Supplemental material

Here we provide the derivation of the results in Section 3.

In Kasza and Forbes[13], it is shown that if the block exchangeable model is fit to data generated according to the discrete time decay model (with an equal number of participants m in each of K clusters over T periods), then the following relationships will hold:

$$
\begin{aligned}
E[\hat{\sigma}^2_{2\epsilon}] &= \sigma^2_{3\epsilon} \\
E[\hat{\sigma}^2_{\gamma}] &= \sigma^2_{3\alpha}\left(\frac{T}{T-1} - \frac{1}{T(T-1)}\sum_{t=1}^{T}\sum_{s=1}^{T} r_d^{|t-s|}\right) \\
E[\hat{\sigma}^2_{2\alpha}] &= \sigma^2_{3\alpha}\left(\frac{1}{T(T-1)}\sum_{t=1}^{T}\sum_{s=1}^{T} r_d^{|t-s|} - \frac{1}{T-1}\right).
\end{aligned}
$$

Hence, by replacing estimators of variance components with their expected values, the within-period intracluster correlation estimated under the block exchangeable model can be written as:

$$
\hat{\rho_b} = \frac{E[\hat{\sigma}^2_{\gamma}] + E[\hat{\sigma}^2_{2\alpha}]}{E[\hat{\sigma}^2_{\gamma}] + E[\hat{\sigma}^2_{2\alpha}] + E[\hat{\sigma}^2_{2\epsilon}]} = \frac{\sigma^2_{3\alpha}}{\sigma^2_{3\alpha} + \sigma^2_{3\epsilon}} = \rho_d.
$$

Similarly, the cluster autocorrelation estimated under the block exchangeable model can be written as:

$$
\hat{r_b} = \frac{E[\hat{\sigma}^2_{2\alpha}]}{E[\hat{\sigma}^2_{\gamma}] + E[\hat{\sigma}^2_{2\alpha}]} = \frac{\sigma^2_{3\alpha}\left(\frac{1}{T(T-1)}\sum_{t=1}^{T}\sum_{s=1}^{T} r_d^{|t-s|} - \frac{1}{T-1}\right)}{\sigma^2_{3\alpha}} = \frac{1}{T(T-1)}\sum_{t=1}^{T}\sum_{s=1}^{T} r_d^{|t-s|} - \frac{1}{T-1}.
$$

Therefore an estimate of the block-exchangeable r_b can be obtained from the discrete time decay model by the simple arithmetic average of all the off-diagonal elements of the matrix Σ from Equation (3).

To find a value of r_d that is aligned with the estimated value from the block exchangeable model, $\hat{r_b}$, we thus propose solving the following equation:

$$
\sum_{t=1}^{T}\sum_{s=1}^{T} r_d^{|t-s|} - T(T-1)\hat{r_b} - T = 0
$$

The double sum $\sum_{t=1}^{T}\sum_{s=1}^{T} r_d^{|t-s|}$ is the sum over all elements of the correlation matrix Σ from Equation (3), hence the equation is given by:

$$
T + 2(T-1)r_d + 2(T-2)r_d^2 + \cdots + 2(T-(T-2))r_d^{T-2} + 2(T-(T-1))r_d^{T-1} - T(T-1)\hat{r_b} - T = 0
$$

which, with some minor rearranging, gives the following polynomial in r_d to be solved, yielding Equation (6) in the main text:

$$
(T-1)r_d + (T-2)r_d^2 + \cdots + 2r_d^{T-2} + r_d^{T-1} - \frac{T(T-1)}{2}\hat{r_b} = 0.
$$

In Kasza and Forbes[13] it is also shown that if the exchangeable model is fit to data generated according to the discrete time decay model (with an equal number of participants m in each of K clusters over T periods), then the following relationships will hold:

$$E[\hat{\sigma}^2_{1\epsilon}] = \sigma^2_{3\epsilon} + \sigma^2_{3\alpha}\left(\frac{(K-1)Tm}{KTm-K-T+1} - \frac{m(K-1)}{T(KTm-K-T+1)}\sum_{t=1}^{T}\sum_{s=1}^{T} r_d^{|t-s|}\right)$$
$$E[\hat{\sigma}^2_{1\alpha}] = \sigma^2_{3\alpha}\left(\frac{Km-1}{T(KTm-K-T+1)}\sum_{t=1}^{T}\sum_{s=1}^{T} r_d^{|t-s|} - \frac{K-1}{KTm-K-T+1}\right).$$

