*b*	*a*	*b*
0.5	0.3	50	−0.075(0.048)	0.122(0.057)	0.177(0.153)	0.191(0.147)
		150	−0.046(0.033)	0.084(0.039)	0.156(0.128)	0.188(0.130)
		250	−0.012(0.020)	0.034(0.026)	0.133(0.118)	0.184(0.119)
		350	−0.003(0.007)	0.017(0.005)	0.113(0.109)	0.176(0.108)
	0.5	50	−0.089(0.074)	0.101(0.031)	0.138(0.128)	0.187(0.131)
		150	−0.063(0.061)	0.077(0.023)	0.119(0.111)	0.163(0.124)
		250	−0.031(0.037)	0.052(0.019)	0.117(0.108)	0.154(0.118)
		350	−0.019(0.012)	0.021(0.011)	0.112(0.103)	0.137(0.110)
0.7	0.7	50	−0.067(0.066)	0.045(0.034)	0.129(0.124)	0.156(0.129)
		150	−0.042(0.042)	0.027(0.022)	0.113(0.116)	0.141(0.126)
		250	−0.021(0.028)	0.012(0.013)	0.108(0.111)	0.131(0.119)
		350	−0.009(0.010)	0.003(0.001)	0.102(0.108)	0.120(0.108)
	0.9	50	−0.069(0.061)	0.055(0.044)	0.133(0.130)	0.144(0.136)
		150	−0.044(0.045)	0.038(0.031)	0.124(0.121)	0.131(0.122)
		250	−0.032(0.029)	0.027(0.016)	0.111(0.113)	0.124(0.112)
		350	−0.018(0.015)	0.011(0.009)	0.105(0.109)	0.112(0.105)
0.9	1.5	50	−0.059(0.067)	0.035(0.021)	0.141(0.110)	0.143(0.124)
		150	−0.035(0.051)	0.024(0.015)	0.125(0.104)	0.129(0.120)
		250	−0.024(0.032)	0.015(0.011)	0.114(0.101)	0.113(0.112)
		350	−0.013(0.017)	0.007(0.004)	0.110(0.100)	0.105(0.108)
	2.5	50	−0.051(0.044)	0.034(0.023)	0.150(0.112)	0.125(0.118)
		150	−0.032(0.031)	0.020(0.015)	0.133(0.110)	0.111(0.115)
		250	−0.018(0.021)	0.011(0.010)	0.127(0.105)	0.104(0.113)
		350	−0.005(0.012)	0.004(0.007)	0.114(0.101)	0.100(0.103)

(3) Compute the biases and mean-squared errors (MSEs), where

$$\text{bias} = \frac{1}{10000}\sum_{j=1}^{10000}\left(\widehat{\alpha}_j - \alpha\right) \quad \text{and} \quad \text{MSE} = \frac{1}{10000}\sum_{j=1}^{10000}\left(\widehat{\alpha}_j - \alpha\right)^2.$$

(4) The empirical results are given in Table 9.

From Table 9, the following observations can be noted:

(1) The magnitude of bias always decreases to zero as $n \to \infty$.
(2) The MSEs always decrease to zero as $n \to \infty$. This shows the consistency of the estimators.
(3) The performance of the MLE method is better than the PnE method.
(4) Under the MLE method, the estimator of a is slightly negative biased.
(5) The performance of the PnE method is inferior with respect to the MLE method, because it uses only the information of 0's and 1's from the samples and discards all other information.

We have presented results only for $(a, b) = (0.5, 0.3)$, $(0.5, 0.5)$, $(0.7, 0.7)$, $(0.7, 0.9)$, $(0.9, 1.5)$ and $(0.9, 2.5)$. But, the results are similar for other choices for a and b. Some statistical measures are listed in Tables 10–12 for various values of the model parameters.

From Tables 10–12, it is clear that the EDLi distribution is suitable of modeling positive and negative skewness as well as symmetric data sets.

Table 10. Some statistical measures based on the MLEs for initial value $a = 0.5$ and $b = 0.3$.

Sample size	bias		Measures		
n	a	b	Skewness	Kurtosis	Stress–strength$\|_{b_1=b_2=b}^{a_1=a_2=a}$
50	−0.075	0.122	−0.5174	6.724	0.2605
150	−0.046	0.084	−0.4103	6.847	0.2602
250	−0.012	0.034	−0.2217	6.147	0.2565
350	−0.003	0.017	−0.0110	6.218	0.2528

Table 11. Some statistical measures based on the MLEs for initial value $a = b = 0.7$.

