Long-term memory induced correction to Arrhenius law
Supplementary Information

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In this Supplementary Information, we provide

1. calculation details to obtain the solution of the GLE equation (without target) [Section A].
2. a detailed derivation of the equations of the non-Markovian theory [Section B]
3. calculation details for the asymptotic analysis in the rare event limit $L \to \infty$ [Section C].
4. Details on simulations and additional simulation data to check the Gaussian behavior of trajectories in the future of first passage events and our scaling arguments [Section D].
5. A note on the exactness of the approach for weakly non-Markovian processes [Section E].

A. Solution of the Generalized Langevin Equation (without absorbing target)

Here, we consider the dynamics given by the overdamped GLE

$$\int_0^t dt' K(|t-t'|)\dot{x}(t') = -k\, x(t) + \xi(t), \qquad \langle \xi(t)\xi(t')\rangle = k_B\mathcal{T}\ K(|t-t'|). \tag{S1}$$

In absence of target, the solution of this equation is well known [1], it is reminded here for the sake of completeness. Since the above equation is linear, the resulting process $x(t)$ is Gaussian and is fully characterized by its two first moments. Denoting $\widetilde{f}(s) = \int_0^\infty f(t)e^{-st}dt$ the Laplace transform of a function f, we obtain

$$\widetilde{x}(s) = \frac{\widetilde{\xi}(s) + x(0)\tilde{K}(s)}{s\tilde{K}(s)+k}. \tag{S2}$$

We also write

$$\langle \widetilde{\xi}(s)\widetilde{\xi}(s')\rangle = k_B\mathcal{T}\int_0^\infty dt \int_0^\infty dt' e^{-(st+s't')}K(|t-t'|), \tag{S3}$$

$$= k_B\mathcal{T}\int_0^\infty dt' \int_0^\infty d\tau\ e^{-(s+s')t'-s\tau}K(\tau) + k_B\mathcal{T}\int_0^\infty dt \int_0^\infty d\tau'\ e^{-(s+s')t-s'\tau'}K(\tau'), \tag{S4}$$

$$= k_B\mathcal{T}\frac{\tilde{K}(s)+\tilde{K}(s')}{s+s'}, \tag{S5}$$

where Eq. (S4) has been obtained by setting $t = t'+\tau$ for $t > t'$ and $t' = t+\tau'$ for $t' > t$. Using this result and (S2) yields, for an initially equilibrated initial position $\langle x(0)^2\rangle = k_B\mathcal{T}/k$:

$$\langle \widetilde{x}(s)\widetilde{x}(s')\rangle = \frac{k_B\mathcal{T}}{[s\tilde{K}(s)+k][s'\tilde{K}(s')+k]}\left\{\frac{\tilde{K}(s)+\tilde{K}(s')}{s+s'} + \frac{\tilde{K}(s)\tilde{K}(s')}{k}\right\},$$

$$= \frac{k_B\mathcal{T}}{k(s+s')}\left\{\frac{\tilde{K}(s)}{s\tilde{K}(s)+k} + \frac{\tilde{K}(s')}{s'\tilde{K}(s')+k}\right\}. \tag{S6}$$

We may recognize that if one sets

$$\langle x(t)x(t')\rangle = l^2\phi(|t-t'|), \tag{S7}$$

then, using the same procedure as in Eq. (S4),

$$\langle \tilde{x}(s)\tilde{x}(s')\rangle = \frac{l^2}{s+s'}[\tilde{\phi}(s)+\tilde{\phi}(s')]. \tag{S8}$$

Comparing the above equation with (S6) leads to

$$\tilde{\phi}(s) = \frac{\tilde{K}(s)}{s\tilde{K}(s)+k}. \tag{S9}$$

This formula is valid for arbitrary kernel. For the power-law kernel (3) of the main text, we obtain $\tilde{K} = K_\alpha/s^{1-\alpha}$ and $\phi(t)$ is a Mittag-Leffler function :

$$\tilde{\phi}(s) = \frac{K_\alpha s^\alpha}{s[s^\alpha K_\alpha + k]}, \qquad \phi(t) = E_\alpha\left(-\left[\frac{t}{\tau_d}\right]^\alpha\right). \tag{S10}$$

