A fractional–order model with different strains of COVID-19

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ARTICLE INFO

ABSTRACT

This study examines the dynamics of COVID-19 variants using a Caputo–Fabrizio fractional order model. The reproduction ratio R_0 and equilibrium solutions are determined. The purpose of this article is to use a non-integer order derivative in order to present information about the model solutions, uniqueness, and existence using a fixed point theory. A detailed analysis of the existence and uniqueness of the model solution is conducted using fixed point theory. For the computation of the iterative solution of the model, the fractional Adams–Bashforth method is used. Using the estimated values of the model parameters, numerical results are used to support the significance of the fractional-order derivative. The graphs provide useful information about the complexity of the model, and provide reliable information about the model for any case, integer or non-integer. Also, we demonstrate that any variant with the largest basic reproduction ratio will automatically outperform the other variant.

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1. Introduction

Pathogen mutation has been a common phenomenon in disease spreading. Typical example can be seen from the emergence of H1N1 influenza virus in Mexico and the USA in the year 2009. H1N1 is the mutation of the seasonal influenza. Dengue fever, HIV, Tuberculosis, and some other sexually transmitted diseases come to existence as a result of more than one pathogen variants. Many researchers studied the dynamical nature of the pathogen–host interactions with more than one variant [1–4]. It is also shown basic reproduction ratio decides which variant outperforms the other [5]. Possibility of mutation,co-infection, and exponential growth of the host population were studied [6–9].

The global transmission and replication of SARSCOV-2, the causative agent of COVID-19 disease, gives rise to the mutations of the virus. This may alter the virus' mode of transmission, the vaccines' effectiveness and the severity of disease. Many variants surfaced, some of which have been identified by World Health Organization (WHO) as variants of concern (VOC). This is as result of the risks they pose and their ability to impact the effectiveness of the available vaccine [10–17].

The generalization of classical integer calculus is the Fractional calculus. Due to hereditary properties and provision of a good description of the memory fractional order derivatives and fractional integrals play important role in the study of fractional calculus. Nowadays, FO differential equations are frequently used in exploring the dynamics of many real life phenomena [18–24]. Caputo–Fabrizio (CF) fractional derivative fractional order derivative was developed in 2015. This fractional order derivative is based on exponential kernel and the detail on the operator can be found in [25]. Many problems used Caputo–Fabrizio derivative to model problems in various fields [26–28]. The fundamental differences

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among the fractional derivatives are their different kernels which can be selected to meet the requirements of different applications. For example, the main differences between the Caputo fractional derivative, the Caputo–Fabrizio derivative, and the Atangana–Baleanu fractional derivative are that the Caputo derivative is defined using a power law, the Caputo–Fabrizio derivative is defined using an exponential decay law, and the Atangana–Baleanu derivative is defined using a Mittag-Leffler law [29–31]. Atangana found that the power law derivative of the Riemann–Liouville fractional derivative or the Caputo–Fabrizio fractional derivative provides noisy information due to its specific memory properties. However, the Caputo–Fabrizio fractional derivative gives less noise than the Riemann–Liouville [32–34]. Hence, in this research we choose Caputo–Fabrizio fractional derivative.

Here, we consider two variants of COVID-19 in which one variant is a mutation of the other. A mutation is the sudden change in the genetic makeup that occurs either due to mistakes when DNA is copied or as a result of environmental factors. In this research new variant is assumed to be as a result of changes in the proteins that made up old variant. Due to the recent progress on fractional calculus and its wide applications, we intend to formulate and analyzed our model with Caputo–Fabrizio fractional derivative. The primary goal of this article is to use a fresh non-integer order derivative to study the model of COVID-19, to present information about the model solution's, uniqueness and existence using a fixed point theory. It is also in our interest to formally examine the mathematical implications of linking the various infectious compartments in a sufficiently general manner.

The paper is divided into six sections: Section 1 is an introduction, Section 2 is a glossary of terms, Section 3 is the model formulation, Section 4 is a study of the existence and uniqueness of the model's solution, Section 5 is a study of the numerical scheme and numerical simulations of the model, and Section 6 is the paper's conclusion.

2. Definitions

Definition 1 ([29]). Caputo–Fabrizio fractional derivative for $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1)$ is defined as;

$$D_t^\alpha (f(t)) = \frac{M(\alpha)}{1 - \alpha} \int_a^t f^I(x) \, exp\left[-\alpha \frac{t - x}{1 - \alpha}\right] dx.$$

$M(\alpha)$ is the normalized function that satisfies $M(0) = M(1) = 1$. When $f^I \notin H^1(a, b)$ the above definition is reduced to;

$$D_t^\alpha (f(t)) = \frac{M(\alpha)}{1 - \alpha} \int_a^t (f(t) - f(x)) \, exp\left[-\alpha \frac{t - x}{1 - \alpha}\right] dx.$$

Definition 2 ([29]). Let $0 < \alpha < 1$, and consider;

$$D_t^\alpha (f(t)) = g(t),$$

then the corresponding $\alpha-$ fractional order integral is given as;

$$I_t^\alpha (f(t)) = \frac{2(1 - \alpha)}{(2 - \alpha) M(\alpha)} g(t) + \frac{2\alpha}{(2 - \alpha) M(\alpha)} \int_0^t g(s) \, ds, \quad t \geq 0.$$

3. Formulation of the model

The model consists of six compartments; Susceptible $S(t)$, Exposed $E(t)$, Infected with new variant $I_n(t)$, Infected with old variant $I_o(t)$, Hospitalized individuals $H(t)$, and Recovered individuals $R(t)$. The total population $N(t)$ is defined as;

$$N(t) = S(t) + E(t) + I_n(t) + I_o(t) + H(t) + R(t).$$

The model is described by the system of Caputo–Fabrizio fractional order differential equations of order α below. We modify the fractional operator via an auxiliary parameter $\Lambda > 0$ to avoid dimensional mismatching.