Sample size	bias		Measures		
n	a	b	Skewness	Kurtosis	Stress–strength$\|_{b_1=b_2=b}^{a_1=a_2=a}$
50	−0.067	0.045	0.5547	5.998	0.3879
150	−0.042	0.027	0.5190	6.104	0.3748
250	−0.021	0.012	0.3536	6.475	0.3712
350	−0.009	0.003	0.2748	6.365	0.3698

Table 12. Some statistical measures based on the MLEs for initial value $a = 0.9$ and $b = 1.5$.

Sample size	bias		Measures		
n	a	b	Skewness	Kurtosis	Stress–strength$\|_{b_1=b_2=b}^{a_1=a_2=a}$
50	−0.059	0.035	1.2581	6.3254	0.5785
150	−0.035	0.024	1.3258	6.2354	0.5365
250	−0.024	0.015	1.3254	6.1201	0.4587
350	−0.013	0.007	1.2580	6.1978	0.4101

Table 13. The competitive models of the EDLi distribution.

Model	Abbreviation	Author(s)
Discrete Lindley	DLi	Gómez-Déniz and Calderín-Ojeda [20]
Discrete Lindley with two parameters	DLi II	Bakouch *et al.* [3]
Geometric	Geo	Gómez-Déniz [19]
Discrete generalized exponential type II	DGE II	Nekoukhou *et al.* [36]
Discrete Rayleigh	DR	Roy [43]
Discrete Weibull	DW	Toshio and Shunji [45]
Discrete Pareto	DPa	Krishna and Pundir [25]
Discrete Burr-XII	DB-XII	Para and Jan [40]
Discrete Burr	DBu	Krishna and Pundir [25]
Discrete Lomax	DLo	Para and Jan [40]
Poisson	Poi	Poisson [42]

6. Data fitting and testing of hypothesis

In this section, we illustrate the importance of the EDLi distribution using four real data sets. Two of the data sets consist of count. The other two data sets consist of times. The competitive models of the EDLi distribution are listed in Table 13.

The first data set (I): represents number of carious teeth among the four deciduous molars in a sample of 100 children aged 10 and 11 years (see Krishna and Pundir [25]).

Table 14. The MLEs with their corresponding Se and C.I for data set I.

Parameter →	a			b		
Model ↓	MLE	Se	C.I	MLE	Se	C.I
EDLi	0.379	0.065	[0.252,0.506]	0.543	0.158	[0.234,0.852]
DLi	0.274	0.029	[0.217,0.331]	–	–	–
DLi II	0.401	0.263	[0,0.916]	0.001	0.652	[0,1.279]
Geo	0.401	0.038	[0.327,0.475]	–	–	–
DGE II	0.468	0.072	[0.327,0.609]	0.718	0.206	[0.314,1.122]
DR	0.665	0.029	[0.608,0.722]	–	–	–
DW	0.374	0.049	[0.278,0.470]	0.895	0.119	[0.662,1.128]
DPa	0.184	0.032	[0.121,0.247]	–	–	–
Poi	0.670	0.082	[0.509,0.831]	–	–	–

We shall compare the fits of the EDLi distribution with some competitive models such as DLi, DLi II, Geo, DGE II, DR, DW, DPa and Poi distributions.

The second data set (*II*) *:* represents the counts of cysts of kidneys using steroids. This data set originated from a study [7]. We shall compare the fits of the EDLi distribution with some competitive models such as DLi, DLi II, Geo, DR, DW, DB-XII, DLo and Poi distributions.

The third data set (*III*)*:* represents the waiting times (in minutes) before service of 100 Bank customers (see [17]). We approximated this data to the nearest minute. We shall compare the fits of the EDLi distribution with some competitive models such as DLi, DLi II, Geo, DBu and DPa distributions.

The fourth data set (*IV*)*:* represents 40 observations of time-to-failure (10^3h) of turbocharger of one type of engine (see [46]). We approximated this data to the nearest hour. We shall compare the fits of the EDLi distribution with some competitive models such as DLi, Geo, DPa, DGE II and DBu distributions.

