Supplementary Materials for "Adaptive Covariance Estimation of Non-stationary Processes and its Application to Infer Dynamic Connectivity from fMRI"

Zening Fu, *Student Member*, *IEEE*, Shing Chow Chan, *Member*, *IEEE*, Xin Di, Bharat Biswal, Zhiguo Zhang, *Member*, *IEEE*

APPENDIX I: SIMULATION RESULTS ON TIME-VARYING VARIANCE ESTIMATION

The asymptotic analysis for LPR-based time-varying covariance estimation can be easily extended to time-varying variance estimation, which underlies the applicability of the LPR-ICI method on time-varying variance estimation. Here, we tested the performance of the LPR-ICI method on time-varying variance estimation. The simulated signals are generated as zero mean Gaussian process with time-varying variance. Other testing parameters are exactly the same as those in Section V-A. We first test the LPR-ICI method using simulated signals with jumping variance. Note that, in this simulation, the zero mean random noise is added to the variance and the SNR is calculated as the ratio between the power of ground-truth variance and the additive noise. It can be seen from Fig. A1 that the LPR-ICI method can achieve good performance for both slowly-varying variance and fast-varying variance by using adaptive windows.

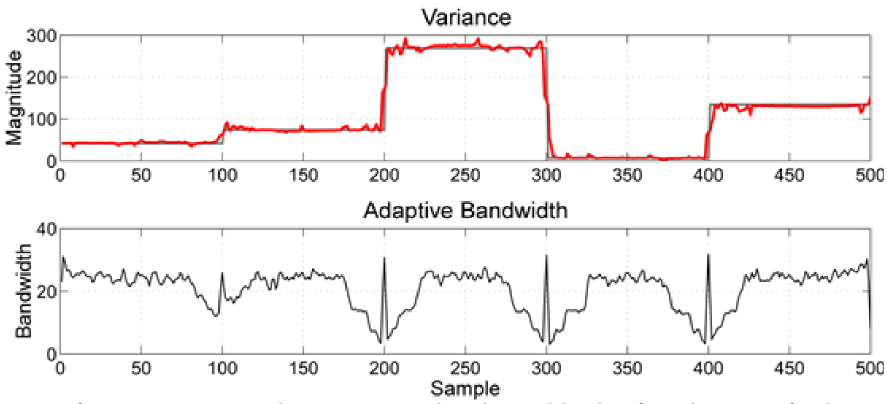

Fig. A1. Estimation of time-varying variance between two signals with a model order of 1 and an SNR of 20dB. Upper: the black line denotes the true variances and the red line denotes the estimated variance; Lower: variable bandwidths used in LPR-ICI.

We further test the performance of the LPR-ICI method on more general signals with randomly varying variance. Four cutoff frequencies f_c: 0.005, 0.01, 0.02, and 0.05, are used to simulate different degrees of variations in time-varying variance. Other testing parameters are exactly the same as those in Section V-B. Table A-II lists the EMSE values of different methods averaged over 50 independent Monte-Carlo realizations. It can be seen that the LPR-ICI method can provide good estimation results, which are close to the best results from an "optimal" fixed bandwidth, in all testing scenarios.

TABLE A-I
EMSEs of Different Covariance Estimation Methods for Two Signals with Randomly Varying Variance (Unit: Decibels)

