Supporting information for 'Covariate adjustment in continuous biomarker assessment' by

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S1. Simulation Study with Discrete Covariates

We use the same setting as the simulation study in Janes and Pepe (2009), where covariate Z is a discrete scalar. As discussed in the introduction section of main manuscript, our point estimator reduces to the nonparametric estimator considered by Janes and Pepe (2009) when covariate takes finite values. However, our inference procedure, especially the sample-based variance estimator, is different from the proposal in Janes and Pepe (2009) and is the focus of the evaluation here.

The biomarker from cases M_1 and controls M_0 follows normal distribution conditional on Z: $M_0 \sim N(0,1)$ and $M_1 \sim N(0.9,1)$ if $Z=0$, $M_0 \sim N(0.2,1)$ and $M_1 \sim N(0.9,1)$ if $Z=1$. We consider both specificity at controlled sensitivity level and sensitivity at controlled specificity level in this study, since specificity at controlled senstivity level is of interest for this paper whereas the reverse way allows us to directly compare with the performance in Janes and Pepe (2009). The true covariate-adjusted specificity is 0.20, 0.31, 0.48, and 0.60 under controlled sensitivity level 0.95, 0.90, 0.85 and 0.80. The true covariate-adjusted sensitivity is 0.21, 0.33, 0.50, 0.80 under controlled specificity level 0.95, 0.90, 0.85 and 0.80. We implement both sample-based standard error and bootstrap-derived standard error. With mean and SE of estimators, we construct Wald-type confidence intervals as well as logit-transformed confidence interval (Pepe et al. 2003, page 102). Janes and Pepe (2009) reported that logit-transformed confidence interval can improve coverage when controlled specificity is close to 0 or 1.

Table S1 and S2 present the simulation results from this setting. Our standard error estimators are close to the standard deviation obtained in both tables, indicating that our proposed inference procedures are effective. Comparing Table S1 and the results in Janes and Pepe (2009), our sample-based inference procedure achieves better covarage rate, especially when controlled specificity is 0.95. Although Table S2 focus on estimating specificity under controlled sensitivity and Janes and Pepe (2009) is the reverse way, the evaluation metrics such as percentage bias and confidence interval coverage rates are comparable between the two studies. We again find that our proposed method has similar or even higher coverage rate comparing with Janes and Pepe (2009).

S2. Proof of Theorems

Proof of Theorem 1. We first establish the consistency and asymptotic normality of $\widehat{\boldsymbol{\beta}}$. These results for quantile regression have been established by, for example, Koenker (2005, section 4.1.1 and theorem 4.1) under fixed design. Although we consider random design, among other assumptions, similar arguments follow through to give the consistency of $\widehat{\boldsymbol{\beta}}$ and

$$n_1^{1/2}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \quad \overset{d}{\to} \quad N\Big(0,\ \rho_0(1-\rho_0)\boldsymbol{D}_1^{-1}\boldsymbol{D}_0\boldsymbol{D}_1^{-1}\Big), \tag{1}$$

Table S1: Results under the simulation setting with discrete covariate values. Sensitivity ρ_0 under controlled specificity ϕ_0 is estimated and presented.

$n_1 = n_0$	Bias	SD	Sample-based			Boostrap-based		
			SE	Cov	LCov	SE	Cov	LCov
			$\phi_0 = 0.95,\ \rho_0 = 0.21$					
100	310	885	1190	93.00	92.66	905	92.02	90.48
200	166	642	795	91.52	91.74	671	93.78	92.94
500	93	416	467	91.72	91.52	429	93.56	93.10
1000	44	293	324	93.16	93.02	304	93.88	93.74
			$\phi_0 = 0.90,\ \rho_0 = 0.33$					
100	200	920	1126	92.50	93.30	962	93.78	94.20
200	95	670	770	92.20	92.98	691	94.04	94.38
500	49	424	466	92.86	92.76	438	93.88	93.96
1000	19	300	323	93.68	93.48	309	94.12	94.60
			$\phi_0 = 0.85,\ \rho_0 = 0.50$					
100	108	872	1017	92.42	93.84	919	93.98	95.80
200	39	640	706	92.58	93.46	657	93.96	94.88
500	20	400	434	93.46	93.62	416	93.74	94.54
1000	2	284	300	94.00	94.20	291	94.34	94.56
			$\phi_0 = 0.50,\ \rho_0 = 0.80$					
100	-15	607	683	93.70	95.22	640	95.10	96.12
200	-13	421	469	94.32	95.34	444	94.70	95.80
500	0	270	286	94.24	94.94	276	94.82	95.04
1000	-16	192	200	95.04	95.04	195	95.10	94.96

