PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES

Volume 33 February 15, 1947 Number 2

COMPLETE CONVERGENCE AND THE LAW OF LARGE NUMBERS

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Communicated January 7, 1947

1. We begin by listing some standard definitions in the theory of probability. A *probability space* is a set Ω of elements ω together with a σ-field m of subsets of Ω on which is defined a completely additive measure P such that $P(\Omega) = 1$. A real-valued P-measurable function $X = X(\omega)$ is a *random variable*, and the function $F(x) = P\{X \leq x\}$, where $\{ \}$ denotes the set of all ω such that the relation within the braces holds, is the *distribution function* of X. The sets of a sequence $A_1, A_2, \ldots$ are *independent* if for every finite set $i_1, \ldots, i_n$ of distinct integers, $P(\prod_{r=1}^{n} A_{i_r}) = \prod_{r=1}^{n} P(A_{i_r})$, and the random variables of a sequence $X_1, X_2, \ldots$ are independent if, for every sequence $x_1, x_2, \ldots$ of real numbers, the sets $\{X_1 \leq x_1\}, \{X_2 \leq x_2\}, \ldots$ are independent.

For purposes of comparison we list the following modes in which a sequence

$$X_1, X_2, \ldots \tag{1}$$

of random variables defined on Ω may converge to 0.

(*i*) The sequence (1) converges to 0 *in probability* if for every $\epsilon > 0$,

$$\lim_{n \to \infty} P\{|X_n| > \epsilon\} = 0.$$

(*ii*) The sequence (1) converges to 0 *with probability* 1 if for every $\epsilon > 0$,

$$\lim_{n \to \infty} P\{\{|X_n| > \epsilon\} + \{|X_{n+1}| > \epsilon\} + \ldots\} 0.$$

It is easily seen that this is equivalent to the usual condition, $P\{\lim_{n \to \infty} X_n = 0\} = 1$, and that (*ii*) implies (*i*) but not conversely.

2. We shall be concerned with a third mode of convergence, which, for want of a better name, we call *complete.*

(*iii*) The sequence (1) converges to 0 *completely* if for every $\epsilon > 0$,

$$\lim_{n\to\infty} [P\{|X_n| > \epsilon\} + P\{|X_{n+1}| > \epsilon\} + \ldots] = 0.$$

Clearly, (*iii*) implies (*ii*). The example: Ω = unit interval $0 < \omega < 1$, P = Lebesgue measure, $X_n = 1$ for $0 < \omega < \frac{1}{n}$ and 0 otherwise, shows that (*ii*) does not imply (*iii*).

Let us call two sequences of random variables $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$, defined, respectively, on probability spaces Ω and Ω_1, *F-equivalent,* if for every n the distribution function of Y_n is identical with that of X_n. Definitions (*i*) and (*iii*) are invariant under F-equivalence, while (*ii*) is not. However, *a sequence* $X_1, X_2, \ldots$ *of random variables converges to* 0 *completely if and only if every F-equivalent sequence converges to* 0 *with probability* 1. The necessity is obvious; to prove sufficiently consider a sequence $Y_1, Y_2, \ldots$ of independent random variables F-equivalent to the given sequence. If the sequence $Y_1, Y_2, \ldots$ converges to 0 with probability 1 then for any $\epsilon > 0$, $P\{\limsup_{n\to\infty}\{|Y_n| > \epsilon\}\} = 0$. Since the sets $\{|Y| > \epsilon\}$ are independent, it follows from a theorem of Borel-Cantelli[1] that $\sum_{n=1}^{\infty} P\{|Y_n| > \epsilon\} = \sum_{n=1}^{\infty} P\{|X_n| > \epsilon\} < \infty$.

It follows from this proof that if $X_1, X_2, \ldots$ is a sequence of *independent* random variables, then definitions (*ii*) and (*iii*) are equivalent.