Methods	f_c = 0.005			
	SNR = 0	SNR = 5	SNR = 10	SNR = 20
LPR (p = 1; h = 2)	-34.18	-41.30	-47.32	-57.74
LPR (p = 1; h = 4)	-36.13	-43.95	-50.32	-60.98
LPR (p = 1; h = 8)	-37.33	-45.78	-52.73	-63.41†
LPR (p = 1; h = 16)	-38.06	-46.92†	-53.94†	-59.92
LPR (p = 1; h = 22)	-38.32†	-46.07	-49.28	-50.07
LPR-ICI (p = 0)	-38.48*	-46.93*	-52.33	-57.93
LPR-ICI (p = 1)	-38.42	-46.90	-52.96*	-62.46*
LPR-ICI (p = 2)	-35.41	-42.76	-49.07	-59.58
LPR w. plug-in (p = 1)	-37.82	-46.17	-52.43	-57.08
Methods	f_c = 0.01			
	SNR = 0	SNR = 5	SNR = 10	SNR = 20
LPR (p = 1; h = 2)	-34.40	-41.67	-47.44	-57.76
LPR (p = 1; h = 4)	-36.34	-44.24	-50.34	-60.89
LPR (p = 1; h = 8)	-37.50	-46.11	-52.62	-62.66†
LPR (p = 1; h = 16)	-38.09†	-46.97†	-52.95†	-57.44
LPR (p = 1; h = 22)	-37.97	-45.02	-47.72	-48.59
LPR-ICI (p = 0)	-38.68*	-47.16*	-51.95	-57.87
LPR-ICI (p = 1)	-38.37	-47.01	-52.84*	-60.84*
LPR-ICI (p = 2)	-35.56	-43.17	-49.18	-59.60
LPR w. plug-in (p = 1)	-38.03	-46.51	-52.55	-60.22
Methods	f_c = 0.02			
	SNR = 0	SNR = 5	SNR = 10	SNR = 20
LPR (p = 1; h = 2)	-18.56	-25.52	-31.32	-41.71
LPR (p = 1; h = 4)	-20.58	-28.11	-34.31	-44.07†
LPR (p = 1; h = 8)	-21.72	-29.51†	-34.84†	-38.58
LPR (p = 1; h = 16)	-21.77†	-27.39	-29.18	-29.66
LPR (p = 1; h = 22)	-20.96	-24.27	-24.88	-24.96
LPR-ICI (p = 0)	-22.14	-28.66	-32.36	-36.12
LPR-ICI (p = 1)	-22.18*	-28.98*	-33.44*	-40.31
LPR-ICI (p = 2)	-19.71	-26.93	-32.91	-43.23*
LPR w. plug-in (p = 1)	-21.66	-29.07	-34.76	-43.28
Methods	f_c = 0.05			
	SNR = 0	SNR = 5	SNR = 10	SNR = 20
LPR (p = 1; h = 2)	-12.53	-19.44	-25.53	-34.86†
LPR (p = 1; h = 4)	-14.45	-21.31†	-25.87†	-28.69
LPR (p = 1; h = 8)	-14.61†	-18.86	-20.07	-20.40
LPR (p = 1; h = 16)	-13.74	-16.27	-16.71	-16.80
LPR (p = 1; h = 22)	-13.13	-15.17	-15.49	-15.55
LPR-ICI (p = 0)	-14.11	-17.62	-19.43	-20.81
LPR-ICI (p = 1)	-14.25*	-18.30	-21.26	-25.38
LPR-ICI (p = 2)	-13.46	-20.25*	-25.73*	-30.99*
LPR w. plug-in (p = 1)	-14.20	-20.62	-26.08	-34.24

The minimum EMSE of LPR over all fixed bandwidths under one certain SNR is marked with †. The minimum EMSE of LPR-ICI over all model orders under one certain SMR is marked with *.

Appendix II: Comparison of Elapsed Time between Different Methods

The elapsed time of different methods is compared in this appendix. The testing signals (two signals with randomly varying covariance) are from Section V-B of the manuscript and the elapsed time is listed in Table A-I is averaged over 50 independent Monte-Carlo realizations. The configuration of the computer used in the simulation is: Intel Core 5-2400 CPU @ 3.10GHz, 4GB RAM. It can be seen from this table that: (1) the elapsed time of the LPR method is increased with the bandwidth used; (2) the elapsed time of the LPR-ICI method is generally increased with the model order (although this relationship doesn't always hold because the adaptive bandwidths selected under different model orders are different); (3) the LPR-ICI method has a higher complexity than LPR with a fixed window size; (4) the plug-in method ([A1] and [A2]) has a significantly longer computational time than the ICI method.

TABLE A-II
Elapsed Time of Different Covariance Estimation Methods for Two Signals with Randomly Varying Covariance (Unit: Second)

Methods	f_c = 0.005			
	SNR = 0	SNR = 5	SNR = 10	SNR = 20
LPR (*p* = 1; *h* = 2)	29.79	28.25	29.12	29.47
LPR (*p* = 1; *h* = 4)	30.69	28.77	29.96	30.15
LPR (*p* = 1; *h* = 8)	31.53	30.80	30.75	31.45
LPR (*p* = 1; *h* = 16)	33.84	31.55	32.29	33.61
LPR (*p* = 1; *h* = 22)	45.44	43.54	44.80	43.08
LPR-ICI (*p* = 0)	91.94	89.68	89.59	88.24
LPR-ICI (*p* = 1)	96.12	94.56	95.69	93.50
LPR-ICI (*p* = 2)	96.01	93.34	93.74	93.53
LPR w. plug-in (*p* = 1)	1370.92	1248.77	1154.41	1054.69