For the first, second and fourth data sets, the fitted models are compared using some criteria namely, the maximized log-likelihood ($-L$), Akaike information criterion (AIC), corrected Akaike information criterion ($CAIC$), Bayesian information criterion (BIC), Hannan-Quinn information criterion ($HQIC$), Chi-square (χ^2) and its p-value. But, for the third data set, the fitted models are compared using Kolmogorov-Smirnov (K-S) statistic and its p-value.

For the data set I, the MLEs with their corresponding standard errors (Se) and confidence interval (C.I) are listed in Table 14. Table 10 shows the $-L$, AIC, CAIC, BIC, HQIC, χ^2, degree of freedom (D.F), observed frequency (O.Fr), expected frequency (E.Fr) and p-values.

From Table 15, it is clear that the EDLi distribution is the best distribution among all tested distributions, because it has the smallest value among $-L$, AIC, CAIC, BIC, HQIC and χ^2, as well as it has the largest p-value. Figure 4 shows the fitted PMFs for data set I, which support the results in Table 15.

For the data set II, the MLEs and goodness of fit test are reported in Tables 16 and 17, respectively.

From Table 17, it is clear that the EDLi distribution is the best distribution among all tested models. Figure 5 shows the fitted PMFs for data set II, which support the results in Table 17.

For the data set III, the MLEs and goodness of fit test are listed in Table 18.

Table 15. The goodness of fit test for data set I.

		E.Fr								
X	O.Fr	EDLi	DLi	DLi II	Geo	DGE II	DR	DW	DPa	Poi
0	64	63.57	57.13	59.88	59.88	63.51	33.50	62.58	69.04	51.17
1	17	19.75	26.88	24.02	24.02	20.19	46.94	21.35	15.37	34.28
2	10	9.09	10.45	9.64	9.64	8.81	17.01	8.85	6.01	11.49
3	6	4.19	3.71	3.87	3.87	4.01	2.39	3.88	3.01	2.57
≥ 4	3	3.4	1.83	2.59	2.59	3.48	0.16	3.34	6.57	0.49
Total	100	100	100	100	100	100	100	100	100	100
$-L$		111.45	113.68	112.47	112.4735	111.80	142.61	112.10	116.83	121.05
AIC		226.91	229.36	228.95	226.95	227.61	287.21	228.20	235.66	244.09
CAIC		227.03	229.39	229.07	227.99	227.73	287.25	228.32	235.70	244.14
BIC		232.12	232.96	234.16	232.56	232.82	289.82	233.41	238.27	246.70
HQIC		229.02	230.41	231.06	230.00	229.72	288.26	230.30	236.72	245.15
χ^2		0.739	6.638	3.347	3.347	0.973	66.07	1.507	3.225	23.65
D.F		1	2	1	2	1	2	1	2	2
p-value		0.390	0.036	0.067	0.188	0.324	< 0.0001	0.219	0.199	< 0.0001

