A review of spatial causal inference methods for environmental and epidemiological applications

Supplementary Materials

Appendix A.1: CAR and SAR covariance models

In Section 2, we define the CAR and SAR models for individual observations, and in this section we provide the induced joint distribution of the spatial process at all N locations. If $\mathbf{U} \sim \text{CAR}(\rho, \sigma)$ then the joint distribution of $\mathbf{U}$ defined by the full conditional distributions given in Section 2.1 is multivariate normal with mean zero and covariance $\Sigma_{CAR}(\rho, \sigma) = \sigma^2(\mathbf{M} - \rho\mathbf{W})^{-1}$, where $\mathbf{M}$ is diagonal with the i^{th} diagonal element m_i (the number of regions neighboring region i) and $\mathbf{W}$ has (i, k) element equal one if regions i and k are adjacent and zero otherwise.

Similarly, the SAR model in (5) can be solved for $\mathbf{Y} = (Y_1, ..., Y_N)^T$ to show that the induced joint distribution is

$$\mathbf{Y} = \mathbf{A}\beta + \mathbf{X}\boldsymbol{\gamma} + \boldsymbol{\varepsilon} \quad \text{where} \quad \boldsymbol{\varepsilon} \sim \text{Normal}\left\{\mathbf{0}, \sigma^2(\mathbf{I}_N - \psi\mathbf{C})^{-1}(\mathbf{I}_N - \psi\mathbf{C})^{-1}\right\} \tag{33}$$

with the (i, k) element of $\mathbf{C}$ is $1/m_i$ if regions i and k are adjacent and 0 otherwise, so that, e.g., $\mathbf{CY} = (\bar{Y}_1, ..., \bar{Y}_N)^T$ is the vector of neighborhood means.

Appendix A.2

Consider the true data-generating model $\mathbf{Y}|\mathbf{A}, \mathbf{U} \sim \text{Normal}(\beta\mathbf{A} + \mathbf{U}, \tau^2\mathbf{I}_n)$, $\mathbf{U}|\mathbf{A} \sim \text{Normal}(\phi\mathbf{A}, \boldsymbol{\Sigma}_1)$ and $\mathbf{A} \sim \text{Normal}(0, \boldsymbol{\Sigma}_2)$. In this model the treatment variable and spatial process are correlated unless $\phi = 0$. If the assumed model is $\mathbf{Y}|\mathbf{A}, \mathbf{U} \sim \text{Normal}(\beta\mathbf{A} + \mathbf{U}, \tau^2\mathbf{I}_n)$ and $\mathbf{U}|\mathbf{A} \sim \text{Normal}(0, \boldsymbol{\Omega})$, or equivalently $\mathbf{Y}|\mathbf{A} \sim \text{Normal}(\beta\mathbf{A}, \boldsymbol{\Sigma})$ where $\boldsymbol{\Sigma} = \tau^2\mathbf{I}_n + \boldsymbol{\Omega}$, then the generalized least squares (and posterior mean under flat prior) estimator is $\hat{\beta}(\mathbf{A}, \mathbf{Y}) = (\mathbf{A}^T\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\boldsymbol{\Sigma}^{-1}\mathbf{Y}$. The expected value of this estimator under the true data-generating model is $\beta + \phi$ for any assumed covariance model $\boldsymbol{\Sigma}$, including the model that excludes $\mathbf{U}$ by setting $\boldsymbol{\Omega} = 0$.