Hence, by replacing estimators of variance components with their expected values, the intracluster correlation estimated under the exchangeable model can be written as:

$$\hat{\rho_e} = \frac{E[\hat{\sigma}^2_{1\alpha}]}{E[\hat{\sigma}^2_{1\alpha}] + E[\hat{\sigma}^2_{1\epsilon}]} = \frac{\sigma^2_{3\alpha}\left(\frac{Km-1}{T(KTm-K-T+1)}\sum_{t=1}^{T}\sum_{s=1}^{T} r_d^{|t-s|} - \frac{K-1}{KTm-K-T+1}\right)}{\sigma^2_{3\epsilon} + \sigma^2_{3\alpha}\left(\frac{(K-1)(Tm-1)}{KTm-K-T+1} + \frac{m-1}{T(KTm-K-T+1)}\sum_{t=1}^{T}\sum_{s=1}^{T} r_d^{|t-s|}\right)}.$$

We will assume that $\sigma^2_{3\epsilon} + \sigma^2_{3\alpha} = 1$, so that $\sigma^2_{3\epsilon} = 1 - \sigma^2_{3\alpha}$, and $\rho_d = \sigma^2_{3\alpha}$. Substituting this into the equation above gives:

$$\hat{\rho_e} = \frac{\rho_d\left(\frac{Km-1}{T(KTm-K-T+1)}\sum_{t=1}^{T}\sum_{s=1}^{T} r_d^{|t-s|} - \frac{K-1}{KTm-K-T+1}\right)}{1 + \rho_d\left(\frac{K+T-Tm}{KTm-K-T+1} + \frac{m-1}{T(KTm-K-T+1)}\sum_{t=1}^{T}\sum_{s=1}^{T} r_d^{|t-s|}\right)}.$$

Rearranging this equation gives the following polynomial in r_d, which we propose solving to find values of r_d and ρ_d that are aligned with the estimate of the aggregate intracluster correlation $\hat{\rho}_e$, yielding Equation (7) in the main text:

$$(T-1)r_d + (T-2)r_d^2 + (T-3)r_d^3 + \cdots + 2r_d^{T-2} + r_d^{T-1}$$
$$+ \frac{T}{2}\left(1 - \frac{\hat{\rho}_e}{\rho_d}\frac{KTm-K-T+1}{Km-1-\hat{\rho}_e m + \hat{\rho}_e} - \frac{(K-1+\hat{\rho}_e K + \hat{\rho}_e T - \hat{\rho}_e Tm)}{Km-1-\hat{\rho}_e m+\hat{\rho}_e}\right) = 0.$$

For this equation to be equivalent to the approximation in Equation (8), we need

$$\frac{KTm-K-T+1}{Km-1-\hat{\rho}_e m+\hat{\rho}_e} - \frac{\rho_d}{\hat{\rho}_e}\frac{(K-1+\hat{\rho}_e K+\hat{\rho}_e T-\hat{\rho}_e Tm)}{Km-1-\hat{\rho}_e m+\hat{\rho}_e}$$

to be approximately equal to T. Rearranging this gives

$$T\frac{Km-1-\rho_d+\rho_d m}{Km-1-\hat{\rho}_e m+\hat{\rho}_e} + \frac{(1-K)(\hat{\rho}_e+\rho_d) - \rho_d\hat{\rho}_e K}{\hat{\rho}_e(Km-1-\hat{\rho}_e m+\hat{\rho}_e)},$$

which will be close to T when Km is large relative to $\rho_d m$, $\hat{\rho}_e m$ and K.

Finally, Kasza and Forbes [13] shows that if the exchangeable model is fit to data generated according to the block exchangeable model (with an equal number of participants m in each of K clusters over T periods), then the following relationships will hold:

$$E[\hat{\sigma}^2_{1\epsilon}] = \sigma^2_{2\epsilon} + \sigma^2_\gamma \frac{(KT-K-T+1)m}{KTm-K-T+1}$$
$$E[\hat{\sigma}^2_{1\alpha}] = \sigma^2_{2\alpha} + \sigma^2_\gamma \frac{K(m-1)}{KTm-K-T+1}.$$

Hence, by replacing estimators of variance components with their expected values, the intracluster correlation estimated under the exchangeable model can be written as:

$$\hat{\rho_e} = \frac{E[\hat{\sigma}^2_{1\alpha}]}{E[\hat{\sigma}^2_{1\alpha}] + E[\hat{\sigma}^2_{1\epsilon}]} = \frac{\sigma^2_{2\alpha}(KTm-K-T+1) + \sigma^2_\gamma K(m-1)}{KTm-K-T+1+\sigma^2_\gamma(m-1)(1-T)}.$$

We will assume that $\sigma^2_{2\epsilon} + \sigma^2_{2\alpha} + \sigma^2_\gamma = 1$, so that $\sigma^2_{2\alpha} = r_b\rho_b$, and $\sigma^2_\gamma = \rho_b(1-r_b)$. Substituting this into the equation above gives Equation 9 in the main text.

References

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