$$
\begin{aligned}
\Lambda^{\alpha-1}{}_0^{CF}D_t^\alpha S(t) &= \lambda - \beta_1 SI_n - \beta_2 SI_o - \mu S, \\
\Lambda^{\alpha-1}{}_0^{CF}D_t^\alpha E(t) &= \beta_1 SI_n + \beta_2 SI_o - \alpha_1 E - \alpha_2 E - \mu E, \\
\Lambda^{\alpha-1}{}_0^{CF}D_t^\alpha I_n(t) &= \alpha_1 E - \gamma_1 I_n - \mu I_n - d_1 I_n, \\
\Lambda^{\alpha-1}{}_0^{CF}D_t^\alpha I_o(t) &= \alpha_2 E - \gamma_2 I_o - \mu I_o - d_2 I_o, \\
\Lambda^{\alpha-1}{}_0^{CF}D_t^\alpha H(t) &= \gamma_1 I_n + \gamma_2 I_o - (\mu + \Phi + d_3) H, \\
\Lambda^{\alpha-1}{}_0^{CF}D_t^\alpha R(t) &= \Phi H - \mu R,
\end{aligned}
\tag{1}
$$

with the following initial conditions;

$$S(0) = a_1, E(0) = a_2, I_n(0) = a_3, I_0(0) = a_4, H(0) = a_5, R(0) = a_6.$$

The meaning of parameters involved in the model is given in Table 1.

Table 1
Meaning of parameters.

Parameter	Meaning
λ	Recruitment rate
μ	Natural death rate
β_1	Effective contact rate with new variant
β_2	Effective contact rate with old variant
α_1	Progression from Exposed class to Infective with New variant
α_2	Progression from Exposed class to Infective with Old variant
γ_1	Hospitalization rate due to new variant
γ_2	Hospitalization rate due to old variant
d_1	Death rate due to new variant
d_2	Death rate due to old variant
d_3	Death rate in the hospital
Φ	Recovery rate in the hospital

3.1. Equilibria and basic reproduction number

The equilibrium solutions are obtained by solving the following system of equations;

$$\,_{0}^{CF}D_t^\alpha S(t) = \,_{0}^{CF}D_t^\alpha E(t) = \,_{0}^{CF}D_t^\alpha I_n(t) = \,_{0}^{CF}D_t^\alpha I_o(t) = \,_{0}^{CF}D_t^\alpha H(t) = \,_{0}^{CF}D_t^\alpha R(t) = 0.$$

Four equilibrium solutions are obtained; Disease free equilibrium (E_0), Endemic with respect to new variant (E_1), Endemic with respect to old variant (E_2), and Endemic with respect to both variants (E_3).

1. The Disease free equilibrium (E_0) is given as;

$$E_0 = \left(\frac{\lambda}{\mu}, 0, 0, 0, 0, 0 \right),$$

where $S_0 = \frac{\lambda}{\mu}$.

2. Endemic with respect to new variant (E_1) is given as;

$$E_1 = (S^1, E^1, I_n{}^1, I_o{}^1, H^1, R^1)$$

where,

$$S^1 = \frac{\lambda}{\beta_1 I_n{}^1 + \mu}, E^1 = \frac{\left[\beta_1 \lambda - \alpha_1(\beta_1 I_n{}^1 + \mu) \right] I_n{}^1}{\mu \left(\beta_1 I_n{}^1 + \mu \right)}, I_o{}^1 = 0, H^1 = \frac{\gamma_1 I_n{}^1}{\mu + d_3 + \Phi}$$

$$R^1 = \frac{\Phi \gamma_1 I_n{}^1}{\mu(\mu + d_3 + \Phi)}, \text{ and } I_n{}^1 = \frac{\mu}{\beta_1} \left[\frac{\lambda \alpha_1 \beta_1}{\mu^2(\gamma_1 + \mu + d_1 + \alpha_1^2 \beta_1)} - 1 \right].$$

Let,

$$R_1 = \frac{\lambda \alpha_1 \beta_1}{\mu^2(\gamma_1 + \mu + d_1 + \alpha_1^2 \beta_1)}.$$

Therefore, this equilibrium only exists if $R_1 > 1$.

3. Endemic equilibrium with respect to old variant (E_2) is given as;

$$E_2 = \left(S^2, E^2, I_n{}^2, I_o{}^2, H^2, R^2 \right),$$

where,

$$S^2 = \frac{\lambda}{\beta_1 I_o{}^2 + \mu}, E^2 = \frac{\left[\beta_2 \lambda - \alpha_2(\beta_2 I_o{}^2 + \mu) \right] I_o{}^2}{\mu \left(\beta_2 I_o{}^2 + \mu \right)}, I_n{}^2 = 0, H^2 = \frac{\gamma_2 I_o{}^2}{\mu + d_3 + \Phi}$$

$$R^2 = \frac{\Phi \gamma_2 I_o{}^2}{\mu(\mu + d_3 + \Phi)}, \text{ and } I_o{}^2 = \frac{\mu}{\beta_2} \left[\frac{\lambda \alpha_2 \beta_2}{\mu^2(\gamma_2 + \mu + d_2 + \alpha_2^2 \beta_2)} - 1 \right].$$

Let,

$$R_2 = \frac{\lambda \alpha_2 \beta_2}{\mu^2(\gamma_2 + \mu + d_2 + \alpha_2^2 \beta_2)}.$$

Therefore, this equilibrium only exists if $R_2 > 1$.