Appendix A.3: Details of Schnell and Papadogeorgou (2020)

Schnell and Papadogeorgou (2020) provided a set of assumptions to identify the unmeasured confounding bias $\text{E}(U_i|\mathbf{A})$. They assume a joint distribution for ($\mathbf{U}$, $\mathbf{A}$) that is multivariate normal with mean zero and covariance

$$\text{Cov}\begin{pmatrix}\mathbf{U}\\ \mathbf{A}\end{pmatrix} = \begin{pmatrix}\mathbf{Q}_U & \mathbf{Q}_{UA}\\ \mathbf{Q}_{UA}^T & \mathbf{Q}_A\end{pmatrix}^{-1},$$

where $Q_j = \sigma_j^{-2}(\mathbf{M} - \rho_j\mathbf{W})$ for $j \in \{U, A\}$ and $\mathbf{Q}_{UA} = -\rho\sigma_U\sigma_A\mathbf{M}$. Two assumptions are encoded in $\mathbf{Q}_{UA}$: (1) a cross-Markov relationship such that conditional on all other locations' treatments $\mathbf{A}_{-i}$, the local treatment A_i is only correlated with the local confounder U_i (e.g., Reich et al., 2007), and (2) the conditional correlation between A_i and U_i is constant in space. The confounding bias $B(\mathbf{A}) = \text{E}(\mathbf{U}|\mathbf{A}) = -\mathbf{Q}_U^{-1}\mathbf{Q}_{UA}\mathbf{A}$ is mitigated by fitting a spatial model with confounder adjustment,

$$\begin{aligned}\mathbf{Y} &= \mathbf{A}\beta - B(\mathbf{A}) + \mathbf{X}\boldsymbol{\gamma} + \boldsymbol{e} \quad \text{where} \quad \boldsymbol{e} \sim \text{Normal}\left\{\mathbf{0}, \mathbf{Q}_U^{-1} + \tau^2\mathbf{I}_N\right\}\\ \mathbf{A} &\sim \text{Normal}\left[\mathbf{0}, \sigma_A^2\left\{(\mathbf{M} - \rho_A\mathbf{W}) - \rho^2\mathbf{M}^T(\mathbf{M} - \rho_U\mathbf{W})^{-1}\mathbf{M}\right\}^{-1}\right].\end{aligned}$$

Appendix A.4: Additional simulation results

In this section, we conduct additional simulations based on the COVID-19 and $PM_{2.5}$ data analyzed in Section 2.9. The data are generated using the $p = 15$ real standardized confounding variables $\mathbf{X}_i$ (e.g., climate and socioeconomic variables), many of which have a strong spatial pattern. The treatment variable is generated as $A_i|e_i \overset{indep}{\sim} \text{Bernoulli}(e_i)$ where $\text{logit}(e_i) = \mathbf{X}_i\boldsymbol{\alpha} + V_i$, $\boldsymbol{\alpha} = (0.2, ..., 0.2)^T$ and $\mathbf{V} \sim \text{CAR}(\rho_V, 1)$. Given the treatment variable, the response variable is generated as $Y_i = A_i\beta + \mathbf{X}_i\boldsymbol{\gamma} + 0.5V_i + U_i + \varepsilon_i$ where $\beta = 0.5$, $\boldsymbol{\gamma} = (0.1, ..., 0.1)^T$, $\mathbf{U} \sim \text{CAR}(\rho_U, 1)$ and $\varepsilon_i \overset{iid}{\sim} \text{Normal}(0, 1)$. We consider two scenarios by varying $\rho_U = \rho_V \in \{0.90, 0.99\}$.

The simulated data are analyzed using the same methods as in the main simulation of Section 2.8. Notably, the methods do not make use of $\mathbf{X}_i$ and thus $\mathbf{X}_i$ is a missing spatial confounding variable. The results are plotted in Figure 7 and are consist with those in Section 2.8, i.e., the joint and spatial propensity score methods have the best performance.

Figure 7: **Additional simulation study results**. The boxplots summarize the sampling distribution of the causal estimates across datasets and the solid line at 0.5 is the true value. The scenarios vary by the spatial dependence parameter of the confounder (ρ_u) and treatment (ρ_v) variables, and whether the joint model is misspecified. The competing methods are defined in Section 2.8. The empirical coverage of 95% credible intervals for the causal effect are given above the model labels.