SUPPLEMENTARY MATERIAL

Relationship between weights, auto and cross-correlation:

Following the notation in the Methods section, let $v_1(t)$ and $v_2(t)$ denote the speeds of the input and focal fish, respectively, for $t = 1,..., T$. We hypothesize that the speed of the focal fish can be predicted from that of the input fish

$$v_2(t) = [W \star v_1](t) + b$$

where the $\star$ operator denotes the discrete convolution between a filter W of length D and v_1

$$[W \star v_1](t) = \sum_{l=0}^{D} W(l) v_1(t - l) + b.$$

Next, we define the normalized discrete cross-correlation function between signals $v_p(t)$ and $v_q(t)$ for $p, q = 1, 2$

$$R_{pq}(\tau) \equiv \frac{1}{\sigma_p \sigma_q T} \sum_{t=1}^{T} \left[v_p(t) - \mu_p\right]\left[v_q(t + \tau) - \mu_q\right] = \frac{1}{\sigma_p \sigma_q T} \sum_{t=1}^{T} \left[v_p(t) - \mu_p\right] v_q(t + \tau)$$

where we expand and use the definition of the mean $\mu_p \equiv \frac{1}{T} \sum_{t=1}^{T} v_p(t)$ to arrive at the final equality. If $p = q$, then we define $R_p(\tau) \equiv R_{pp}(\tau)$ as the autocorrelation function. We let $v_q(t + \tau) = 0$ if index $t + \tau$ is outside the range of v_q.

Inserting the expression for $v_2(t + \tau)$ into the cross-correlation function, we find

$$R_{21}(\tau) = \frac{\sigma_1}{\sigma_2} \sum_{l=0}^{D} W(l) R_1(l - \tau) = \frac{\sigma_1}{\sigma_2} [W * R_1](\tau)$$

demonstrating that unique filters W can be identified that relate state-conditioned autocorrelated movement from some input signal to the cross-correlation with the focal signal.

M-step update equations for unconstrained weights, constant biases:

We now consider multiple linear models for states $k = 1,..., K$ using a hidden Markov model (HMM) framework. Following Bishop & Nasrabadi 2006, we optimize the expected complete-data log likelihood (ECLL)

$$Q(\Theta, \Theta^{(t-1)}) = \sum_{k=1}^{K} \gamma(z_{1k}) \ln \pi^{(k)} + \sum_{t=2}^{T} \sum_{j=1}^{K} \sum_{k=1}^{K} \xi(z_{t-1,j}, z_{tk}) \ln A_{jk} + \sum_{t=1}^{T} \sum_{j=1}^{K} \gamma(z_{tk}) \ln p(\vec{y}_t | \vec{x}_t, \phi^{(k)})$$

where the posteriors $\gamma(z_{tk})$ and joint posteriors $\xi(z_{t-1,j}, z_{tk})$ are efficiently calculated in the E-step of the Expectation-Maximization (EM) algorithm. The observation model is defined by $p(\vec{y}_t | \vec{x}_t, \phi^{(k)})$ which gives the probability of observing the output $\vec{y}_t$ given the input $\vec{x}_t$ and observation model parameters $\phi^{(k)}$.

We denote the speed of the input and focal fish as $v_1(n)$ and $v_2(n)$, respectively, for $n = 1,..., T + 2L$ and construct the $D = 2L + 1$ dimensional input and output vectors

$$\vec{x}_t = [v_1(t), \ldots, v_1(t + L), \ldots, v_1(t + 2L)]^{\top}$$

$$\vec{y}_t = [v_2(t), \ldots, v_2(t+L), \ldots, v_2(t+2L)]^\top$$

for $t = 1, \ldots, T$. Each vector captures the symmetrically time-lagged speeds about index $t + L$.