4. Endemic with respect to both variants (E_3) is given as;

$$E_3 = \left(S^3, E^3, I_n{}^3, I_o{}^3, H^3, R^3 \right),$$

where,

$$S^3 = \frac{\lambda \left(\gamma_1 + \mu + d_1\right)\left(\gamma_2 + \mu + d_2\right)}{\left[\alpha_1\beta_1 \left(\gamma_2 + \mu + d_2\right) + \alpha_2\beta_2 \left(\gamma_1 + \mu + d_1\right)\right]E^3 + \mu \left(\gamma_1 + \mu + d_1\right)\left(\gamma_2 + \mu + d_2\right)},$$

$$I_n^{\ 3} = \frac{\alpha_1 E^3}{\gamma_1 + \mu + d_1}, I_o^{\ 3} = \frac{\alpha_2 E^3}{\gamma_2 + \mu + d_2}, H^3 = \frac{\left[\alpha_1\gamma_1 \left(\gamma_2 + \mu + d_2\right) + \alpha_2\gamma_2 \left(\gamma_1 + \mu + d_1\right)\right]}{\left(\mu + d_3 + \Phi\right)\left(\gamma_1 + \mu + d_1\right)\left(\gamma_2 + \mu + d_2\right)}E^3,$$

$$R^3 = \frac{\Phi \left[\alpha_1\gamma_1 \left(\gamma_2 + \mu + d_2\right) + \alpha_2\gamma_2 \left(\gamma_1 + \mu + d_1\right)\right]}{\mu(\mu + d_3 + \Phi)\left(\gamma_1 + \mu + d_1\right)\left(\gamma_2 + \mu + d_2\right)}E^3, \ and$$

$$E^3 = \frac{\lambda}{\alpha_1 + \alpha_2 + \mu} - \frac{\mu \left(\gamma_1 + \mu + d_1\right)\left(\gamma_2 + \mu + d_2\right)}{\alpha_1\beta_1 \left(\gamma_2 + \mu + d_2\right) + \alpha_2\beta_2 \left(\gamma_1 + \mu + d_1\right)}.$$

Therefore, this equilibrium only exists if $R_1 + R_2 > 1$.

By applying the next generation matrix method presented in [28], basic reproduction ratio (R_0) is obtained to be;

$$R_0 = R_1 + R_2.$$

4. Existence and uniqueness of solution

Here we apply fixed-point results to show the existence and uniqueness of the solutions of model (1). Let (1) be re-written in the following form;

$$\begin{aligned}
{}_0^{CF}D_t^\alpha S\left(t\right) &= F_1\left(t, S\right), \\
{}_0^{CF}D_t^\alpha E\left(t\right) &= F_2\left(t, E\right), \\
{}_0^{CF}D_t^\alpha I_n\left(t\right) &= F_3\left(t, I_n\right), \\
{}_0^{CF}D_t^\alpha I_o\left(t\right) &= F_4\left(t, I_o\right), \\
{}_0^{CF}D_t^\alpha H\left(t\right) &= F_5\left(t, H\right), \\
{}_0^{CF}D_t^\alpha R\left(t\right) &= F_6\left(t, R\right).
\end{aligned}$$

We apply fundamental theorem of Integration to write the system in fractional Volterra form [24]. Applying Caputo–Fabrizio integral operator definition 2, the system above becomes integral equation of Volterra type, with $0 < \alpha < 1$.

$$S\left(t\right) - S\left(0\right) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}F_1\left(t, S\right) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t F_1\left(\eta, S\right)d\eta,$$

$$E\left(t\right) - E\left(0\right) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}F_2\left(t, E\right) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t F_2\left(\eta, E\right)d\eta,$$

$$I_n\left(t\right) - I_n\left(0\right) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}F_3\left(t, I_n\right) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t F_3\left(\eta, I_n\right)d\eta,$$

$$I_o\left(t\right) - I_o\left(0\right) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}F_4\left(t, I_o\right) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t F_4\left(\eta, I_o\right)d\eta,$$

$$H\left(t\right) - H\left(0\right) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}F_5\left(t, H\right) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t F_5\left(\eta, H\right)d\eta,$$

$$R\left(t\right) - R\left(0\right) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}F_6\left(t, R\right) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t F_6\left(\eta, R\right)d\eta.$$

Next, is to prove that the kernels $F_1, \dots, F_6$ satisfy Lipchitz continuity and subsequently contraction. The following theorem takes care of that;

Theorem 1. *The kernel F_1 is Lipschitz. Moreover it satisfies contraction if the following inequality is satisfied;*

$$0 \le \left(\beta_1 k_1 + \beta_2 k_2 + \mu\right) < 1.$$

Proof. Consider S and S_1, then

$$\begin{aligned}
\left\|F_1\left(t, S\right) - F_1\left(t, S_1\right)\right\| &= \left\|-\beta_1 I_n \left(S\left(t\right) - S_1(t)\right) - \beta_2 I_o \left(S\left(t\right) - S_1(t)\right) - \mu(S\left(t\right) - S_1(t))\right\| \\
&\le \beta_1 \left\|I_n(t)\right\| \left\|S\left(t\right) - S_1(t)\right\| + \beta_2 \left\|I_o(t)\right\| \left\|S\left(t\right) - S_1(t)\right\| + \mu \left\|S\left(t\right) - S_1(t)\right\| \\
&\le \left\{\beta_1 k_1 + \beta_2 k_2 + \mu\right\} \left\|S\left(t\right) - S_1(t)\right\| \\
&\le L_1 \left\|S\left(t\right) - S_1(t)\right\|,
\end{aligned} \tag{2}$$

where, $L_1 = \beta_1 k_1 + \beta_2 k_2 + \mu$, $k_1 \geq \|I_n(t)\|$, $k_2 \geq \|I_o(t)\|$, k_1 and k_2 are bounded functions. This implies;

$$\|F_1(t, S) - F_1(t, S_1)\| \leq L_1 \|S(t) - S_1(t)\|.$$

Hence F_1 is Lipschitz continuous. In addition if $0 \leq (\beta_1 k_1 + \beta_2 k_2 + \mu) < 1$, then we have a contraction.