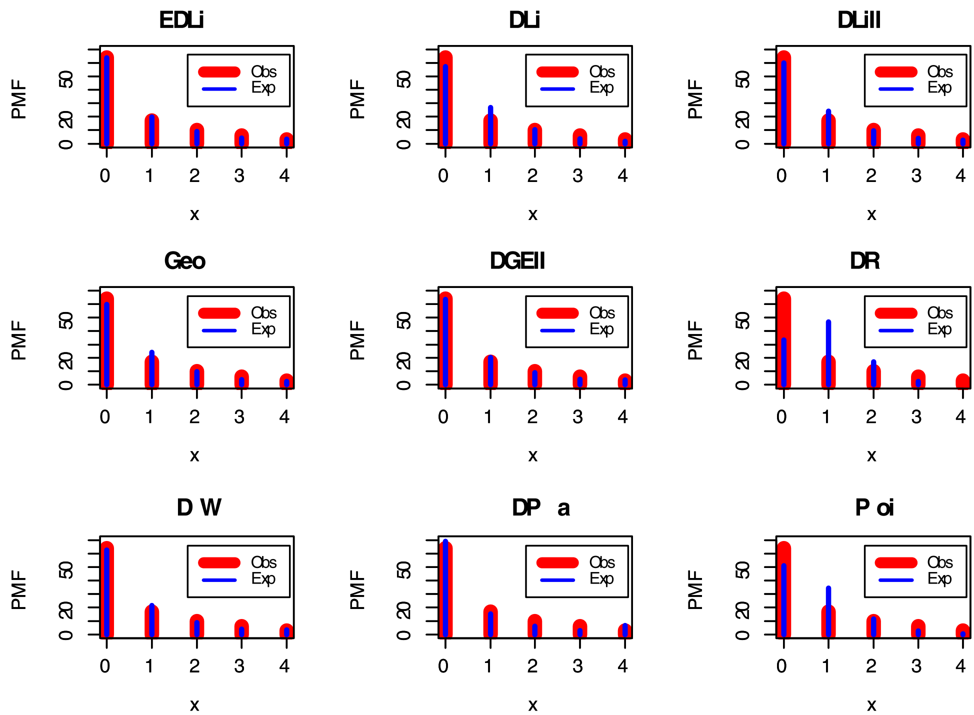

Figure 4. The fitted PMFs for data set I.

From Table 18, it is clear that the EDLi distribution is the best distribution among all tested models. Figures 6 and 7 show the estimated CDFs and P–P plots for data set III, which support the results in Table 18.

For the data set IV , the MLEs and goodness of fit test are reported in Tables 19 and 20, respectively.

Table 16. The MLEs with their corresponding Se and C.I for data set II.

Parameter →	a			b			c		
Model ↓	MLE	Se	C.I	MLE	Se	C.I	MLE	Se	C.I
EDLi	0.672	0.048	[0.578,0.766]	0.264	0.056	[0.154,0.374]	–	–	–
DLi	0.436	0.026	[0.385,0.487]	–	–	–	–	–	–
DLi II	0.581	0.045	[0.494,0.669]	0.001	0.058	[0,0.115]	–	–	–
Geo	0.582	0.030	[0.523,0.641]	–	–	–	–	–	–
DR	0.900	0.009	[0.882,0.918]	–	–	–	–	–	–
DW	0.421	0.047	[0.329,0.513]	0.629	0.073	[0.456,0.772]	–	–	–
DB-XII	0.003	0.002	[0,0.00692]	12.75	5.060	[2.832,22.67]	0.720	0.087	[0.549,0.891]
DLo	0.150	0.098	[0,0.342]	1.830	0.950	[0,3.692]	–	–	–
Poi	1.390	0.112	[1.17,1.609]	–	–	–	–	–	–

Table 17. The goodness of fit test for data set II.

		E.Fr								
X	O.Fr	EDLi	DLi	DLi II	Geo	DR	DW	DB-XII	DLo	Poi
0	65	64.97	40.25	46.03	45.98	11	63.64	63.32	61.89	27.42
1	14	14.39	29.83	26.77	26.76	26.83	17.45	18.19	21.01	38.08
2	10	9.01	18.36	15.57	15.57	29.55	9.3	9.29	9.65	26.47
3	6	6.14	10.35	9.05	9.06	22.23	5.68	5.49	5.24	12.26
4	4	4.33	5.53	5.27	5.28	12.49	3.73	3.52	3.17	4.26
5	2	3.10	2.86	3.06	3.07	5.42	2.56	2.39	2.06	1.18
6	2	2.24	1.44	1.78	1.79	1.85	1.82	1.69	1.42	0.27
7	2	1.62	0.71	1.04	1.04	0.52	1.32	1.23	1.02	0.05
8	1	1.18	0.35	0.60	0.61	0.11	0.98	0.92	0.76	0.01
9	1	0.85	0.17	0.35	0.35	0.02	0.74	0.70	0.58	0
10	1	0.62	0.08	0.20	0.21	0	0.57	0.55	0.46	0
11	2	1.55	0.07	0.28	0.28	0	2.21	2.71	2.74	0
Total	110	110	110	110	110	110	110	110	110	110
−L		166.9	189.1	178.8	178.8	277.8	167.9	168.8	170.5	246.2
AIC		337.9	380.2	361.5	359.5	557.6	339.9	343.5	344.9	494.4
CAIC		338.0	380.3	361.6	359.6	557.6	340.1	343.8	345.1	494.5
BIC		343.3	382.9	366.9	362.2	560.3	345.4	351.6	350.4	497.1
HQIC		340.1	381.3	363.7	360.6	558.7	342.2	346.8	347.2	495.5
χ^2		0.507	43.48	22.89	22.84	321.1	1.04	2.469	3.316	294.1
D.F		3	4	3	4	4	3	3	3	4
p.value		0.917	< 0.01	< 0.001	< 0.01	< 0.01	0.792	0.480	0.345	< 0.01

From Table 20, it is clear that the EDLi distribution is the best distribution among all tested models. Figure 8 shows the estimated PMFs for data set IV, which support the results in Table 20.