We consider an input-driven spherical-Gaussian observation model with covariance $\Sigma^{(k)} = \sigma^2 I$ tied across states

$$p(\vec{y}_t | \vec{x}_t, W^{(k)}, b^{(k)}, \sigma^2) = N(\vec{y}_t | W^{(k)}\vec{x}_t + \vec{b}^{(k)}, \sigma^2 I)$$

The contribution of the observation model to the ECCL is

$$Q(W^{(k)}, b^{(k)}, \sigma) = -\frac{1}{2}\sum_{t=1}^{T}\sum_{k=1}^{K}\gamma(z_{tk})[\, D\ln 2\pi + D\ln\sigma^2 + \frac{1}{\sigma^2}||\vec{y}_t - W^{(k)}\vec{x}_t - \vec{b}^{(k)}||^2\,]$$

For applications within this paper, we constrain the biases to be a constant vector $\vec{b}^{(k)} = b^{(k)}\vec{1}$ and the $D \times D$ weights matrix $W^{(k)}$ to have a symmetric Toeplitz structure. The constraints placed on $W^{(k)}$ require the M-step be carried out using a numerical optimization algorithm. We refer to this model as a constrained linear-model HMM (cLM-HMM).

To gain insight into the role of the weights matrix, and to see how the Toeplitz structure naturally arises, we consider maximization of the unconstrained model parameters. Maximization of the ECLL with respect to the biases reveals

$$\vec{b}^{(k)} = \frac{\sum_{t=1}^{T}\gamma(z_{tk})[\vec{y}_t - W^{(k)}\vec{x}_t]}{\sum_{t=1}^{T}\gamma(z_{tk})} \equiv \vec{\mu}_2^{(k)} - W^{(k)}\vec{\mu}_1^{(k)} \approx \mu_2^{(k)}\vec{1} - \mu_1^{(k)}W^{(k)}\vec{1}$$

where we define the i^{th} element of the state-conditioned mean vector $[\vec{\mu}_p^{(k)}]_i = \mu_p^{(k)}(i) \equiv \frac{\sum_{t=1}^{T}\gamma(z_{tk})v_p(t+i)}{\sum_{t=1}^{T}\gamma(z_{tk})}$ for $i = 0, \ldots, 2L$ and $p = 1, 2$. We note $\mu_p^{(k)}(i) \approx \mu_p^{(k)}(j)$ for all $i \neq j$ when $T >> L$ barring any significant outliers within the first and last $2L$ steps. We refer to this criteria as the short-lag regime. For simplicity, we define the state-conditioned mean $\mu_p^{(k)} \equiv \mu_p^{(k)}(L)$ and approximate $\vec{\mu}_p^{(k)} \approx \mu_p^{(k)}\vec{1}$. Similarly, we define the state-conditioned standard deviation $\sigma_p^{(k)}(i) = \left(\frac{\sum_{t=1}^{T}\gamma(z_{tk})[v_p(t+i) - \mu_p^{(k)}(i)]}{\sum_{t=1}^{T}\gamma(z_{tk})}\right)^{1/2}$ and let $\sigma_p^{(k)} \equiv \sigma_p^{(k)}(L)$.

Maximization of the ECLL with respect to the unconstrained weights matrix results in solving $W^{(k)}B^{(k)} = C^{(k)}$ where

$$B^{(k)} \equiv \sum_{t=1}^{T}\gamma(z_{tk})[\vec{x}_t - \vec{\mu}_1^{(k)}]\,\vec{x}_t^\top$$

$$C^{(k)} \equiv \sum_{t=1}^{T}\gamma(z_{tk})[\vec{y}_t - \vec{\mu}_2^{(k)}]\,\vec{x}_t^\top$$

Each entry $i, j = 0, \ldots, 2L$ of the respective matrices can be expressed as

$$B_{ij}^{(k)} \approx \sum_{t=1}^{T}\gamma(z_{tk})[v_1(t+i) - \mu_1^{(k)}]v_1(t+j) = \sum_{t=1+i-L}^{T+i-L}\gamma(z_{t-i+L,k})[v_1(t+L) - \mu_1^{(k)}]v_1(t+L+j-i)$$

$$C_{ij}^{(k)} \approx \sum_{t=1}^{T}\gamma(z_{tk})[v_2(t+i) - \mu_2^{(k)}]v_1(t+j) = \sum_{t=1+i-L}^{T+i-L}\gamma(z_{t-i+L,k})[v_2(t+L) - \mu_2^{(k)}]v_1(t+L+j-i)$$

where we approximated $\vec{\mu}_p^{(k)} \approx \mu_p^{(k)}\vec{1}$ to arrive at the first equality and re-indexed in the last equality. Next, we define the normalized, state-conditioned cross-correlation between signals v_p and v_q