In the same manner, we show the Lipschitz continuity and subsequent contraction of $F_2, \ldots, F_6$;

$$\|F_2(t, E) - F_2(t, E_1)\| \leq L_2 \|E(t) - E_1(t)\|,$$
$$\|F_3(t, I_n) - F_3(t, I_{n1})\| \leq L_3 \|I_n(t) - I_{n1}(t)\|,$$
$$\|F_4(t, I_o) - F_4(t, I_{o1})\| \leq L_4 \|I_o(t) - I_{o1}(t)\|,$$
$$\|F_5(t, H) - F_5(t, H_1)\| \leq L_5 \|H(t) - H_1(t)\|,$$
$$\|F_6(t, R) - F_6(t, R_1)\| \leq L_6 \|R(t) - R_1(t)\|.$$

Recursively, the difference between successive terms in (1) is given as;

$$\pi_{1n}(t) = S_n(t) - S_{n-1}(t)$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} (F_1(t, S_{n-1}) - F_1(t, S_{n-2})) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t (F_1(\eta, S_{n-1}) - F_1(\eta, S_{n-2}))\, d\eta,$$

$$\pi_{2n}(t) = E_n(t) - E_{n-1}(t)$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} (F_2(t, E_{n-1}) - F_2(t, E_{n-2})) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t (F_2(\eta, E_{n-1}) - F_2(\eta, E_{n-2}))\, d\eta,$$

$$\pi_{3n}(t) = I_{nn}(t) - I_{nn-1}(t)$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} (F_3(t, I_{nn-1}) - F_3(t, I_{nn-2}))$$
$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t (F_3(\eta, I_{nn-1}) - F_3(\eta, I_{nn-2}))\, d\eta,$$

$$\pi_{4n}(t) = I_{on}(t) - I_{on-1}(t)$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} (F_4(t, I_{on-1}) - F_4(t, I_{on-2})) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t (F_4(\eta, I_{on-1}) - F_4(\eta, I_{on-2}))\, d\eta,$$

$$\pi_{5n}(t) = H_n(t) - H_{n-1}(t)$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} (F_5(t, H_{n-1}) - F_5(t, H_{n-2})) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t (F_5(\eta, H_{n-1}) - F_5(\eta, H_{n-2}))\, d\eta,$$

$$\pi_{6n}(t) = R_n(t) - R_{n-1}(t)$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} (F_6(t, R_{n-1}) - F_6(t, R_{n-2})) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t (F_6(\eta, R_{n-1}) - F_6(\eta, R_{n-2}))\, d\eta,$$

with the following initial conditions;

$$S_0(t) = S(0), E_0(t) = E(0), I_{n0}(t) = I_n(0), I_{o0}(t) = I_0(0), H_0(t) = H(0), R_0(t) = R(0). \tag{3}$$

Considering π_{1n} and taking norm, we get;

$$\|\pi_{1n}(t)\| = \|S_n(t) - S_{n-1}(t)\|$$
$$= \left\| \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} (F_1(t, S_{n-1}) - F_1(t, S_{n-2})) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t (F_1(\eta, S_{n-1}) - F_1(\eta, S_{n-2}))\, d\eta \right\|.$$

Applying triangular inequality, we get;

$$\|S_n(t) - S_{n-1}(t)\|$$
$$= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \|F_1(t, S_{n-1}) - F_1(t, S_{n-2})\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \left\| \int_0^t (F_1(\eta, S_{n-1}) - F_1(\eta, S_{n-2}))\, d\eta \right\|$$

From (2), we get

$$\|S_n(t) - S_{n-1}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 \|S_{n-1} - S_{n-2}\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_1 \int_0^t \|S_{n-1} - S_{n-2}\|\, d\eta.$$

This implies,

$$\|\pi_{1n}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 \|\pi_{1n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_1 \int_0^t \|\pi_{1n-1}(t)\|\, d\eta.$$

In the same manner,

$$\|\pi_{2n}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_2 \|\pi_{2n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_2 \int_0^t \|\pi_{2n-1}(t)\| \, d\eta,$$

$$\|\pi_{3n}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_3 \|\pi_{3n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_3 \int_0^t \|\pi_{3n-1}(t)\| \, d\eta,$$

$$\|\pi_{4n}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_4 \|\pi_{4n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_4 \int_0^t \|\pi_{4n-1}(t)\| \, d\eta,$$

$$\|\pi_{5n}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_5 \|\pi_{5n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_5 \int_0^t \|\pi_{5n-1}(t)\| \, d\eta,$$

$$\|\pi_{6n}(t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_6 \|\pi_{6n-1}(t)\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_6 \int_0^t \|\pi_{6n-1}(t)\| \, d\eta.$$

Hence, we can write;

$$S_n(t) = \sum_{i=1}^{n} \pi_{1i}(t), \, E_n(t) = \sum_{i=1}^{n} \pi_{2i}(t), \, I_{nn}(t) = \sum_{i=1}^{n} \pi_{3i}(t), \, I_{on}(t) = \sum_{i=1}^{n} \pi_{4i}(t),$$

$$H_n(t) = \sum_{i=1}^{n} \pi_{5i}(t), \, and \, R_n(t) = \sum_{i=1}^{n} \pi_{6i}(t).$$

The following theorem confirms the existence of the solution.