The mean and covariance of the process when $x(0) = x_0$ is fixed can be obtained by using general formulas on conditional means and covariances for Gaussian processes, see e.g. chapter 3 in Ref. [2] :

$$\mathbb{E}(A|Y=y) = \mathbb{E}(A) - \frac{\mathrm{Cov}(A,Y)}{\mathrm{Var}(A)}(\mathbb{E}(Y)-y), \tag{S11}$$

$$\mathrm{Cov}(A,B|Y=y) = \mathrm{Cov}(A,B) - \frac{\mathrm{Cov}(A,Y)\mathrm{Cov}(B,Y)}{\mathrm{Var}(A)}. \tag{S12}$$

These formulas relate conditional averages and covariances to non-conditional ones, here $\mathbb{E}(A|Y=y)$ is the average of the variable A given that the variable Y takes the value y, and $\mathrm{Cov}(A,B|Y=y)$ is the covariance of A, B given that $Y = y$. Using these formulas, the average and the covariance of the process $x(t)$ conditional to $x(0) = x_0$ read

$$m_0(t) \equiv \mathbb{E}(x(t)|x(0)=x_0) = x_0\phi(t), \tag{S13}$$

$$\sigma(t,t') \equiv \mathrm{Cov}(x(t),x(t')|x(0)=0) = l^2[\phi(|t-t'|) - \phi(t)\phi(t')]. \tag{S14}$$

We can also check these expressions by using directly (S2).

B. Derivation of the equations of the non-Markovian theory [Eqs. (7,8,9,10,11)]

Here we derive the equations that will give access to the mean first passage time (mean FPT) to $x = L$, when the stochastic process starts at x_0 at $t = 0$. Let us start with a two-point generalized version of the renewal equation :

$$p(L,t;x_1,t+t_1) = \int_0^t dt' F(t') p(L,t;x_1,t+t_1|\mathrm{FPT}=t'). \tag{S15}$$

This exact equation comes from the fact that, if x is observed at position L at t, since the process is non-smooth, it means that L was reached for the first time at some time t', and the above equation is obtained by partitioning the event of observing (L, x_1) at times $t, t+t_1$ over the value of the FPT. Here, $p(L,t;x_1,t+t_1)$ is the joint probability density function (pdf) of observing $x = L$ at time t and the position $x = x_1$ at a later time $t + t_1$. The fact that the initial position is fixed is implicitly understood in this notation. Next, $p(L,t;x_1,t+t_1|\mathrm{FPT}=t')$ represents the probability density of observing $x = L$ at time t and $x = x_1$ at a later time $t+t_1$ given that the FPT is t'. Note that, as originally noted in Ref. [3], for non-Markovian processes, it is necessary to keep the information that the target was reached at t' for the first time in the propagators, this condition is different from the condition that $x(t') = L$ which would hold for Markovian processes.

Now, we introduce the process in the future of the FPT, $x_\pi(t) \equiv x(t+\mathrm{FPT})$ and we denote as $p_\pi(y,t)$ its pdf at time t (after the FPT). By definition,

$$p_\pi(L,t;x_1,t+t_1) = \int_0^\infty d\tau F(\tau) p(L,t+\tau;x_1,t+t_1+\tau|\mathrm{FPT}=\tau). \tag{S16}$$

We also define the stationary probability density of observing $x = L$ at some time and x_1 after a time t_1 has elapsed :

$$p_s(L;x_1,t_1) \equiv \lim_{t\to\infty} p(L,t;x_1,t+t_1). \tag{S17}$$

We now consider Eq. (S15), where we substract $p_s(L;x_1,t_1)$ on both sides, leading to

$$p(L,t;x_1,t+t_1)-p_s(L;x_1,t_1)=\int_0^t dt' F(t')[p(L,t;x_1,t+t_1\text{FPT}=t')-p_s(L;x_1,t_1)]$$
$$-\int_t^\infty d\tau F(t')p_s(L;x_1,t_1), \tag{S18}$$

where we have used the fact that $\int_0^\infty dt F(t)=1$. To proceed further, we remark that

$$\int_0^\infty dt \int_t^\infty dt' F(t') = \int_0^\infty dt' \int_0^{t'} dt F(t') = \int_0^\infty dt' t' F(t') = \langle T\rangle. \tag{S19}$$