$$R_{pq}^{(k)}(\tau) \equiv \frac{1}{\beta_{pq}^{(k)}} \sum_{t=1}^{T} \gamma(z_{tk})[v_p(t + L) - \mu_p^{(k)}] v_q(t + L + \tau)$$

where the normalization constant $\beta_{pq}^{(k)} \equiv \sigma_p^{(k)} \sigma_q^{(k)} \sum_{t=1}^{T} \gamma(z_{tk})$. If $p = q$, we define $R_p^{(k)}(\tau) \equiv R_{pp}^{(k)}(\tau)$ as the autocorrelation function. Using this expression, we can approximate

$$B_{ij}^{(k)} \approx \beta_1^{(k)} R_1^{(k)}(j - i)$$

$$C_{ij}^{(k)} \approx \beta_{21}^{(k)} R_{21}^{(k)}(j - i)$$

which is valid in the short-lag regime when the duration of each state is typically longer than the lag L. We can further express $C_{ij}^{(k)} = \sum_{l=0}^{2L} W_{il}^{(k)} B_{lj}^{(k)}$. Combining expressions, we find

$$R_{21}^{(k)}(j - i) \approx \frac{\sigma_1^{(k)}}{\sigma_2^{(k)}} [\vec{w}_i^{(k)} * R_1^{(k)}](j)$$

where $\vec{w}_i^{(k)}$ denotes the i^{th} row of the weights matrix. The shift invariance of this expression with respect to indices i and j implies that each row $\vec{w}_{i+1}^{(k)}$ must be a one-step time-shifted version of the preceding row $\vec{w}_i^{(k)}$. This results in a weights matrix with approximately Toeplitz structure.

Dual Fit Model

To simultaneously fit both permutations of the selected input and output signals, we construct the column-stacked $D = 4L + 2$ dimensional dual input $\vec{x}_t^{(2)} = \begin{bmatrix} \vec{x}_t^{\top} & \vec{y}_t^{\top} \end{bmatrix}^{\top}$ and output $\vec{y}_t^{(2)} = \begin{bmatrix} \vec{y}_t^{\top} & \vec{x}_t^{\top} \end{bmatrix}^{\top}$ vectors where the single input $\vec{x}_t$ and output $\vec{y}_t$ vectors are as previously defined. We define the dual-observation model

$$p(\vec{y}_t^{(2)} | \vec{x}_t^{(2)}, \phi^{(k)}) = p(\vec{y}_t | \vec{x}_t, \phi^{(k)}) p(\vec{x}_t | \vec{y}_t, \phi^{(k)}) = N(\vec{y}_t | W^{(k)} \vec{x}_t + b^{(k)} \vec{1}, \sigma^2 I)\, N(\vec{x}_t | W^{(k)} \vec{y}_t + b^{(k)} \vec{1}, \sigma^2 I)$$

resulting in the contribution to the ECLL

$$Q(W^{(k)}, b^{(k)}, \sigma) = -\frac{1}{2} \sum_{t=1}^{T} \sum_{k=1}^{K} \gamma(z_{tk}) [\, 2D \ln 2\pi + 2D \ln \sigma^2 + \frac{1}{\sigma^2} || \vec{y}_t - W^{(k)} \vec{x}_t - b^{(k)} \vec{1} ||^2 + \frac{1}{\sigma^2} || \vec{x}_t - W^{(k)} \vec{y}_t - b^{(k)} \vec{1} ||^2 \,]$$