Bias, $(\widehat{\rho}-\rho)\times 10^4$; SD, standard deviation ($\times 10^4$); SE, mean standard error ($\times 10^4$); Cov (%) and LCov (%), coverage rates of 95% confidence interval and logit-transformed confidence interval.

Table S2: Results under the simulation setting with discrete covariate values. Specificity ρ_0 under controlled sensitivity ϕ_0 is estimated and presented.

$n_1 = n_0$	Bias	SD	Sample-based			Boostrap-based		
			SE	Cov	LCov	SE	Cov	LCov
			$\rho_0 = 0.95,\ \phi_0 = 0.19$					
100	202	819	1071	90.76	93.08	839	93.12	91.46
200	94	570	724	91.04	92.30	608	94.94	94.58
500	32	372	417	90.28	91.70	382	94.20	94.16
1000	22	261	287	91.36	91.74	270	94.58	94.66
			$\rho_0 = 0.90,\ \phi_0 = 0.30$					
100	144	871	1117	90.86	92.34	909	94.26	95.28
200	42	612	730	90.94	92.02	651	94.68	95.26
500	27	389	439	91.76	92.28	410	94.80	94.98
1000	6	282	303	92.08	92.36	290	93.98	94.30
			$\rho_0 = 0.85,\ \phi_0 = 0.47$					
100	42	886	1061	91.10	92.36	912	94.18	95.66
200	16	632	709	90.88	91.74	647	93.84	94.68
500	10	398	436	92.48	92.68	411	94.32	94.66
1000	-6	286	301	92.58	92.78	288	94.22	94.32
			$\rho_0 = 0.80,\ \phi_0 = 0.59$					
100	1	832	971	91.22	92.28	863	94.48	95.84
200	-8	588	660	91.40	92.04	612	94.52	95.50
500	0	372	402	92.62	92.90	384	94.72	95.12
1000	-3	265	280	93.46	93.42	270	94.74	94.76

Bias, $(\widehat{\phi}-\phi)\times 10^4$; SD, standard deviation ($\times 10^4$); SE, mean standard error ($\times 10^4$); Cov (%) and LCov (%), coverage rates of 95% confidence interval and logit-transformed confidence interval.

where $\boldsymbol{D}_0 = E\widetilde{\boldsymbol{Z}}_1^{\otimes 2}$ and $\boldsymbol{D}_1 = E\{F_1^{'}(\widetilde{\boldsymbol{Z}}_1^T\boldsymbol{\beta}_0)\widetilde{\boldsymbol{Z}}_1^{\otimes 2}\}$.

Next we turn to $\widehat{\phi}$. By Condition 4a and the consistency of $\widehat{\boldsymbol{\beta}}$, the consistency of $\widehat{\phi}$ can be easily established. For asymptotic normality, we have