We also note the following equalities :

$$\int_0^\infty dt \int_0^t dt' F(t')[p(L,t;x_1,t+t_1|\text{FPT}=t')-p_s(L;x_1,t_1)],$$
$$=\int_0^\infty dt' \int_{t'}^\infty dt\ F(t')\ [p(L,t;x_1,t+t_1|\text{FPT}=t')-p_s(L;x_1,t_1)], \tag{S20}$$
$$=\int_0^\infty dt' \int_0^\infty du\ F(t')\ [p(L,t'+u;x_1,t'+t_1+u|\text{FPT}=t')-p_s(L;x_1,t_1)], \tag{S21}$$
$$=\int_0^\infty du \int_0^\infty dt'\ F(t')\ [p(L,t'+u;x_1,t'+t_1+u|\text{FPT}=t')-p_s(L;x_1,t_1)], \tag{S22}$$
$$=\int_0^\infty du\ [p_\pi(L,u;x_1,u+t_1)-p_s(L;x_1,t_1)], \tag{S23}$$

where the successive calculation steps are : (i) the inversion of the order of integration for the variables (t,t') in Eq. (S20), (ii) the change of variable $t=u+t'$ in Eq. (S21), (iii) again a change in the order of integration between the variables u,t' in (S22), and (iv) finally the use of the definition (S16) to simplify the integral. Next, using Eqs. (S19) and (S23), we see that that integrating Eq. (S18) over t leads to

$$\int_0^\infty dt\ [p_\pi(L,t;x_1,t+t_1)-p(L,t;x_1,t+t_1)]=\langle T\rangle p_s(L;x_1,t+t_1). \tag{S24}$$

This equation is general and exact, as soon as p_s exists, for any continuous non-smooth stochastic process (even non-Gaussian). Integrating over x_1 leads to a general expression for the mean FPT :

$$\langle T\rangle p_s(L)=\int_0^\infty dt\ [p_\pi(L,t)-p(L,t)]. \tag{S25}$$

Next, we write $p_\pi(L,t;x_1,t+t_1)=p_\pi(L,t)p_\pi(x_1,t+t_1|L,t)$ (this is Bayes' formula). Using this, multiplying Eq. (S24) by x_1 and integrating over x_1 yields

$$\int_0^\infty dt[p_\pi(L,t)m_\pi^*(t+t_1|L,t)-p(L,t)m_0^*(t+t_1|L,t)]=\langle T\rangle p_s(L)m_s^*(t_1|L,0), \tag{S26}$$

where $m_\pi^*(t+t_1|L,t)$ is the conditional average of $x_\pi(t+t_1)$ given that $x_\pi(t)=L$, and (similarly) $m_0^*(t+t_1|L,t)$ is the conditional average of $x(t+t_1)$ given that $x(t)=L$. Finally, $m_s^*(t_1)$ is the average of $x(t_1)$ given that the system is equilibrated at $t=0$, with the condition $x(0)=L$. Combining Eqs. (S25) and (S26), we obtain

$$\int_0^\infty dt\{p_\pi(L,t)[m_\pi^*(t+t_1|L,t)-m_s^*(t_1)]-p(L,t)[m_0(t+t_1|L,t)-m_s^*(t_1)]\}=0. \tag{S27}$$

To proceed further, we assume that, in the future of the FPT, the process $x_\pi(t)$ is Gaussian, with a mean $m_\pi(t)$ and a covariance $\sigma_\pi(t,t')\simeq\sigma(t,t')$ that is approximated by the stationary covariance conditioned to $x = 0$ at $t = 0$. The next step consists in using the above equations as closure relations to determine the mean FPT.

We now write explicit expressions for m_π^*, m_0^*, m_s^*. Using the general formula (S11) for conditional averages, where we use $A = x_\pi(t)$, $Y = x_\pi(t+t_1)$ and $y = L$, we obtain

$$m_\pi^*(t+t_1|L,t) = m_\pi(t+t_1) - \frac{\sigma(t+t_1,t)}{\psi(t)}[m_\pi(t) - L], \tag{S28}$$

where $\psi(t) = \sigma(t,t) = l^2[1-\phi(t)^2]$ is the mean square displacement of the process $x(t)$ conditioned to $x(0) = 0$. Similarly, applying again Eq. (S11) for $A = x(t)$, $Y = x(t+t_1)$ and $y = L$, we obtain

$$m_0^*(t+t_1|L,t) = m_0(t+t_1) - \frac{\sigma(t+t_1,t)}{\psi(t)}[m_0(t) - L]. \tag{S29}$$