SUPPLEMENTAL FIGURES

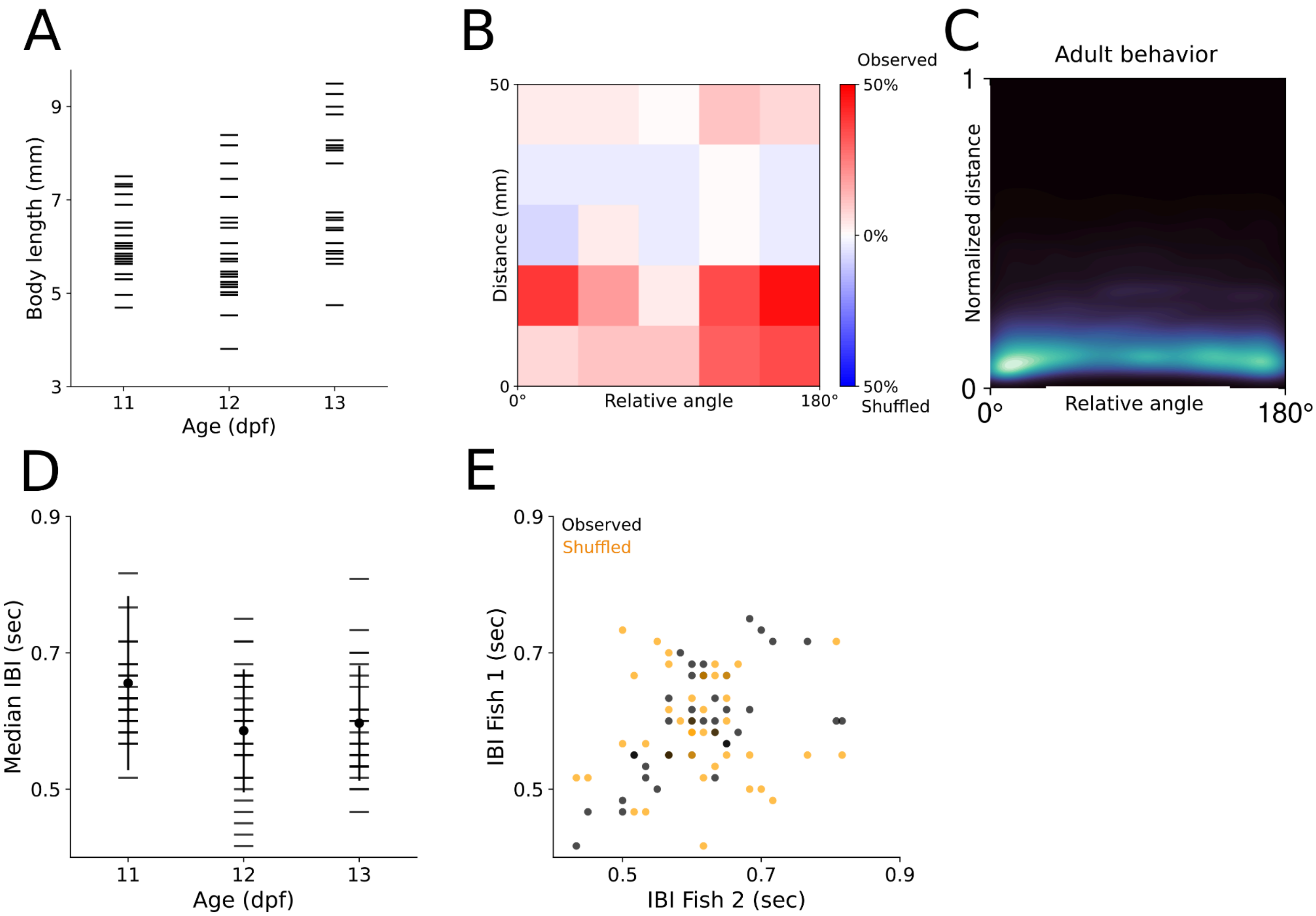

Figure S1. A.) Body length by age. B.) Statistical quantification of 2-dimensional histograms of distance and relative angle. Colors represent the percentage of pairs that are greater than the 95th percentile of the opposing (observed/shuffled) distribution, showing a significant bias in the observed data towards close and parallel/anti-aligned configurations **C.)** 2D histogram of distance and relative angle for adult pairs of zebrafish. **D.)** Median interbout interval as a function of age. **E.)** Median interbout intervals for each fish plotted against the median interbout interval of their partner. Shuffled data is presented in orange. Linear correlations for observed (R^2 = .321, $p < 0.001$) and shuffled (R^2 = .002, p = 0.739).