$$\begin{aligned} n_0^{1/2}(\widehat{\phi} - \phi_0) &= n_0^{-1/2}\sum_{i=1}^{n_0}\{I(M_{0i} \le \widetilde{\boldsymbol{Z}}_{0i}^T\widehat{\boldsymbol{\beta}}) - \Pr(M_0 \le \widetilde{\boldsymbol{Z}}_0^T\boldsymbol{\beta}_0)\} \\ &= n_0^{-1/2}\sum_{i=1}^{n_0}\{I(M_{0i} \le \widetilde{\boldsymbol{Z}}_{0i}^T\widehat{\boldsymbol{\beta}}) - \Pr(M_0 \le \widetilde{\boldsymbol{Z}}_0^T\boldsymbol{\beta}|\boldsymbol{\beta} = \widehat{\boldsymbol{\beta}})\} \\ &\quad + n_0^{1/2}\{\Pr(M_0 \le \widetilde{\boldsymbol{Z}}_0^T\boldsymbol{\beta}|\boldsymbol{\beta} = \widehat{\boldsymbol{\beta}}) - \Pr(M_0 \le \widetilde{\boldsymbol{Z}}_0^T\boldsymbol{\beta}_0)\} \\ &\equiv A_n(\widehat{\boldsymbol{\beta}}) + B_n. \end{aligned}$$

Since $F_0(t;\boldsymbol{z})$ is differentiable at $\widetilde{\boldsymbol{Z}}_0^T\boldsymbol{\beta}_0$, in light of (1), Delta method leads to

$$B_n \ \stackrel{d}{\to}\ N\Big(0,\ c\rho_0(1-\rho_0)\boldsymbol{D}_2^T\boldsymbol{D}_1^{-1}\boldsymbol{D}_0\boldsymbol{D}_1^{-1}\boldsymbol{D}_2\Big), \tag{2}$$

where $\boldsymbol{D}_2 = E\{F_0^{'}(\widetilde{\boldsymbol{Z}}_0^T\boldsymbol{\beta}_0)\widetilde{\boldsymbol{Z}}_0\}$. Meanwhile, $A_n(\widehat{\boldsymbol{\beta}})$ can be written as

$$A_n(\boldsymbol{\beta}_0) + \{A_n(\widehat{\boldsymbol{\beta}}) - A_n(\boldsymbol{\beta}_0)\}, \tag{3}$$

where $A_n(\boldsymbol{\beta}_0) = n_0^{-1/2}\sum_{i=1}^{n_0}\{I(M_{0i} \le \widetilde{\boldsymbol{Z}}_{0i}^T\boldsymbol{\beta}_0) - \phi_0\}$. By central limit theorem,

$$A_n(\boldsymbol{\beta}_0) \ \stackrel{d}{\to}\ N\big(0, \phi_0(1-\phi_0)\big). \tag{4}$$

On the other hand,

$$\begin{aligned} E[\{A_n(\widehat{\boldsymbol{\beta}}) - A_n(\boldsymbol{\beta}_0)\}^2] &= n_0^{-1}\sum_{i=1}^{n_0}E\{I(M_{0i} \le \widetilde{\boldsymbol{Z}}_{0i}^T\widehat{\boldsymbol{\beta}}) - I(M_{0i} \le \widetilde{\boldsymbol{Z}}_{0i}^T\boldsymbol{\beta}_0) \\ &\qquad - \Pr(M_0 \le \widetilde{\boldsymbol{Z}}_0^T\widehat{\boldsymbol{\beta}}|\widehat{\boldsymbol{\beta}}) + \Pr(M_0 \le \widetilde{\boldsymbol{Z}}_0^T\boldsymbol{\beta}_0)\}^2 \\ &= E\Big(E\big[\{I(M_0 \le \widetilde{\boldsymbol{Z}}_0^T\widehat{\boldsymbol{\beta}}) - I(M_0 \le \widetilde{\boldsymbol{Z}}_0^T\boldsymbol{\beta}_0) \\ &\qquad - F_0(\widetilde{\boldsymbol{Z}}_0^T\widehat{\boldsymbol{\beta}}) - F_0(\widetilde{\boldsymbol{Z}}_0^T\boldsymbol{\beta}_0)\}^2|\widetilde{\boldsymbol{Z}}_0^T\widehat{\boldsymbol{\beta}}\big]\Big) \\ &\le E|I(M_0 \le \widetilde{\boldsymbol{Z}}_0^T\widehat{\boldsymbol{\beta}}) - I(M_0 \le \widetilde{\boldsymbol{Z}}_0^T\boldsymbol{\beta}_0)| \\ &\le E|F_{0|\boldsymbol{z}}(\widetilde{\boldsymbol{Z}}_0^T\widehat{\boldsymbol{\beta}}) - F_{0|\boldsymbol{z}}(\widetilde{\boldsymbol{Z}}_0^T\boldsymbol{\beta}_0)|. \end{aligned}$$