3. Let the random variables X_n in (1) be independent with the same distribution function $F(x) = P\{X_n \leq x\}$ and such that the expectation $E(X_n) = \int_{-\infty}^{\infty} x dF(x) = 0$. The *strong law of large numbers* for identically distributed random variables states that the sequence of random variables $Y_1, Y_2, \ldots$, where for each n

$$Y_n = (X_1 + \ldots + X_n)/n \tag{2}$$

converges to 0 with probability 1. We shall show in Theorems 1 and 2 that under the same hypotheses the sequence (2) need not converge to 0 completely, but that it will do so under the further hypothesis that $\int_{-\infty}^{\infty} x^2 d F(x) < \infty$.

4. THEOREM 1. *Let (1) be a sequence of independent random variables with the same distribution function* $F(x)$ *and such that*

$$\int_{-\infty}^{\infty} x dF(x) = 0, \quad \sigma^2 = \int_{-\infty}^{\infty} x^2 dF(x) < \infty. \tag{3}$$

Then the sequence (2) converges to 0 completely; i.e., the series

$$\sum_{n=1}^{\infty} P\{|Y_n| > \epsilon\} \tag{4}$$

converges for every $\epsilon > 0$.

Proof. We shall prove the theorem for $\epsilon = 2$. This is no restriction since we can always consider $\frac{2}{\epsilon}X_n$ instead of X_n. Moreover, we may assume that $\sigma^2 > 0$.

Let $f(t) = \int_{-\infty}^{\infty} e^{itx}dF(x)$ be the characteristic function of the distribution $F(x)$. From (3) it follows that constants α, α', α'' exist such that

$$|1 - f(t)| \leq \alpha t^2, \quad |f'(t)| \leq \alpha' t, \quad |f''(t)| \leq \alpha''. \tag{5}$$

Choose and fix a positive δ so small that for $|t| \leq 4\delta$ the following conditions (6) and (7) are satisfied:

$$|\sin {}^1/_2 t| \geq Bt, \; |(1 - f(t))^2 - 4(1 - f(t)) \sin^2 {}^1/_2 t + 4 \sin^2 {}^1/_2 t| \geq Ct^2, \tag{6}$$

where B and C are constants,

$$|f(\,)| \neq 1 \qquad \text{except at } t = 0. \tag{7}$$

Let Z be a random variable distributed with the density $3(2\pi)^{-1}x^{-4} \sin^4 x$ and hence the characteristic function

$$\varphi(t) = \begin{cases} 1 - {}^3/_8 t^2 + {}^3/_{32}|t|^3, & |t| \leq 2, \\ {}^1/_{32}(4 - |t|)^2, & 2 \leq |t| \leq 4, \\ 0 & 4 \leq |t|. \end{cases} \tag{8}$$

We regard Z as independent of Y_n and use addition in this sense. Since

$$P\{|Y_n| > 2\} \leq P\left\{\left| Y_n + \frac{Z}{n\delta}\right| > 1\right\} + P\left\{\left|\frac{Z}{n\delta}\right| > 1\right\} =$$
$$P\left\{\left|\frac{Z}{n\delta}\right| \leq 1\right\} - P\left\{\left| Y_n + \frac{Z}{n\delta}\right| \leq 1\right\} + 2P\left\{\left|\frac{Z}{n\delta}\right| > 1\right\},$$

and since

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{Z}{n\delta}\right| > 1\right\} \leq \frac{3}{\pi} \sum_{n=1}^{\infty} \int_{n\delta}^{\infty} \frac{dx}{x^4} = \frac{1}{\pi\delta^3} \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty,$$

it is sufficient to prove that

$$\sum_{n=1}^{N-1}\left[P\left\{\left|\frac{Z}{n\delta}\right| \leq 1\right\} - P\left\{\left| Y_n + \frac{Z}{n\delta}\right| \leq 1\right\}\right] = 0(1), \tag{9}$$

where $0(1)$ always denotes a quantity bounded with respect to N.