Taking the limit $t \to \infty$ in the above formula enables us to identify m_s^* :

$$m_s^*(t_1|L,0) = L\phi(t_1). \tag{S30}$$

We also note that, for Gaussian propagators,

$$p_\pi(L,t) = \frac{e^{-[L-m_\pi(t)]^2/2\psi(t)}}{\sqrt{2\pi\psi(t)}}, \qquad p(L,t) = \frac{e^{-[L-m_0(t)]^2/2\psi(t)}}{\sqrt{2\pi\psi(t)}}. \tag{S31}$$

Collecting these results, the closure equation (S27) for $m_\pi(t)$ becomes

$$\begin{aligned}\mathcal{H}(\tau) \equiv \int_0^\infty dt \Bigg\{ &\frac{e^{-[L-m_\pi(t)]^2/2\psi(t)}}{[\psi(t)]^{1/2}} \left[m_\pi(t+\tau) - [m_\pi(t) - L]\frac{\phi(\tau) - \phi(t)\phi(t+\tau)}{1-\phi^2(t)} - L\phi(\tau) \right] \\ &- \frac{e^{-[L-x_0\phi(t)]^2/2\psi(t)}}{[\psi(t)]^{1/2}} \left[x_0\phi(t+\tau) - [x_0\phi(t) - L]\frac{\phi(\tau) - \phi(t)\phi(t+\tau)}{1-\phi^2(t)} - L\phi(\tau) \right] \Bigg\} = 0, \end{aligned} \tag{S32}$$

and the expression (S25) for the mean FPT becomes

$$\langle T \rangle p_s(L) = \int_0^\infty dt \left\{ \frac{e^{-[L-m_\pi(t)]^2/2\psi(t)}}{[2\pi\psi(t)]^{1/2}} - \frac{e^{-(L-x_0\phi(t))^2/2\psi(t)}}{[2\pi\psi(t)]^{1/2}} \right\}. \tag{S33}$$

Behavior of $m_\pi(t)$ at large times and consequence for the mean FPT

We note that, for large times, $\phi(t)$ becomes a small quantity for large times. Then we see that the second line of the integrande in Eq. (S32) behaves as

$$\frac{e^{-(L-x_0\phi(t))^2/2\psi(t)}}{[2\pi\psi(t)]^{1/2}} \left[x_0\phi(t+\tau) - [x_0\phi(t) - L]\frac{\phi(\tau) - \phi(t)\phi(t+\tau)}{1-\phi^2(t)} - L\phi(\tau) \right] \underset{t\to\infty}{\simeq} x_0(1-\phi(\tau))\phi(t). \tag{S34}$$

Since $\phi(t) \simeq A/t^{2H}$ and $H < 1/2$, we see that these terms have to be compensated so that the integral (S32) exists ; this implies that

$$m_\pi(t) \underset{t\to\infty}{\simeq} x_0\ \phi(t), \tag{S35}$$

and this equality should hold at all orders of t^{-a} with $a < 1$. If the behavior (S35) holds then the mean FPT predicted by Eq. (S33) is finite.

C. Asymptotic analysis in the rare event limit, $L \to \infty$

Here, we analyze the structure of the solution $m_\pi(t, L)$ in the limit $L \to \infty$. As mentioned in the main text, a natural length scale for the dynamics near the top of the potential is $l^* = k_B \mathcal{T}/F$, where $F = kL$ is the slope of the potential. Hence $l^* = l^2/L$. The associated time scale t^* is the time at which $\psi(t^*)$ is of order l^*, this leads to $t^* = (l^*/\sqrt{\kappa})^{1/H}$. This suggests the ansatz

$$m_\pi(t, L) \simeq L - \frac{l^2}{L} f(t/t^*), \qquad t^* = \left(\frac{l^2}{L\sqrt{\kappa}}\right)^{1/H}. \tag{S36}$$