By Markov's inequality, $A_n(\widehat{\boldsymbol{\beta}}) - A_n(\boldsymbol{\beta}_0) \stackrel{d}{\to} 0$.
Together with (2) and (4), Slutsky's theorem yields the result. □

Proof of Theorem 2. Start with the cases, and write

$$\begin{aligned} \Psi_n(\boldsymbol{\beta}, \rho) &= n_1^{-1}\sum_{i=1}^{n}\widetilde{\boldsymbol{Z}}_{1i}\{I(M_{1i} > \widetilde{\boldsymbol{Z}}_{1i}^T\boldsymbol{\beta}) - \rho\}, \\ \Psi(\boldsymbol{\beta}, \rho) &= E\big[\widetilde{\boldsymbol{Z}}_1\{I(M_1 > \widetilde{\boldsymbol{Z}}_1^T\boldsymbol{\beta}) - \rho\}\big]. \end{aligned}$$

It is known that $\{I(M_1 > \widetilde{\boldsymbol{Z}}_1^T\boldsymbol{\beta}) : \boldsymbol{\beta} \in \mathbb{R}^{\boldsymbol{p}}\}$ is Donsker (e.g. Kosorok 2007, lemma 9.12). Furthermore, $\widetilde{\boldsymbol{Z}}_1$ is bounded by Condition 2. By permanence property of the Donsker class, $\{\widetilde{\boldsymbol{Z}}_1 I(M_1 > \widetilde{\boldsymbol{Z}}_1^T\boldsymbol{\beta}) : \boldsymbol{\beta} \in \mathbb{R}^{\boldsymbol{p}}\}$ is Donsker. Since Donsker implies Glivenko-Cantelli, it follows that, almost surely

$$\sup_{\boldsymbol{\beta}, \rho\in[\rho_1,\rho_2]} ||\Psi_n(\boldsymbol{\beta}, \rho) - \Psi(\boldsymbol{\beta}, \rho)|| = o(1).$$

Thus, $||\Psi\{\widehat{\boldsymbol{\beta}}(\rho),\rho\}|| \le ||\Psi_n\{\widehat{\boldsymbol{\beta}}(\rho),\rho\}|| + ||\Psi_n\{\widehat{\boldsymbol{\beta}}(\rho),\rho\} - \Psi\{\widehat{\boldsymbol{\beta}}(\rho),\rho\}||$ leads to, almost surely,

$$\sup_{\rho\in[\rho_1,\rho_2]} ||\Psi\{\widehat{\boldsymbol{\beta}}(\rho),\rho\}|| = o(1).$$

It remains to be shown that, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\sup_{\rho\in[\rho_1,\rho_2]} ||\Psi\{\boldsymbol{\beta}(\rho),\rho\}|| < \delta$, then $\sup_{\rho\in[\rho_1,\rho_2]} ||\boldsymbol{\beta}(\rho) - \boldsymbol{\beta}_0(\rho)|| < \epsilon$. Suppose that this is not true. Thus, for each $\delta > 0$, there exists (ζ,ν) such that $||\Psi(\zeta,\nu) - \Psi\{\boldsymbol{\beta}_0(\nu),\nu\}|| < \delta$ and $||\zeta - \boldsymbol{\beta}_0(\nu)|| > c$ for some constant $c > 0$. Then, there exists a subsequence of (ζ,ν) that converges to, say, (ζ_0,ν_0), which implies that $\zeta_0 \ne \boldsymbol{\beta}_0(\nu_0)$ also solves $\Psi(\boldsymbol{\beta},\nu_0)$. This contradicts the fact that $\boldsymbol{\beta}_0(\rho)$ is the unique solution of $\Psi(\beta,\rho)$ for all $\rho \in [\rho_1,\rho_2]$, as guaranteed by Condition 3 and 4a. Therefore,