The characteristic function $f^n\left(\frac{t}{n}\right) \varphi\left(\frac{t}{n\delta}\right)$ of $Y_n + \frac{Z}{n\delta}$ vanishes for $|t| > 4n\delta$; hence by a well-known inversion formula,[2]

$$P\left\{\left| Y_n + \frac{Z}{n\delta}\right| \leq 1\right\} = \frac{1}{\pi}\int_{-4n\delta}^{4n\delta} f^n\left(\frac{t}{n}\right)\varphi\left(\frac{t}{n\delta}\right)\frac{\sin t}{t}dt =$$
$$\frac{1}{\pi}\int_{-4\delta}^{4\delta} f^n(t)\,\varphi\left(\frac{t}{\delta}\right)\frac{\sin nt}{t}dt.$$

Also,

$$P\left\{\left|\frac{Z}{n\delta}\right| \leq 1\right\} = \frac{1}{\pi}\int_{-4\delta}^{4\delta}\varphi\left(\frac{t}{\delta}\right)\frac{\sin nt}{t}dt.$$

Hence, by subtraction the left side of (9) is equal to

$$\frac{1}{\pi}\int_{-4\delta}^{4\delta}\frac{1}{t}\varphi\left(\frac{t}{\delta}\right)\sum_{n=1}^{N-1}(1 - f^n(t))\sin nt\, dt = \frac{1}{\pi}A_N, \text{ say.} \tag{10}$$

From now on we write f for $f(t)$. Direct computation gives the result

$$\sum_{n=1}^{N-1}(1 - f^n)\sin nt = \frac{(1-f)^2 \sin {}^1/_2Nt \sin {}^1/_2(N-1)t}{q(t)\sin {}^1/_2t}$$
$$+ \frac{(1-f)\sin t}{q(t)} - \frac{4(1-f)\sin {}^1/_2t \sin {}^1/_2Nt \sin {}^1/_2(N-1)t}{q(t)}$$
$$+ \frac{(1-f)f^N \sin Nt}{q(t)} - \frac{2(1-f^{N+1})\sin {}^1/_2t \cos (N - {}^1/_2)t}{q(t)}, \tag{11}$$

where

$$q(t) = (f - e^{it})(f - e^{-it}) = (1-f)^2 - 4(1-f)\sin^2 {}^1/_2t + 4\sin^2 {}^1/_2t. \tag{12}$$

By (6) we have $|q(t)| \geq Ct^2$, $|q(t)\sin {}^1/_2t| \geq C'|t|^3$, where C and C' are constants. Hence when (11) is substituted into (10) and the first inequality of (5) is used, we see that the first three terms merely contribute 0(1). Consequently,

$$A_N = \int_{-4\delta}^{4\delta}\varphi\left(\frac{t}{\delta}\right)\frac{(1-f)f^N \sin Nt - 2(1-f^{N+1})\sin {}^1/_2t\cos (N - {}^1/_2)t}{tq(t)}dt$$
$$+ 0(1). \tag{13}$$

For $\delta \leq |t| \leq 4\delta$ we have, by (7), $|f| \neq 1$, hence $q(t) = |f - e^{it}|^2 \geq (1 - |f|)^2 \geq a > 0$. Therefore the part of the integral in (13) extended over the range $\delta \leq |t| \leq 4\delta$ is 0(1), so that (13) holds with 4δ replaced by δ in the two limits of integration. For $|t| \leq \delta$, however, $\varphi\left(\frac{t}{\delta}\right) = 1 - \frac{3t^2}{8\delta^2} + \frac{3|t|^3}{32\delta^3}$, and the terms with t^2 and $|t|^3$ are easily seen to contribute 0(1). Hence,

$$A_N = \int_{-\delta}^{\delta} \frac{(1-f)f^N \sin Nt - 2(1 - f^{N+1}) \sin {}^1/_2 t \cos (N - {}^1/_2)t}{tq(t)} dt + 0(1). \quad (14)$$