Theorem 2. *The solution of the model exists if we can find t_1 in which the following inequality holds;*

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_i + \frac{2\alpha t_1}{(2-\alpha)M(\alpha)} L_i < 1, \quad i = 1, \ldots, 6.$$

Proof. By applying recursive technique on (2) and (3), we obtain;

$$\|\pi_{1n}(t)\| \leq \|S_n(0)\| \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 + \frac{2\alpha t}{(2-\alpha)M(\alpha)} L_1 \right]^n,$$

$$\|\pi_{2n}(t)\| \leq \|E_n(0)\| \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_2 + \frac{2\alpha t}{(2-\alpha)M(\alpha)} L_2 \right]^n,$$

$$\|\pi_{3n}(t)\| \leq \|I_{nn}(0)\| \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_3 + \frac{2\alpha t}{(2-\alpha)M(\alpha)} L_3 \right]^n,$$

$$\|\pi_{4n}(t)\| \leq \|I_{on}(0)\| \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_4 + \frac{2\alpha t}{(2-\alpha)M(\alpha)} L_4 \right]^n,$$

$$\|\pi_{5n}(t)\| \leq \|H_n(0)\| \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_5 + \frac{2\alpha t}{(2-\alpha)M(\alpha)} L_5 \right]^n,$$

$$\|\pi_{6n}(t)\| \leq \|R_n(0)\| \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_6 + \frac{2\alpha t}{(2-\alpha)M(\alpha)} L_6 \right]^n.$$

Therefore, the solutions exist and are continuous. To confirm the functions above construct solutions of (1), we consider;

$$S(t) - S(0) = S_n(t) - B_{1n}(t),$$

$$E(t) - E(0) = E_n(t) - B_{2n}(t),$$

$$I_n(t) - I_n(0) = I_{nn}(t) - B_{3n}(t),$$

$$I_o(t) - I_o(0) = I_{on}(t) - B_{4n}(t),$$

$$H(t) - H(0) = H_n(t) - B_{5n}(t),$$

$$R(t) - R(0) = R_n(t) - B_{6n}(t).$$

Hence,

$$\|B_{1n}(t)\| = \left\| \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} (F_1(t,S) - F_1(t,S_{n-1})) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t (F_1(\eta,S) - F_1(\eta,S_{n-1}))\, d\eta \right\|$$

$$\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \|F_1(t,S) - F_1(t,S_{n-1})\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \left\| \int_0^t (F_1(\eta,S) - F_1(\eta,S_{n-1}))\, d\eta \right\|$$

$$\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 \|S - S_{n-1}\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_1 \|S - S_{n-1}\|\, t.$$

Repeating the same procedure,

$$\|B_{1n}(t)\| \leq \left(\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} t \right)^{n+1} L_1^{n+1} b. \tag{4}$$

At t_1, we have;

$$\|B_{1n}(t)\| \leq \left(\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} t_1 \right)^{n+1} L_1^{n+1} b.$$

Taking limit on (4) as n approaches ∞, we get, $\|B_{1n}(t)\| \to 0$. Similarly, $\|B_{2n}(t)\|$, $\|B_{3n}(t)\|$, $\|B_{4n}(t)\|$, $\|B_{5n}(t)\|$, $\|B_{6n}(t)\|$ $\to 0$.

Lastly, to show the uniqueness of the solutions of the model, we suppose there exist some solutions of the model say; $S^1(t), E^1(t), I_n^1(t), I_o^1(t), H^1(t), R^1(t)$, then

$$S(t) - S^1(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} (F_1(t,S) - F_1(t,S^1)) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t (F_1(\eta,S) - F_1(\eta,S^1))\, d\eta.$$

Taking norm, we get

$$\left\| S(t) - S^1(t) \right\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left\| F_1(t,S) - F_1(t,S^1) \right\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left\| F_1(\eta,S) - F_1(\eta,S^1) \right\| d\eta.$$

Applying the Lipschitz continuity result, we get

$$\left\| S(t) - S^1(t) \right\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 \left\| S(t) - S^1(t) \right\| + \frac{2\alpha L_1 t}{(2-\alpha)M(\alpha)} \left\| S(t) - S^1(t) \right\|.$$

It simplifies to,

$$\left\| S(t) - S^1(t) \right\| \left(1 - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 - \frac{2\alpha L_1 t}{(2-\alpha)M(\alpha)} \right) \leq 0. \tag{5}$$

Theorem 3. *If the condition below holds,*

$$\left(1 - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 - \frac{2\alpha L_1 t}{(2-\alpha)M(\alpha)} \right) > 0,$$

then the solution is unique.

Proof. Consider (5), that is

$$\left\| S(t) - S^1(t) \right\| \left(1 - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 - \frac{2\alpha L_1 t}{(2-\alpha)M(\alpha)} \right) \leq 0,$$

since,

$$\left(1 - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 - \frac{2\alpha L_1 t}{(2-\alpha)M(\alpha)} \right) > 0,$$

then

$$\left\| S(t) - S^1(t) \right\| = 0.$$

This implies,

$$S(t) = S^1(t).$$

This is true for the remaining solutions. Hence, the model solution exists and is unique.