Note that $t^* \to 0$ when $L \to \infty$. Here f is a scaling function that is determined by requiring that $\mathcal{H}(\tau = t^* v)$, where $\mathcal{H}$ is defined in Eq. (S32), vanishes in the limit $L \to \infty$ (at fixed v) :

$$\mathcal{H}(t^* v) \underset{L\to\infty}{\simeq} \frac{l^2 (t^*)^{1-H}}{L\sqrt{\kappa}} \int_0^\infty \frac{du}{u^H} e^{-\frac{f^2(u)}{2u^{2H}}} \left[-f(u+v) + f(u) \frac{u^{2H} + (u+v)^{2H} - v^{2H}}{2u^{2H}} + \frac{v^{2H}}{2} \right] = 0, \tag{S37}$$

where we have used $\phi(\tau) \simeq 1 - \tau^{2H}\kappa/(2l^2)$ for small τ (so that $\psi(\tau) \simeq \kappa\tau^{2H}$). Solving this equation yields the scaling function f. Next, we investigate the behavior of $m_\pi(t)$ at time scales larger than t^*. It is natural to assume that $m_\pi(t)$ admits a regime that varies at the same time scale τ_d as the original dynamics for $x(t)$, which leads us to the ansatz

$$m_\pi(t, L) \simeq \begin{cases} L - \frac{l^2}{L} f(t/t^*) & t = \mathcal{O}(t^*),\ (t \ll \tau_d), \\ L\ \phi_\pi(t) & t = \mathcal{O}(\tau_d),\ (t \gg t^*), \end{cases} \tag{S38}$$

where ϕ_π is a scaling function that is independent of L. The linear term in L in factor of ϕ_π is justified by the fact that the matching with the solution at scale t^* can be achieved with the conditions

$$\phi_\pi(t) \underset{t\to 0}{\simeq} 1 - c\, \frac{\kappa\, t^{2H}}{l^2}, \qquad f(u \to \infty) \simeq c\, u^{2H}, \tag{S39}$$

where c is a numerical constant. The equation for ϕ_π is obtained by looking at the behavior of $\mathcal{H}(\tau)$ for $L \to \infty$ at fixed τ, the integrals can in fact be evaluated at times t^* (all other terms are exponentially small) so that we obtain

$$\mathcal{H}(\tau) \underset{L\to\infty}{\simeq} \frac{(t^*)^{1-H}}{\sqrt{\kappa}} \int_0^\infty du \frac{e^{-f^2(u)/2u^{2H}}}{u^H} \left[L\ \phi_\pi(\tau) - L\ \phi(\tau) \right]. \tag{S40}$$

Since $\mathcal{H}(\tau)$ has to vanish for all τ we conclude that $\phi_\pi = \phi$: thus at this time scale τ_d the average trajectory in the future of the FPT is, at leading order, the same as the trajectory constrained to $x(0) = L$ starting from an equilibrium configuration. However, it is obvious that this behavior (S38) cannot hold at very long times since at this stage it is not possible to connect the long-time behavior of $m_\pi = L\phi(t)$ to the already identified behavior given by Eq. (S35), where $m_\pi(t) \simeq x_0\phi(t)$. Hence, we have to postulate the existence of at least one additional longer time scales. Let us define now T_{RE} as

$$T_{\mathrm{RE}} = e^{L^2/(2l^2)} \frac{(t^*)^{1-H} l}{\kappa^{1/2}} \nu_H, \qquad \nu_H = \int_0^\infty du\ \frac{e^{-f^2(u)/2u^{2H}}}{u^H}. \tag{S41}$$

It turns out that T_{RE} will be the value of the mean FPT at leading order when $L \to \infty$, but since this is not obvious for our long-term memory process we use the above equation as a definition for T_{RE}. Note that $L^2/(2l^2) = E/k_B\mathcal{T}$ is the value of the energy barrier to be crossed to reach the target point. Anticipating the final result, we postulate that T_{RE} is also a characteristic time scale for m_π. Considering this third time scale, the behavior of m_π reads

$$m_\pi(t, L) \simeq \begin{cases} L - \frac{l^2}{L} f(t/t^*) & t = \mathcal{O}(t^*),\ (t \ll \tau_d), \\ L\ \phi(t) & t = \mathcal{O}(\tau_d),\ (t^* \ll t \ll T_{\mathrm{RE}}), \\ \frac{LA}{T_{\mathrm{RE}}^{2H}} \chi\left(\frac{t}{T_{\mathrm{RE}}}\right) & t = \mathcal{O}(T_{\mathrm{RE}}),\ (\tau_d \ll t) \end{cases} \tag{S42}$$