Since

$$\left|\frac{1}{tq(t)} - \frac{1}{t^3}\right| = \left|\frac{t^2 - q(t)}{t^3 q(t)}\right| \leq k \frac{t^4}{|t|^5} = \frac{k}{|t|},$$

where k is a constant, the replacement of $tq(t)$ by t^3 in (14) will make a difference of only 0(1), so that

$$A_N = \int_{-\delta}^{\delta} \frac{(1-f)f^N \sin Nt}{t^3} dt - \int_{-\delta}^{\delta} \frac{2(1 - f^{N+1}) \sin {}^1/_2 t \cos (N - {}^1/_2)t}{t^3} dt + 0(1). \quad (15)$$

Let the two integrals in (15) be denoted by I_N and J_N, respectively. We have

$$I_N = \frac{2}{N} \int_{-\delta}^{\delta} \frac{(1-f)f^N}{t^3} d(\sin^2 {}^1/_2 Nt) = 0(1) + \frac{2}{N} \int_{-\delta}^{\delta} \left\{ \frac{3(1-f)f^N}{t^4} + \frac{f^N f'}{t^3} - \frac{Nf^{N-1}(1-f)f'}{t^3} \right\} \sin^2 {}^1/_2 Nt dt.$$

Using the first two inequalities of (5) we obtain the result

$$|I_N| \leq 0(1) + (6\alpha + 2\alpha') \int_{-\infty}^{\infty} \frac{\sin^2 {}^1/_2 Nt}{Nt^2} dt + 2 \int_{-\infty}^{\infty} \frac{|1-f|\,|f'|}{|t^3|} dt = 0(1),$$

since the integral involving N is independent of N.

To deal with J_N we observe first that in J_N, $\sin {}^1/_2 t$ may be replaced by ${}^1/_2 t$ and f^{N+1} by f^N, the difference thus made being 0(1). Hence

$$J_N = \int_{-\delta}^{\delta} \frac{(1 - f^N) \cos (N - {}^1/_2)t}{t^2} dt + 0(1) = \int_{-\infty}^{\infty} \frac{(1 - f^N) \cos (N - {}^1/_2)t}{t^2} dt + 0(1).$$

We may replace $\cos (N - {}^1/_2)t$ by $\cos Nt$, since

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1 - f^N}{t^2} (\cos (N - {}^1/_2)t - \cos Nt) dt = \frac{1}{\pi} \int_{-\infty}^{\infty} (1 - f^N) \frac{\sin {}^1/_4 t}{{}^1/_4 t} \frac{\sin (N - {}^1/_4)t}{t} dt = P\{|U| \leq N - {}^1/_4\} - P\{|U + NY_N| \leq N - {}^1/_4\} = 0(1),$$

where U is a random variable independent of Y_N and whose characteristic function is $4/t \sin {}^1/_4t$. Hence

$$J_N = 0(1) + \frac{1}{N}\int_{-\infty}^{\infty}\frac{1-f^N}{t^2}\,d\sin Nt = 0(1) + \frac{2}{N^2}\int_{-\infty}^{\infty}\left\{\frac{2(1-f^N)}{t^3} + \frac{Nf^{N-1}f'}{t^2}\right\}d\sin^2 {}^1/_2t$$

$$= 0(1) + \frac{2}{N^2}\int_{-\infty}^{\infty}\left\{\frac{6(1-f^N)}{t^4} + \frac{4Nf^{N-1}f'}{t^3} - \frac{N(N-1)f^{N-2}f'^2}{t^2} - \frac{Nf^{N-1}f''}{t^2}\right\}\sin^2 {}^1/_2Nt dt.$$

Using all the inequalities (5) we have

$$|J_N| \leq 0(1) + (12\alpha + 8\alpha' + 2\alpha'')\int_{-\infty}^{\infty}\frac{\sin^2 {}^1/_2Nt}{Nt^2}dt + 2\int_{-\infty}^{\infty}\frac{|f'|^2}{t^2}dt = 0(1)$$

The proof is now complete.