5. Numerical scheme and numerical simulations

In this section, we give an approximate solution of the Caputo–Fabrizio fractional order model for the dynamics of two-strain COVID-19 model using two-step fractional Adams–Bash forth technique [25]. We use fundamental theorem of Integration to write the system in fractional Volterra form. Consider the first equation in (1),

$$S(t) - S(0) = \frac{1-\alpha}{M(\alpha)} F_1(t, S) + \frac{\alpha}{M(\alpha)} \int_0^t F_1(\eta, S) \, d\eta$$

for $t = t_{j+1}, j = 0, 1, 2, \ldots$, we get

$$S(t_{j+1}) - S_0 = \frac{1-\alpha}{M(\alpha)} F_1(t_j, S_j) + \frac{\alpha}{M(\alpha)} \int_0^{t_{j+1}} F_1(t, S) \, dt.$$

Hence, the difference in successive terms is given as,

$$S_{j+1} - S_j = \frac{1-\alpha}{M(\alpha)} \left\{ F_1(t_j, S_j) - F_1(t_{j-1}, S_{j-1}) \right\} + \frac{\alpha}{M(\alpha)} \int_{t_j}^{t_{j+1}} F_1(t, S) \, dt,$$

over the interval $[t_k, t_{k+1}]$. We can approximate $F_1(t, S)$ interpolation polynomial;

$$P_k(t) \cong \frac{f(t_k, y_k)}{h} (t - t_{k-1}) - \frac{f(t_{k-1}, y_{k-1})}{h} (t - t_k),$$

where $h = t_j - t_{j-1}$. Also

$$\int_{t_j}^{t_{j+1}} F_1(t, S) \, dt = \int_{t_j}^{t_{j+1}} \left(\frac{F_1(t_j, S_j)}{h} (t - t_{j-1}) - \frac{F_1(t_{j-1}, S_{j-1})}{h} (t - t_j) \right) dt$$

$$= \frac{3h}{2} F_1(t_j, S_j) - \frac{h}{2} F_1(t_{j-1}, S_{j-1}).$$

Simplifying, we get

$$S_{j+1} = S_0 + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3h}{2M(\alpha)} \right) F_1(t_j, S_j) - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)} \right) F_1(t_{j-1}, S_{j-1}).$$

Similarly, we get

$$E_{j+1} = E_0 + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3h}{2M(\alpha)} \right) F_2(t_j, E_j) - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)} \right) F_2(t_{j-1}, E_{j-1}),$$

$$I_{n,j+1} = I_{n,0} + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3h}{2M(\alpha)} \right) F_3(t_j, I_{n,j}) - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)} \right) F_3(t_{j-1}, I_{n,j-1}),$$

$$I_{o,j+1} = I_{o,0} + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3h}{2M(\alpha)} \right) F_4(t_j, I_{o,j}) - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)} \right) F_4(t_{j-1}, I_{o,j-1}),$$

$$H_{j+1} = H_0 + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3h}{2M(\alpha)} \right) F_5(t_j, H_j) - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)} \right) F_5(t_{j-1}, H_{j-1}),$$

$$R_{j+1} = R_0 + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3h}{2M(\alpha)} \right) F_6(t_j, R_j) - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)} \right) F_6(t_{j-1}, R_{j-1}).$$

We describe the numerical simulations to study the dynamics of the propose model for various values of $\alpha \in [0, 1]$ and mode parameters. The parameter values used are obtained from [30], and they are; $\lambda = 400$, $\beta_1 = 1.7 \times 10^{-5}$, $\beta_2 = 1.7 \times 10^{-5}$, $\mu = 5 \times 10^{-4}$, $\alpha_1 = 2 \times 10^{-4}$, $\alpha_2 = 2 \times 10^{-4}$, $\gamma_1 = 1.6979 \times 10^{-1}$, $\gamma_2 = 1.6979 \times 10^{-1}$, $d_1 = 9.6 \times 10^{-3}$, $d_2 = 9.6 \times 10^{-3}$, $d_3 = 1 \times 10^{-6}$, $\Phi = 0.06$, $\alpha \in [0, 1]$.

Figs. 1–6, show the influence of the variation in the fractional order α on the biological behavior of the classes of model (1). It is clear from these Figures that the population of Susceptible individuals, Exposed individuals, New variant of COVID-19, Old variant of COVID-19, Hospitalized individuals and Recovered individuals have decreasing effect when α is decreased from $1 - 0.2$.

Fig. 7 compares the dynamics of new and old strain of COVID-19. This figure shows that the two variants can co-exist in the same population when their basic reproduction ratio is the same.

Fig. 8 shows that when $R_1 > R_2$, then the new variant of COVID-19 outperform the old variant which leads to subsequent domination of the old variant by the new variant.

Fig. 9 shows that when $R_2 > R_1$, then the old variant of COVID-19 outperform the new variant which leads to subsequent domination of the new variant by the old variant.
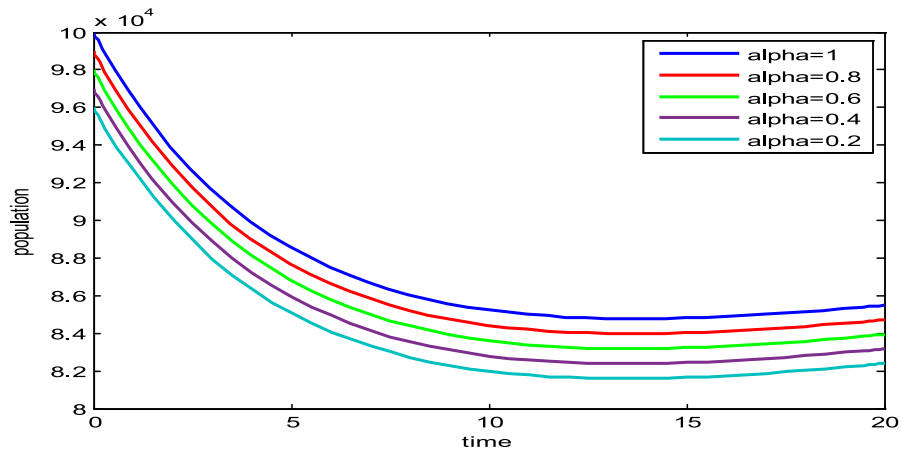
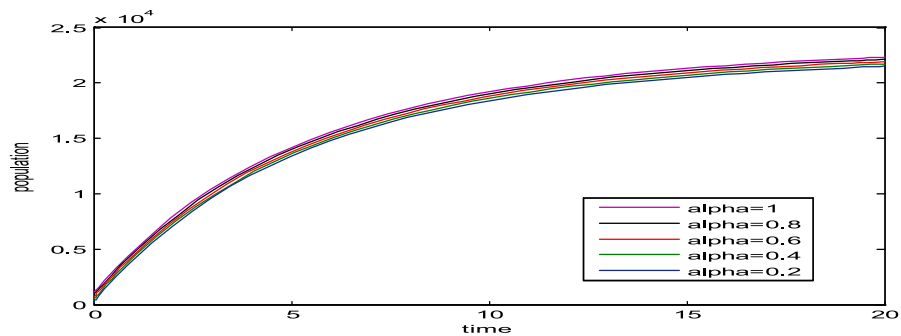**Fig. 1.** Dynamics of Susceptible individuals for various values of α.