Methods	f_c = 0.01			
	SNR = 0	SNR = 5	SNR = 10	SNR = 20
LPR (*p* = 1; *h* = 2)	29.65	28.98	29.36	29.06
LPR (*p* = 1; *h* = 4)	30.49	29.86	29.47	29.41
LPR (*p* = 1; *h* = 8)	31.19	30.60	30.31	31.03
LPR (*p* = 1; *h* = 16)	32.87	32.34	32.03	33.17
LPR (*p* = 1; *h* = 22)	43.42	46.08	39.21	46.89
LPR-ICI (*p* = 0)	94.03	89.05	90.74	88.58
LPR-ICI (*p* = 1)	98.37	95.88	94.42	90.63
LPR-ICI (*p* = 2)	94.18	92.74	95.05	94.07
LPR w. plug-in (*p* = 1)	1268.51	1163.88	1052.02	974.70

Methods	f_c = 0.02			
	SNR = 0	SNR = 5	SNR = 10	SNR = 20
LPR (*p* = 1; *h* = 2)	29.62	29.24	30.07	29.73
LPR (*p* = 1; *h* = 4)	30.33	30.23	30.43	29.99
LPR (*p* = 1; *h* = 8)	31.09	32.25	31.01	30.85
LPR (*p* = 1; *h* = 16)	32.51	32.57	33.59	33.03
LPR (*p* = 1; *h* = 22)	44.16	44.00	42.81	43.60
LPR-ICI (*p* = 0)	91.39	89.20	88.69	87.36
LPR-ICI (*p* = 1)	96.17	99.81	92.83	89.42
LPR-ICI (*p* = 2)	95.62	94.27	97.30	97.23
LPR w. plug-in (*p* = 1)	1209.02	1148.31	1077.56	1051.12

Methods	f_c = 0.05			
	SNR = 0	SNR = 5	SNR = 10	SNR = 20
LPR (*p* = 1; *h* = 2)	30.15	29.73	32.40	30.11
LPR (*p* = 1; *h* = 4)	31.27	30.97	33.62	30.45
LPR (*p* = 1; *h* = 8)	33.21	31.46	34.03	31.32
LPR (*p* = 1; *h* = 16)	34.28	33.38	36.16	33.26
LPR (*p* = 1; *h* = 22)	48.81	43.67	42.57	42.49
LPR-ICI (*p* = 0)	94.53	92.06	88.69	94.23
LPR-ICI (*p* = 1)	96.51	95.24	91.38	89.31
LPR-ICI (*p* = 2)	94.13	97.25	99.12	93.70
LPR w. plug-in (*p* = 1)	1283.32	1187.12	1004.42	940.86

We now explain why the plug-in rule is computationally demanding. The plug-in rule is based on the asymptotic expression of the optimal bandwidth and it consists of two steps: estimation of global bandwidth and estimation of local bandwidth. Since there are many unknown quantities to be estimated in the asymptotic expression of the optimal bandwidth, both global estimation and local estimation are executed iteratively (usually 6-8 iterations in global estimation and 2 iterations in local estimation) to ensure the accuracy and convergence of the estimation of optimal bandwidth. In the plug-in rule, the computational complexity of each iteration for global estimation is $O(N)$, where N is the number of samples of the whole period and the computational complexity of each iteration for local estimation is $O(N_L^{(i)})$, where $N_L^{(i)}$ is the number of samples determined by the bandwidth calculated by the previous iteration step. Suppose there are $\mathcal{I}_{\mathcal{G}}$ global iterations and $\mathcal{I}_{\mathcal{L}}$ local iterations in the plug-in rule, then the total computational complexity of the plug-in rule is $\mathcal{I}_{\mathcal{G}} \cdot O\{N\} + \sum_{i=1}^{\mathcal{I}_{\mathcal{L}}} O(N_L^{(i)})$. On the other hand, the ICI rule used in this manuscript has a computational complexity of $\sum_{\gamma=1}^{\Gamma} O\{N_K^{(\gamma)}\} + O\{N_K^{opt}\}$, where

$\sum_{\gamma=1}^{\Gamma} O\{N_K^{(\gamma)}\}$ is the total complexity to calculate LPR with each bandwidth in the bandwidth set $\tilde{H}$ and $O\{N_K^{opt}\}$ is the complexity to calculate LPR with the optimal bandwidth $\hat{h}_k^{opt}$. Considering that both $N_K^{(\gamma)}$ and N_K^{opt} should be much smaller than N and Γ is usually around 4-5, we can conclude that the plug-in method is more computationally expensive than the ICI method for adaptive bandwidth selection, which is clearly shown in Table A-I.

REFERENCES

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