Now, we want to perform the following test: $H_0 : b = 1$ (DLi) against $H_1 : b \neq 1$ (EDLi). The likelihood ratio test statistic (Λ), D.F and p-values for the DLi distribution are given in Table 21.

We can conclude that H_0 is rejected with 5% level of significance. Hence, the DLi distribution cannot be used for analyzing the data sets considered. So, we prefer the EDLi distribution. Figure 9 shows the HRF and RHRF for data sets using the EDLi model. Since the HRF for data sets III and IV are increasing, then the MRL for these data sets is decreasing. Some statistical measures for data sets using the EDLi model are reported in Table 22 . On the other hand, Table 23 shows the PnEs of the unknown parameters of the EDLi distribution for just data sets I and II, because this method uses only the information

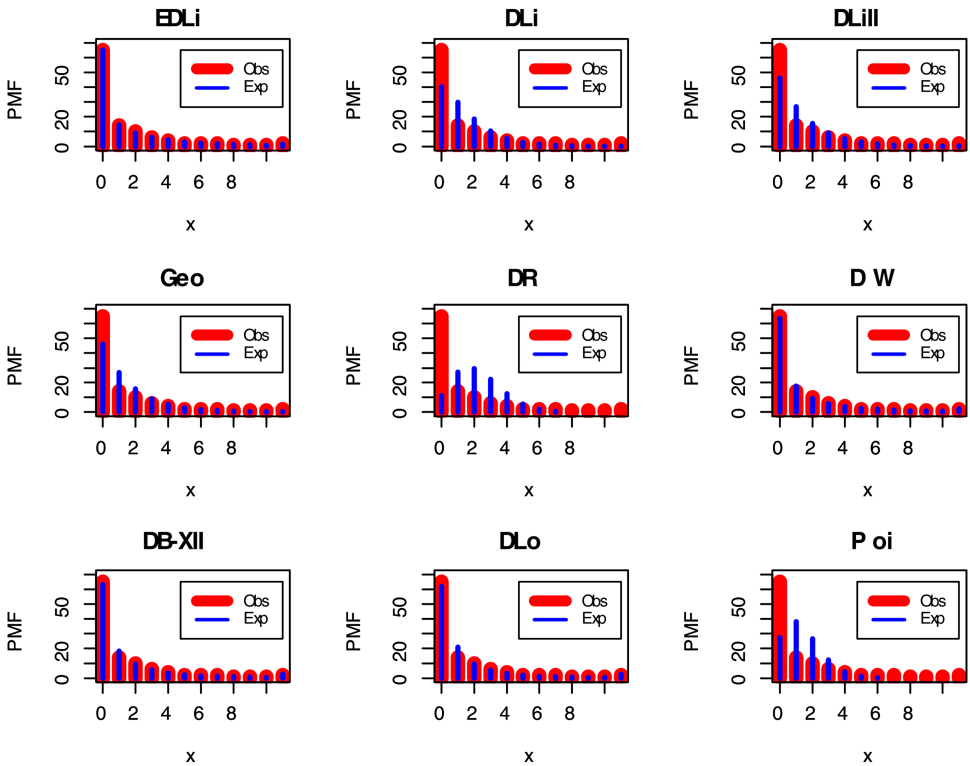

Figure 5. The fitted PMFs for data set II.

