Online Appendix for "Intergenerational Transmission Is Not Sufficient for Positive Long-Term Population Growth"

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Proof for case where $p_{L\to L} < 1$

For compactness in this appendix, write p_L for $p_{L\to L}$ and p_H for $p_{H\to H}$. One condition that is sufficient to ensure that positive LTPG does not take place is if the Markov transition matrix $\mathbf{A}$ is a convergent matrix, or a matrix that converges to the zero matrix as the exponent on the matrix goes to infinity; in that case, overall population would eventually converge to zero. This occurs if the spectral radius of $\mathbf{A}$, or the largest absolute value of its eigenvalues, is less than 1. For a 2x2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the eigenvalues are given by:

$$\lambda = \frac{1}{2}\left(a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}\right)$$

and in the context of our matrix $\mathbf{A}$, this gives us:

$$\begin{aligned}\lambda &= \frac{1}{2}\left(p_H F_H + p_L F_L \pm \sqrt{(p_H F_H + p_L F_L)^2 - 4(p_H F_H p_L F_L - (1-p_L)F_L(1-p_H)F_H)}\right) \\ &= \frac{1}{2}\left(p_H F_H + p_L F_L \pm \sqrt{(p_H F_H + p_L F_L)^2 - 4(p_H + p_L - 1)F_H F_L}\right).\end{aligned}$$

It is possible to show that the term under the square root is always positive: that term decreases with p_L, and yet it is always positive even if $p_L = 1$ (algebra available upon request). Therefore, we do not need to concern ourselves with imaginary numbers, and the largest eigenvalue (in absolute value) will be the one with the plus sign in front of the square root; as a result, the condition for $\mathbf{A}$ to be a convergent matrix is:

$$X \equiv p_H F_H + p_L F_L + \sqrt{(p_H F_H + p_L F_L)^2 - 4(p_H + p_L - 1)F_H F_L} < 2.$$

It is then possible to show that $\frac{\partial X}{\partial p_H} \geq 0$ (algebra available upon request), with a strict inequality unless $p_L = 1$ and $p_H F_H < F_L$; but in the latter case, $X < 2$, and as p_H increases, it will eventually reach a point at which $p_H F_H > F_L$, at which point $\frac{\partial X}{\partial p_H} > 0$. This means that for any combination of values of p_L, F_L, and F_H, there is one unique value of p_H for which $X = 2$, and for any value of p_H below this critical value the matrix $\mathbf{A}$ is convergent.

It remains only for us to characterize this critical value of p_H, which we call $\widehat{p_H}$:

$$\widehat{p_H} F_H + p_L F_L + \sqrt{(\widehat{p_H} F_H + p_L F_L)^2 - 4(\widehat{p_H} + p_L - 1)F_H F_L} = 2$$

$$\therefore (\widehat{p_H} F_H + p_L F_L)^2 - 4(\widehat{p_H} + p_L - 1)F_H F_L = (2 - \widehat{p_H} F_H - p_L F_L)^2$$

$$\therefore (\widehat{p_H} F_H + p_L F_L)^2 - 4(\widehat{p_H} + p_L - 1)F_H F_L = 4\left(1 - \widehat{p_H} F_H - p_L F_L\right) + (\widehat{p_H} F_H + p_L F_L)^2$$

$$\therefore (1 - \widehat{p_H} - p_L)F_H F_L = (1 - \widehat{p_H} F_H - p_L F_L)$$

$$\therefore \widehat{p_H} = \frac{1 - F_H F_L + F_L(F_H - 1)p_L}{F_H(1 - F_L)}.$$

Therefore, if $p_H < \widehat{p_H}$, positive LTPG will not take place, and the population will converge to zero in the long run. If $p_L = 1$ as previously assumed, this condition simplifies to $\widehat{p_H} = \frac{1}{F_H}$, which demonstrates that the earlier result for long-run population decline with $p_L = 1$ is just a special case of this general solution. It is also important to note that, if $F_H F_L > 1$, the condition above could generate negative critical values for $\widehat{p_H}$, in which case it is impossible for $\mathbf{A}$ to be a convergent matrix; however, for any value of $p_L \in [0, 1]$, there exist parameter configurations such that positive LTPG will not take place.