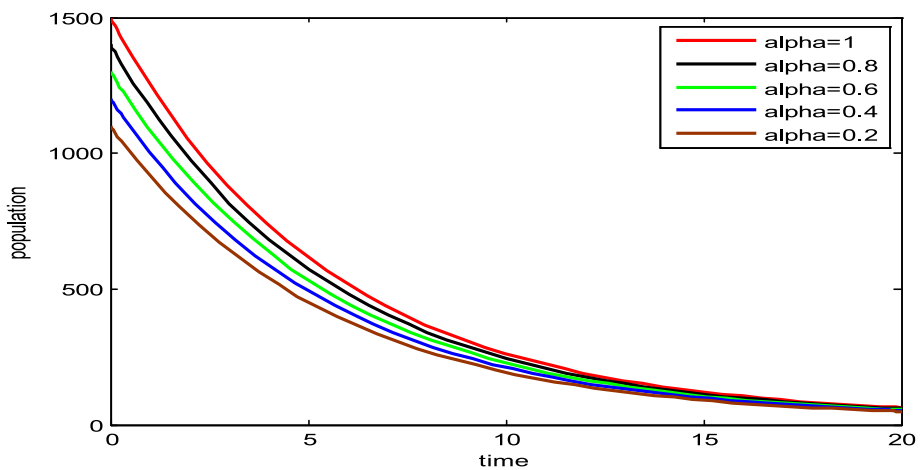**Fig. 2.** Dynamics of Exposed individuals for various values of α.

Fig. 3. Dynamics of new variant of COVID-19 for various values of α.

It worth mentioning here that the fractional derivative with $\alpha \in (0, 1]$ is defined in Caputo–Fabrizio sense, so introducing a convolution integral with a power-law memory kernel benefits in describing memory effects in dynamical systems. The decaying rate of the memory kernel depends on α. A lower value of α corresponding to more slowly-decaying time-correlation functions leads a long memory. Therefore, as $\alpha \rightarrow 1$, the influence of memory decreases.
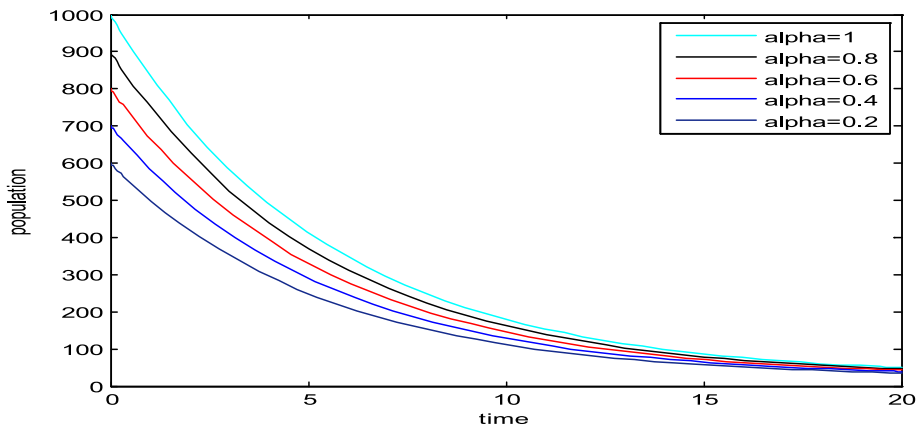
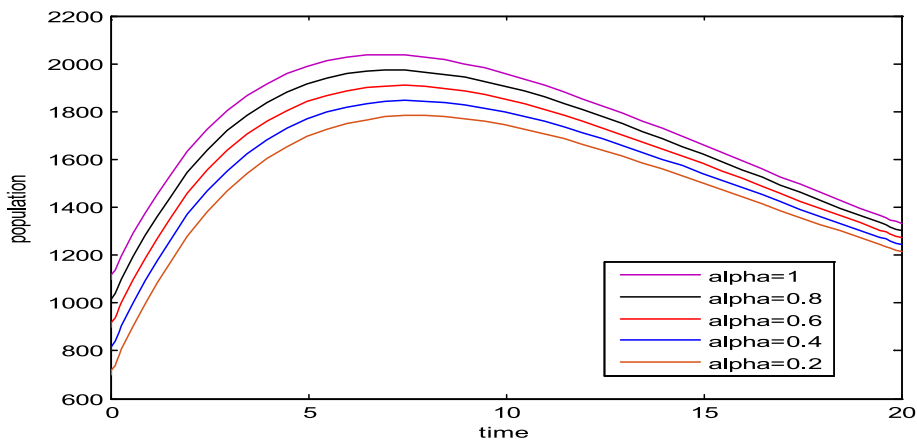**Fig. 4.** Dynamics of old variant of COVID-19 for various values of α.