5. By following the essential steps of the proof of Theorem 1 we obtain the following theorem, the proof of which is omitted from the present communication.

THEOREM 2. *If instead of conditions (3) we have*

$$\int_{-\infty}^{\infty} xdF(x) = 0,\ \int_{-\infty}^{\infty} |x|^a dF(x) < \infty,\ \int_{-\infty}^{\infty} x^2 dF(x) = \infty \quad (16)$$

where a is some constant such that $\frac{1}{2}(1 + 5^{1/2}) \leq a < 2$, then the series (4) diverges for every $\epsilon > 0$. (Example: Let X_n be distributed with the density $|x|^{-3}$ for $|x| \geq 1$ and 0 elsewhere.)

Since the finiteness of the second integral in (16) would seem rather to favor than to oppose the convergence of (4), it may be conjectured that given the first condition of (3), the finiteness of σ^2 is not only sufficient but also necessary for the convergence of (4). We have not been able to prove this.

6. The following generalization[3] of the strong law of large numbers is an immediate consequence of Theorem 1 and the remarks in section 2.

THEOREM 3. *Let $X_n^{(r)}$ ($n = 1, 2, \ldots;\ r = 1\ \ldots, n$) be an array of random variables with the same distribution function $F(x)$ and such that (1) $\int_{-\infty}^{\infty} xdF(x) = 0$, $\int_{-\infty}^{\infty} x^2dF(x) < \infty$, and (2) for each n the random variables $X_1^{(n)}, \ldots. X_n^{(n)}$ are independent. Then the sequence of random variables Y_1, $Y_2, \ldots,$ where for each n, $Y_n = (X_1^{(n)} + \ldots + X_n^{(n)})/n$, converges to 0 with* *probability 1. (Note that we do not assume any relation of dependence or independence between $X_n^{(r)}$ and $X_m^{(s)}$* for $m \neq n$.)

[1] See M. Fréchet, *Recherches theoriques modernes*, Vol. 1, Paris, 1937, p. 27.

[2] See H. Cramér, *Mathematical methods of statistics*, Princeton, 1946, p. 93.

[3] Compare F. P. Cantelli, Considerazioni sulla legge uniforme dei grandi numeri ecc., *Giornale dell 'Istituto Italiano degli Attuari*, IV (1933), pp. 331–332; also H. Cramér, Su un theorema relativo alla leggi uniforme dei grandi numeri, *Ibid.*, V (1934), pp. 1–13.

GREEN'S FUNCTIONS FOR LINEAR DIFFERENTIAL SYSTEMS OF INFINITE ORDER

BY D. V. WIDDER

In these PROCEEDINGS[1] the author sketched a theory whereby a special differential equation of infinite order

$$\frac{\sin \pi D}{\pi} y(x) = \varphi(x) \tag{1}$$

could be solved by use of a Green's function. The operator on the left of this equation is interpreted to mean

$$\lim_{n \to \infty} D\left(1 - \frac{D^2}{1^2}\right) \cdots \left(1 - \frac{D^2}{n^2}\right) y(x),$$

where D is the operation of differentiation with respect to x. In preparing the details of this theory the author discovered that much more general differential equations could be treated by the same method. It is the purpose of the present note to outline this more general theory.

We define an entire function $E(s)$ as follows:

$$E(s) = s \prod_{k=1}^{\infty} \left(1 - \frac{s^2}{a_k^2}\right), \tag{2}$$

where the constants a_k are real and such that

$$0 < a_1 < a_2 < \ldots. \tag{3}$$

$$\sum_{k=1}^{\infty} \frac{1}{a_k^2} < \infty. \tag{4}$$

Consider now the differential system

$$E(D)y(x) = \varphi(x) \tag{5}$$