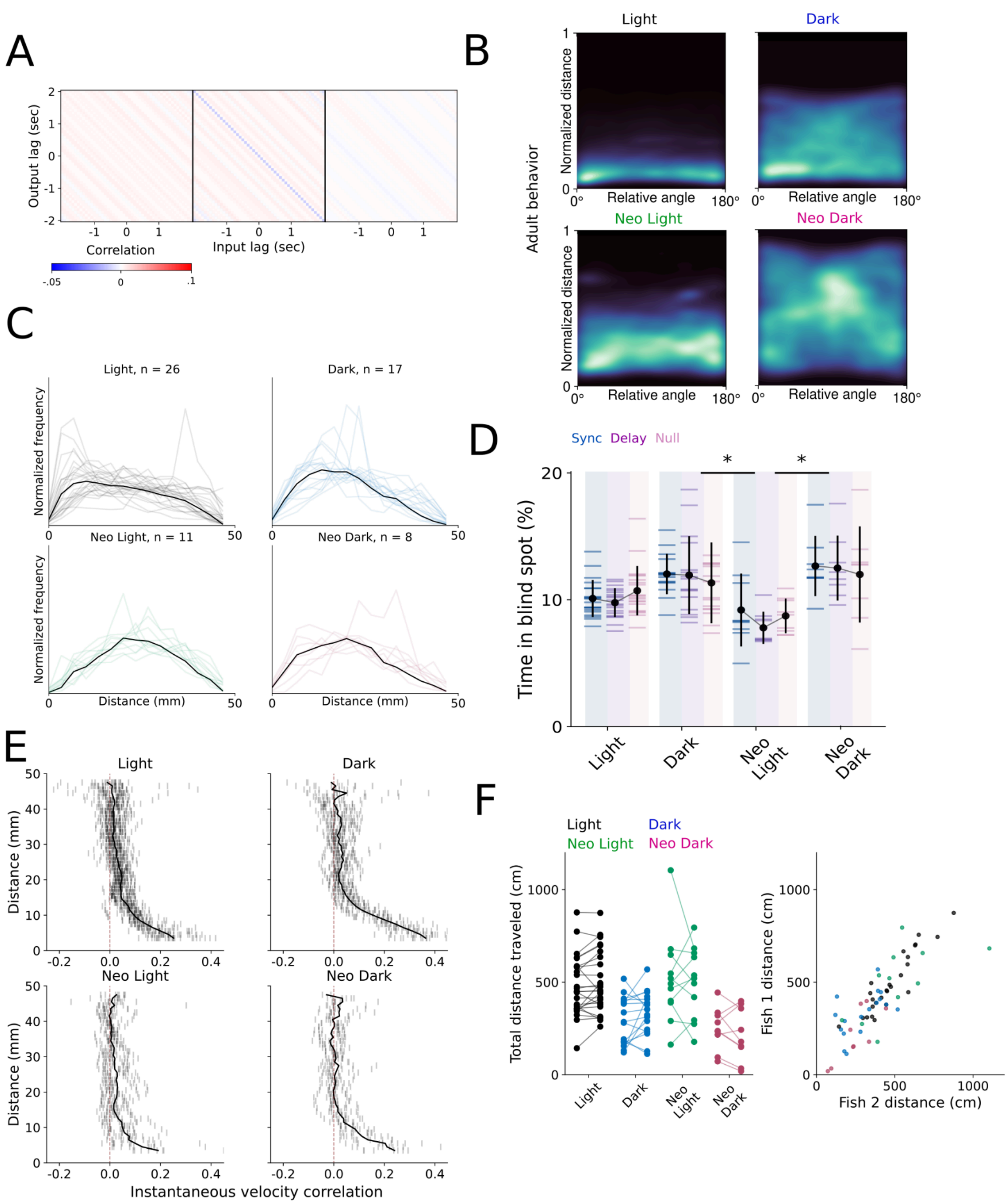

Figure S2. A.) Inferred weights from shuffled dataset. **B.)** 2D histograms of adult zebrafish in all sensory conditions. **C.)** Distance histograms of juvenile zebrafish. **D.)** Percentage of time spent in blind spot across conditions. **E.)** Instantaneous velocity correlations as a function of distance across sensory conditions. **F.)** Total distance traveled across sensory conditions, with lines connecting individuals in the same experiment, and total distance traveled for an individual plotted against the distance traveled by its partner. Asterisks indicate a p value below an alpha of 0.05 corrected for multiple comparisons with a Šidák adjustment.