where χ is a scaling function. The term $LA/T_{\rm RE}^{2H}$ in factor of χ is justified by the fact that the solutions at scales τ_d and $T_{\rm RE}$ are matched (i.e. predict the same value for m_π) at the condition

$$\chi(u) \underset{u\to 0}{\simeq} 1/u^{2H}. \tag{S43}$$

We find the equation for χ by calculating $\mathcal{H}(\tau=\overline{\tau}T_{\rm RE})$ when $L\to\infty$ at fixed $\overline{\tau}$. The key remark is that since $T_{\rm RE}$ is exponentially large with L, the integral (S32) has two contributions : a first one coming from τ of oder t^* and a second one coming from $\tau=O(T_{\rm RE})$. We note that

$$\frac{\phi(\tau)-\phi(t+\tau)\phi(t)}{1-\phi^2(t)} \underset{t\ll\tau_d\ll\tau}{\simeq} \frac{A}{2\tau^{2H}}, \tag{S44}$$

so that, with $\tau=T_{\rm RE}\overline{\tau}$ and $t=ut^*$, we have

$$m_\pi(t+\tau)=m_\pi(ut^*+\overline{\tau}T_{\rm RE})\simeq \frac{LA}{T_{\rm RE}^{2H}}\chi(\overline{\tau}), \tag{S45}$$

$$(m_\pi(t)-L)\frac{\phi(\tau)-\phi(t+\tau)\phi(t)}{1-\phi^2(t)} = -\frac{l^2}{L}\frac{f(u)}{T_{\rm RE}^{2H}}\frac{A}{2\tau^{2H}} \ll m_\pi(t+\tau). \tag{S46}$$

Following these considerations, we evaluate

$$\begin{aligned}\mathcal{H}(\tau=T_{\rm RE}\overline{\tau}) \underset{L\to\infty}{\simeq} & \frac{(t^*)^{1-H}}{\sqrt{\kappa}}\frac{LA}{T_{\rm RE}^{2H}}\int_0^\infty du\frac{e^{-f^2(u)/2u^{2H}}}{u^H}\left[\chi(\overline{\tau})-\frac{1}{\overline{\tau}^{2H}}\right] \\ &+\frac{LA\;T_{\rm RE}}{l\;T_{\rm RE}^{2H}}\int_0^\infty d\overline{t}\;e^{-\frac{L^2}{2l^2}}\left[\chi(\overline{t}+\overline{\tau})-\frac{\tilde{x}_0}{(\overline{t}+\overline{\tau})^{2H}}\right],\end{aligned} \tag{S47}$$

where $\tilde{x}_0=x_0/L$ and one keeps $\tilde{x}_0$ constant when taking the limit $L\to\infty$. Equating this expression to zero and using the definition of $T_{\rm RE}$ in Eq. (S41) we thus obtain

$$\chi(\overline{\tau})-\frac{1}{\overline{\tau}^{2H}}+\int_0^\infty d\overline{t}\;\left[\chi(\overline{t}+\overline{\tau})-\frac{\tilde{x}_0}{(\overline{t}+\overline{\tau})^{2H}}\right]=0. \tag{S48}$$

This equation can be solved by setting $G(\overline{\tau})=\chi(\overline{\tau})-\tilde{x}_0/\overline{\tau}^{2H}$, and differentiating with respect to $\overline{\tau}$:

$$G'(\overline{\tau})+(1-\tilde{x}_0)\frac{2H}{\overline{\tau}^{2H+1}}-G(\overline{\tau})=0, \tag{S49}$$

where one has assumed that $G(\infty)=0$. The only solution that does not diverge exponentially for large arguments is

$$G(\overline{\tau})=(1-\tilde{x}_0)2H\;\Gamma(-2H,\overline{\tau})e^{\overline{\tau}}, \qquad \chi(\overline{\tau})=(1-\tilde{x}_0)2H\;\Gamma(-2H,\overline{\tau})e^{\overline{\tau}}+\frac{\tilde{x}_0}{\overline{\tau}^{2H}}. \tag{S50}$$

where $\Gamma(s,x)=\int_x^\infty t^{s-1}e^{-t}dt$ is the upper incomplete gamma function. We note that, the above expression satisfies the matching condition Eq. (S43), suggesting that our analysis is consistent. We also note that, when $t\to\infty$, the predictions of Eqs. (S42) and (S50) coincide with the behavior (S35). This means that the complete structure of $m_\pi(t)$ has been determined, at all time scales.