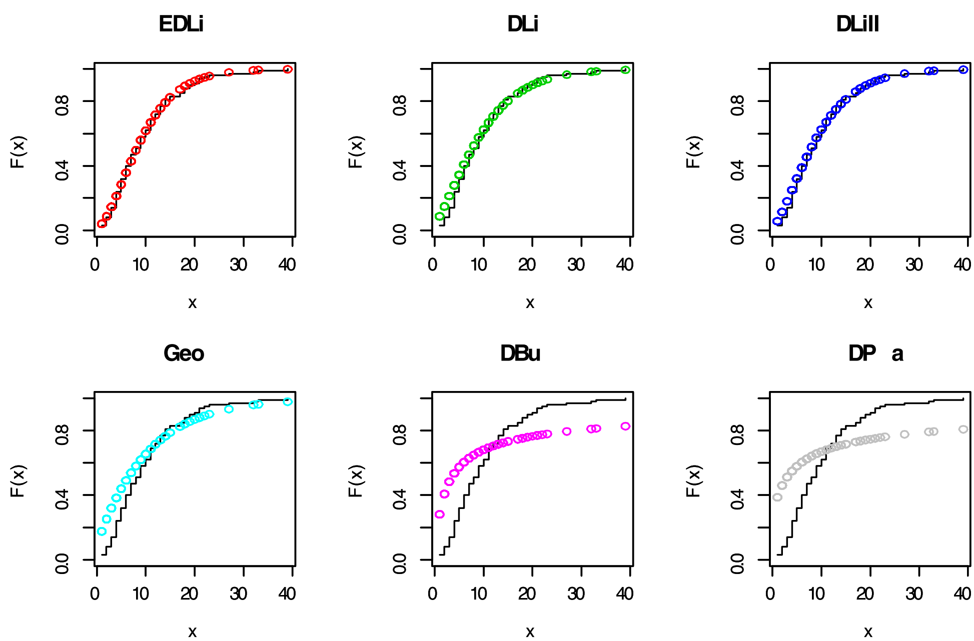

Figure 6. The estimated CDFs for data set III.

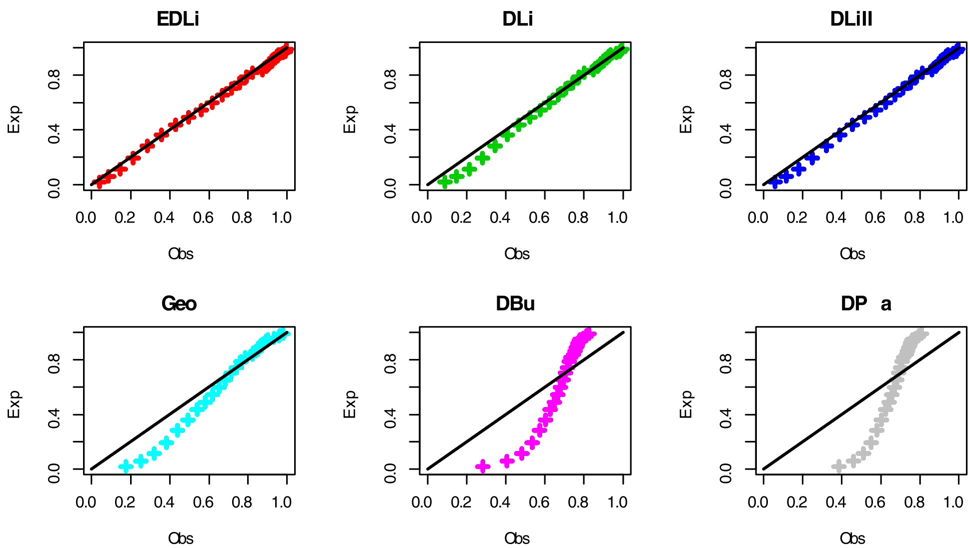

Figure 7. The P–P plots for data set III.

Table 18. The MLEs and goodness of fit test for data set III.

		Model					
Statistic		EDLi	DLi	DLi II	Geo	DBu	DPa
a	MLE(Se)	0.805(0.017)	0.837(0.011)	0.825(0.011)	0.908(0.009)	0.965(0.040)	0.641(0.028)
	C.I	[0.772,0.838]	[0.815,0.859]	[0.803,0.859]	0.890,0.926	[0.887,1]	[0.586,0.696]
b	MLE(Se)	1.518(0.245)	–	2556.9(8.388)	–	13.35(1.566)	–
	C.I	[1.038,1.998]	–	[2540.5,2573.3]	–	[10.28,16.42]	–
$-L$		318.41	321.63	318.65	334.33	386.14	406.45
K–S		0.0736	0.1381	0.1095	0.2414	0.4017	0.4308
p-value		0.6498	0.0441	0.1814	0.000017	1.93×10^{-14}	1.1×10^{-16}
AIC		640.83	645.26	641.31	670.67	776.28	814.91
CAIC		640.95	645.29	641.43	670.71	776.39	814.95
BIC		646.04	647.86	646.52	673.27	781.49	817.51
HQIC		642.94	646.31	643.42	671.72	778.38	815.96

Table 19. The MLEs with their corresponding Se and C.I for data set IV.