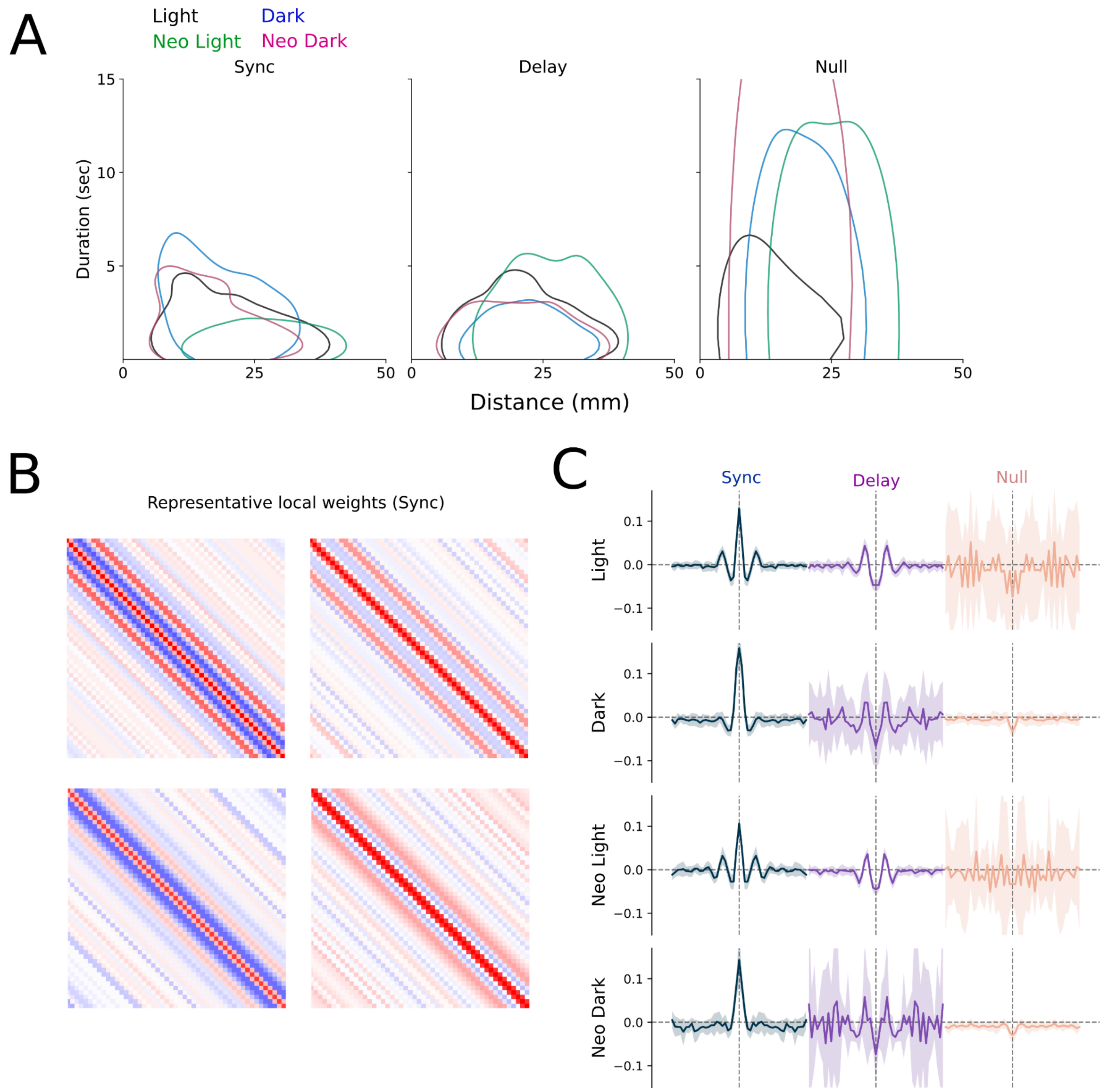

Figure S3. A.) Distributions of state duration by distance. **B.)** Representative local inferred weights for the synchronized state in 4 light pairs. **C.)** Average central row of the local inferred weights matrices for all sensory conditions.