$$\sup_{\rho\in[\rho_1,\rho_2]} ||\widehat{\boldsymbol{\beta}}(\rho) - \boldsymbol{\beta}_0(\rho)|| = o(1)$$

almost surely.

In light of the above Donsker result, for given ρ, $n_1^{1/2}\{\Psi_n(\boldsymbol{\beta},\rho) - \Psi(\boldsymbol{\beta},\rho)\}$ converges weakly to a Gaussian process. Under Conditions 2 and 4b, $n_1^{1/2}\{\Psi_n(\boldsymbol{\beta},\rho) - \Psi(\boldsymbol{\beta},\rho)\}$ is asymptotically uniformly equicontinuous in probability using arguments similar to Huang (2017), appendix. Thus, for any positive sequence $d_n = o(1)$,

$$\sup_{||\boldsymbol{\beta}-\boldsymbol{\beta}'||<d_n,\ \rho\in[\rho_1,\rho_2]} n_1^{1/2}||\Psi_n(\boldsymbol{\beta},\rho) - \Psi_n(\boldsymbol{\beta}',\rho) - \Psi(\boldsymbol{\beta},\rho) + \Psi(\boldsymbol{\beta}',\rho)|| = o_p(1);$$

note that the above expression does not actually involve ρ. Therefore,

$$\sup_{\rho\in[\rho_1,\rho_2]} ||\Psi_n\{\boldsymbol{\beta}_0(\rho),\rho\} + \Psi\{\hat{\boldsymbol{\beta}}(\rho),\rho\}|| = o_p(n_1^{-1/2}).$$

Under Condition 4b, by component-wise Taylor expansion, one can show that, almost surely,

$$\sup_{\rho\in[\rho_1,\rho_2]} \frac{||\Psi\{\hat{\boldsymbol{\beta}}(\rho),\rho\} + E[\widetilde{\boldsymbol{Z}}_1^{\otimes 2} f_1\{\widetilde{\boldsymbol{Z}}_1^T\boldsymbol{\beta}_0(\rho)\widetilde{\boldsymbol{Z}}_1\}]\{\widehat{\boldsymbol{\beta}}(\rho) - \boldsymbol{\beta}_0(\rho)\}||}{||\widehat{\boldsymbol{\beta}}(\rho) - \boldsymbol{\beta}_0(\rho)||} = o(1).$$

Thus,

$$n_1^{1/2}\{\widehat{\boldsymbol{\beta}}(\rho) - \boldsymbol{\beta}_0(\rho)\} = n_1^{1/2}\big(E[\widetilde{\boldsymbol{Z}}_1^{\otimes 2} f_1\{\widetilde{\boldsymbol{Z}}_1^T\boldsymbol{\beta}_0(\rho)\widetilde{\boldsymbol{Z}}_1\}]\big)^{-1}\Psi_n\{\boldsymbol{\beta}_0(\rho),\rho\} + o_p(1),$$

uniformly in $\rho \in [\rho_1,\rho_2]$. Therefore, $n_1^{1/2}\{\widehat{\boldsymbol{\beta}}(\cdot) - \boldsymbol{\beta}(\cdot)\}$ over $[\rho_1,\rho_2]$ converges weakly to a Gaussian process.