Parameter →	*a*			*b*		
Model ↓	MLE	Se	C.I	MLE	Se	C.I
EDLi	0.545	0.037	[0.472,0.618]	10.279	3.733	[2.962,17.596]
DLi	0.768	0.022	[0.725,0.811]	–	–	–
Geo	0.863	0.020	[0.824,0.902]	–	–	–
DPa	0.609	0.047	[0.517,0.701]	–	–	–
DGE II	0.608	0.038	[0.534,0.682]	17.317	6.435	[4.704,29.929]
DBu	0.948	0.203	[0.559,1.00]	10.107	40.983	[0,90.433]

of 0's and 1's from the samples. Therefore, we cannot apply this method on data sets III and IV. From Table 22, it is clear that the EDLi model is a good model to fit the data sets considered.

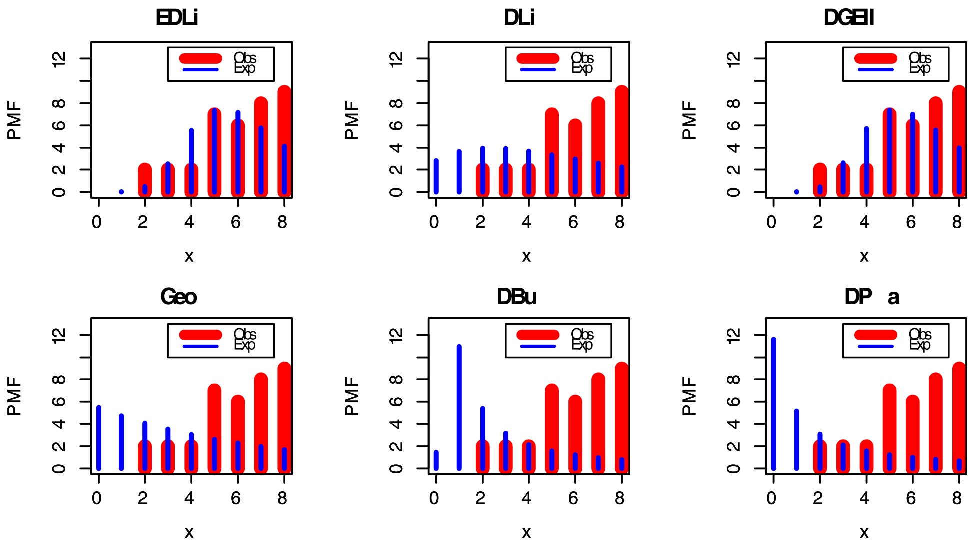

Figure 8. The fitted PMFs for data set IV.

Table 20. The goodness of fit test for data set IV.

		E.Fr					
X	O.Fr	EDLi	DLi	Geo	DPa	DGE II	DBu
0	0	0.00	2.85	5.46	11.62	0.00	1.44
1	0	0.02	3.67	4.72	5.16	0.01	10.96
2	2	0.49	3.96	4.07	3.08	0.46	5.39
3	2	2.53	3.92	3.52	2.11	2.62	3.17
4	2	5.54	3.68	3.04	1.56	5.72	2.14
5	7	7.34	3.35	2.62	1.21	7.37	1.57
6	6	7.15	2.97	2.26	0.98	7.00	1.21
7	8	5.76	2.59	1.95	0.81	5.57	0.97
8	9	4.11	2.23	1.69	0.68	3.99	0.80
9	4	7.05	10.78	10.67	12.79	7.26	12.34
Total	40	40	40	40	**40**	**40**	**40**
−**L**		87.18	107.08	116.78	148.94	87.89	139.77
AIC		178.36	216.17	235.57	299.89	179.78	283.55
CAIC		178.69	216.28	235.68	299.99	180.11	283.87
BIC		181.74	217.86	237.26	301.57	183.16	286.93
HQIC		179.59	216.79	236.18	300.50	181.01	284.77
χ^2		2.13	23.22	38.33	127.16	2.38	99.92
D.F		2	3	3	3	2	2
p.value		0.345	< 0.001	< 0.01	< 0.01	0.304	< 0.01

Table 21. The Λ, D.F and p-values for the DLi distribution.