Fig. 5. Dynamics of hospitalized individuals for various values of α.
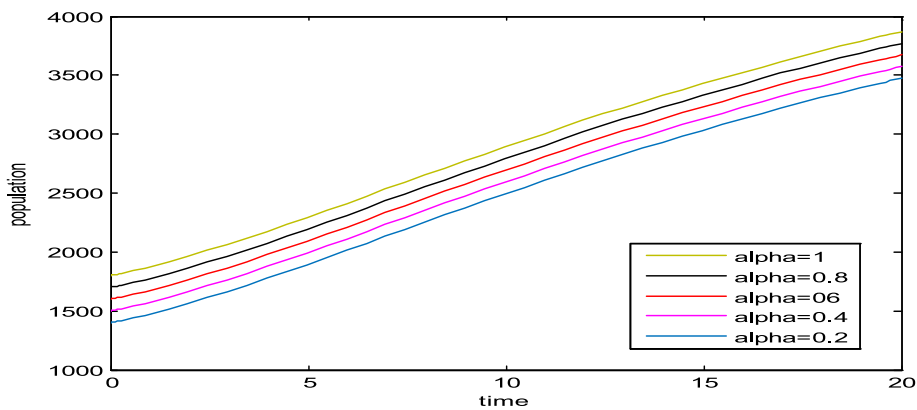

Fig. 6. Dynamics of Recovered individuals for various values of α.

6. Conclusion

The dynamics of COVID-19 variations were explored using a Caputo–Fabrizio fractional-order model. The model's fundamental properties were studied. The next-generation matrix (NGM) approach was used to calculate the basic reproduction ratio R_0. Equilibrium solutions are found by equating system (1) to zero and simultaneously solving the result. Fixed point theory is used to perform a detailed study of the existence and uniqueness of the model solution. The
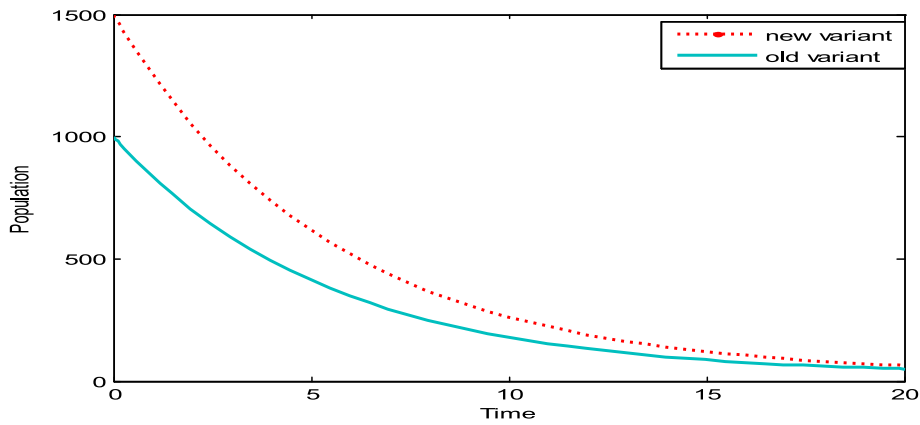
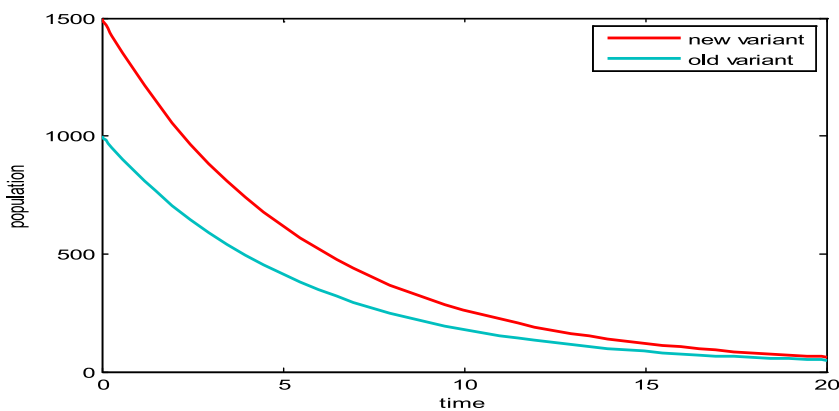**Fig. 7.** Dynamics of new and old variants of COVID-19..

Fig. 8. Dynamics of new and old variants of COVID-19 when $R_1 > R_2$.
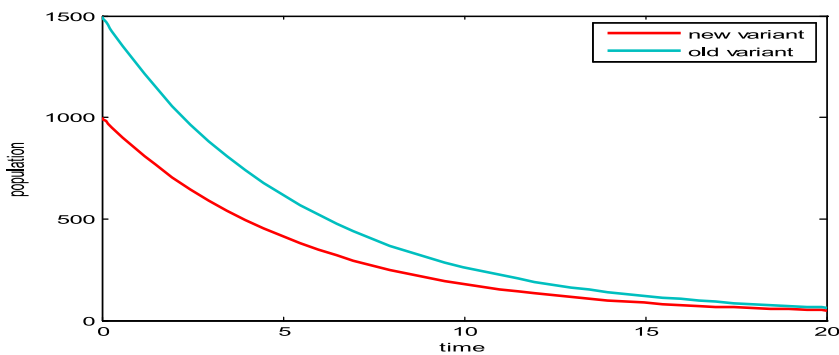

Fig. 9. Dynamics of new and old variants of COVID-19 when $R_2 > R_1$.

iterative solution of the model is computed using the fractional Adams–Bashforth technique. The numerical results are shown using the estimated values of the model parameters to justify the importance of the fractional-order derivative. The graphs provide useful information about the model's complexity and the feasibility of obtaining reliable information about it.

CRediT authorship contribution statement

Isa Abdullahi Baba: Visualization, Investigation, Supervision, Software, Validation, Writing – review & editing. **Fathalla A. Rihan:** Conceptualization, Methodology, Software, Data curation, Writing – original draft.

Declaration of competing interest

Funding

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