Now, we turn to the controls. Write $\Gamma_n(\boldsymbol{\beta}) = n_0^{-1}\sum_{j=1}^{n_0} I(M_{0j} \le \widetilde{\boldsymbol{Z}}_{0j}^T\boldsymbol{\beta})$ and $\Gamma(\boldsymbol{\beta}) = \Pr(M_0 \le \widetilde{\boldsymbol{Z}}_0^T\boldsymbol{\beta})$. Similar arguments as above give

$$\sup_{\boldsymbol{\beta}} |\Gamma_n(\boldsymbol{\beta}) - \Gamma(\boldsymbol{\beta})| = o(1).$$

Thus,

$$\sup_{\rho\in[\rho_1,\rho_2]} |\widehat{\phi}(\rho) - \phi_0(\rho)| \le \sup_{\rho\in[\rho_1,\rho_2]} |\Gamma\{\widehat{\boldsymbol{\beta}}(\rho)\} - \Gamma\{\boldsymbol{\beta}_0(\rho)\}| + o(1) = o(1)$$

almost surely, given the strong consistency of $\widehat{\boldsymbol{\beta}}(\cdot)$ and the continuity of $\Gamma(\boldsymbol{\beta})$. To establish the weak convergence of $\widehat{\phi}(\rho)$, one can show that, for any positive sequence $d_n = o(1)$,

$$\sup_{||\boldsymbol{\beta}-\boldsymbol{\beta}'||<d_n} n_0^{1/2}|\Gamma_n(\boldsymbol{\beta}) - \Gamma_n(\boldsymbol{\beta}') - \Gamma(\boldsymbol{\beta}) + \Gamma(\boldsymbol{\beta}')| = o_p(1),$$

using similar arguments as for the cases. Therefore

$$\begin{aligned} n_0^{1/2}\{\widehat{\phi}(\rho) - \phi_0(\rho)\} &= n_0^{1/2}\big[\Gamma_n\{\widehat{\boldsymbol{\beta}}(\rho)\} - \Gamma\{\boldsymbol{\beta}_0(\rho)\}\big] \\ &= n_0^{1/2}\big[\Gamma_n\{\boldsymbol{\beta}_0(\rho)\} - \Gamma\{\boldsymbol{\beta}_0(\rho)\}\big] + n_0^{1/2}\big[\Gamma\{\widehat{\boldsymbol{\beta}}(\rho)\} - \Gamma\{\boldsymbol{\beta}_0(\rho)\}\big] + o_p(1) \\ &= n_0^{1/2}\big[\Gamma_n\{\boldsymbol{\beta}_0(\rho)\} - \Gamma\{\boldsymbol{\beta}_0(\rho)\}\big] + n_0^{1/2}\Gamma'\{\boldsymbol{\beta}_0(\rho)\}\{\widehat{\boldsymbol{\beta}}(\rho) - \boldsymbol{\beta}_0(\rho)\} + o_p(1) \end{aligned}$$

uniformly in $\rho \in [\rho_1,\rho_2]$. Then, the weak convergence of $\widehat{\phi}(\rho)$ follows. □

S3. Details about two monotonization methods

To recover the monotonicity of the constructed ROC curves, we applied two monotonization methods based on Huang (2017). Using the notations in the main manuscript Section 2, for linear dynamic regression model with covariate $\boldsymbol{z}$ and coefficient $\boldsymbol{\beta}$, the quantile regression model is

$$F_1^{-1}(t;\boldsymbol{z}) = \widetilde{\boldsymbol{z}}^T \boldsymbol{\beta}(t),$$

where $\widetilde{\boldsymbol{z}} = (1, \boldsymbol{z})$. Denote the estimator of $\boldsymbol{\beta}(t)$ by $\widehat{\boldsymbol{\beta}}(t)$ and the estimator for specificity by $\widehat{\phi}(\cdot)$. Note that both $\widehat{\boldsymbol{\beta}}(t)$ and $\widehat{\phi}(\cdot)$ are piece-wise constant and thus we can identify a countable set of breakpoints. Given a starting quantile point τ, let

$$\max\left[t : t < \tau, \sup_{\boldsymbol{z}\in\mathcal{Z}_s} \{\widetilde{\boldsymbol{z}}^T \widehat{\boldsymbol{\beta}}(t) - \widetilde{\boldsymbol{z}}^T \widehat{\boldsymbol{\beta}}(\tau)\} \le 0\right]$$