Data	Λ	D.F	p-value
I	4.46	1	0.035
II	44.41	1	0.00
III	6.44	1	0.011
IV	39.82	1	0.00

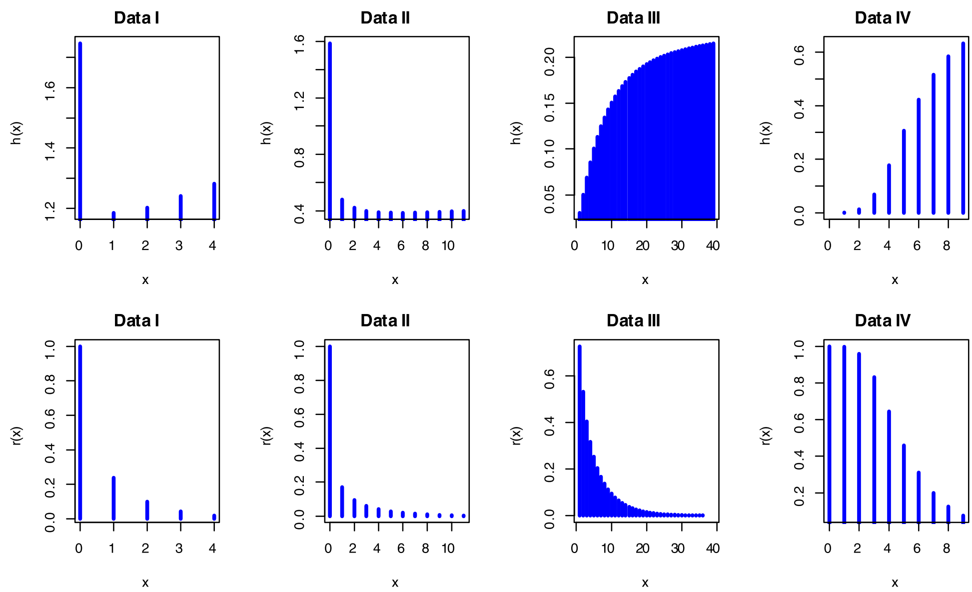

Figure 9. The HRF and RHRF for data sets using the EDLi model.

Table 22. Some statistical measures for data sets using the EDLi model.

Data set	Mean	Variance	Skewness	Kurtosis	$MRL_{i=6}$
I	1.612	2.177	−1.645	−1.607	–
II	2.774	0.734	−17.678	52.703	–
III	4.989	32.065	−0.055	−0.180	9.25
IV	5.135	33.116	−1.209	−2.288	2.59

Table 23. The PnE summaries for the EDLi model from the data.

Data	Parameter	Estimate	χ^2	D.F	*p*-value
I	a	0.466	0.307	1	0.579
	b	0.403			
II	a	0.685	0.463	3	0.927
	b	0.254			

7. Conclusions

In this paper, we have proposed a new two-parameter distribution called the exponentiated discrete Lindley (EDLi) distribution. The proposed distribution is a generalization of the standard Lindley distribution, which evidently provides additional flexibility to analyze real data. Some of its fundamental properties have been discussed in detail. It is found that the hazard rate function can be increasing, decreasing, decreasing–increasing–decreasing, increasing–decreasing–increasing, unimodal, bathtub, and *J*-shaped. The unknown parameters of the EDLi distribution have been estimated by using two methods, namely, the maximum likelihood and proportion methods. Moreover, their long-run performances have been compared through an extensive simulation study. The numerical simulation experiments suggest that the method of maximum likelihood outperforms the proportion method. The flexibility of the EDLi distribution has been empirically proven by using four real-life applications. The EDLi distribution has proven to provide better fits than some other models. Finally, we believe that the proposed model will serve a wide spectrum of applications including biology, reliability and survival analysis.

Disclosure statement

No potential conflict of interest was reported by the authors.

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