be the left nearest monotonicity-respecting neighbor and

$$\min\left[t : t > \tau, \inf_{\boldsymbol{z}\in\mathcal{Z}_s} \{\widetilde{\boldsymbol{z}}^T \widehat{\boldsymbol{\beta}}(t) - \widetilde{\boldsymbol{z}}^T \widehat{\boldsymbol{\beta}}(\tau)\} \ge 0\right]$$

be the right nearest monotonicity respecting neighbor. We denote the collection of all these points, including the starting point τ, by $\mathcal{M}$. Huang (2017) proposed to adopt an adaptive interpolation method to connect the original estimator $\widehat{\boldsymbol{\beta}}(\cdot)$ linearly between adjacent points in the break point set $\mathcal{M}$. Denote the monotonicity-respecting estimator by $\widetilde{\boldsymbol{\beta}}(\cdot)$. For any t between two adjacent points in $\mathcal{M}$, say $\tau_1 < t < \tau_2$, $\widetilde{\boldsymbol{\beta}}(t)$ is constructed by

$$\widetilde{\boldsymbol{\beta}}(t) = \frac{\tau_2 - t}{\tau_2 - \tau_1}\widehat{\boldsymbol{\beta}}(\tau_1) + \frac{t - \tau_1}{\tau_2 - \tau_1}\widehat{\boldsymbol{\beta}}(\tau_2).$$

For $t < min(\mathcal{M})$ we set $\widetilde{\boldsymbol{\beta}}(t) = \widehat{\boldsymbol{\beta}}(min\mathcal{M})$, and for $t > max(\mathcal{M})$ we set $\widetilde{\boldsymbol{\beta}}(t) = \widehat{\boldsymbol{\beta}}(max\mathcal{M})$. The regression-based monotonization method directly applies the above approach on the coefficient estimator $\widehat{\boldsymbol{\beta}}$ of our quantile regression model. The ROC-based monotonization method uses regular quantile regression estimator $\widehat{\boldsymbol{\beta}}(\cdot)$ to obtain the estimated specificities $\widehat{\phi}(\cdot)$, and then applies the adaptive interpolation approach on $\widehat{\phi}(\cdot)$ to obtain $\widetilde{\phi}(\cdot)$.

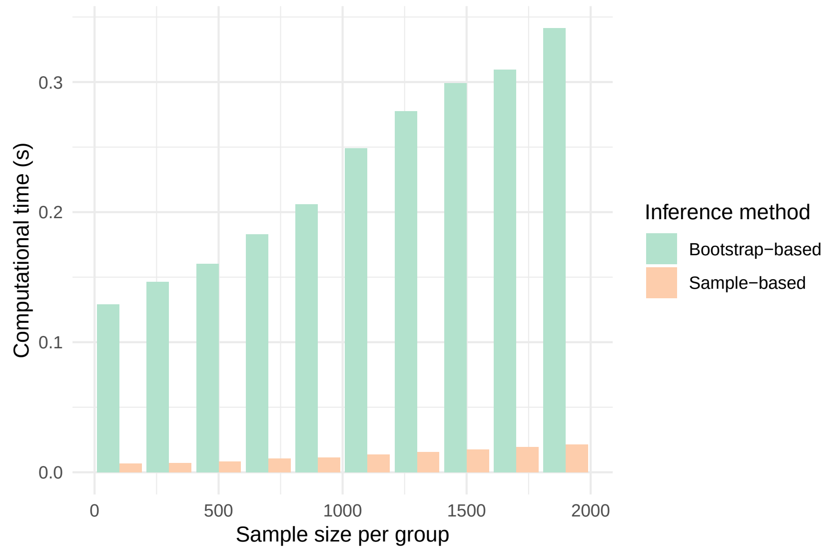

Figure S1: Summarization of the computation time using two inference methods with different sample sizes.

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