To evaluate the mean FPT, we introduce two intermediate time scales ε, λ that satisfy

$$t^*\ll\varepsilon\ll\tau_d\ll\lambda\ll T_{\rm RE}. \tag{S51}$$

The mean FPT is evaluated by splitting the integral (S33) over the three intervals $]0,\varepsilon[$, $]\varepsilon,\lambda[$ and $]\lambda,\infty[$, and by using the appropriate form of m_π in Eq. (S42) for each interval. This leads to

$$\begin{aligned}\langle T\rangle p_s(L)=&\frac{(t^*)^{1-H}}{\sqrt{\kappa}}\int_0^{\varepsilon/t^*}du\frac{e^{-f^2(u)/2u^{2H}}}{[2\pi u^{2H}]^{1/2}}+\int_\varepsilon^\lambda dt\left\{\frac{e^{-[L(1-\phi)]^2/2\psi(t)}}{[2\pi\psi(t)]^{1/2}}-\frac{e^{-(L-x_0\phi(t))^2/2\psi(t)}}{[2\pi\psi(t)]^{1/2}}\right\} \\ &+\frac{ALT_{\rm RE}^{1-2H}}{l^3\sqrt{2\pi}}e^{-L^2/2l^2}\int_{\lambda/T_{\rm RE}}^\infty du\;L\left[\chi(u)-\frac{\tilde{x}_0}{u^{2H}}\right],\end{aligned} \tag{S52}$$

where for $t > \lambda$ we have used the fact that $m_\pi(t) \ll L$ and $x_0\phi \ll L$, and we have set $t = uT_{\rm RE}$. Replacing χ by its value, and taking the limit $\lambda/T_{\rm RE} \to 0$ and $\varepsilon/t^* \to \infty$, we finally obtain

$$\langle T\rangle \simeq T_{RE} + T_{RE}^{1-2H} \times \frac{AL(L-x_0)\Gamma(1-2H)}{l^2}, \tag{S53}$$

which is Eq. (15) in the main text.

D. Details on simulations and additional numerical controls

Here, we present additional numerical results supporting our findings. In Fig. S1 we present additional tests of the validity of the Gaussian approximation and of the stationary covariance approximation. In Fig. S2 we present a test of the scaling behavior of m_π for large L. Last, we report the used values of the time step dt for all simulations of this work in table S1.

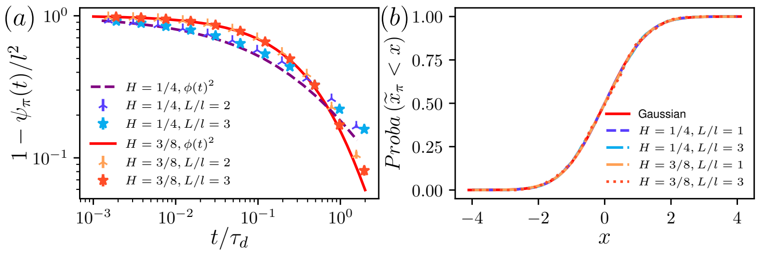

FIGURE S1: (a) Additional check of the stationary covariance hypothesis. Here, $\psi_\pi(t) = \text{var}(x_\pi(t))$ and one represents $1 - \psi_\pi(t)/l^2$ to determine whether the stationary covariance approximation is valid at long times (where $\psi_\pi(t) \to l^2$). Symbols are simulation results (parameter values are indicated in legend) and are compared to $\phi^2(t)$ obtained in the stationary covariance approximation (dashed and full lines). (b) Check of the Gaussian approximation. Here, one represents the cumulative distribution function (CDF) of the rescaled variable $\tilde{x}_\pi(t) = [x_\pi(t) - \langle x_\pi(t)\rangle]/\psi_\pi^{1/2}(t)$. The red line is the CDF of a normalized Gaussian. Other dashed lines represent simulation results, with parameters indicated in the legend. The collapse of the curves suggests that the stochastic process $x_\pi(t)$ is well approximated by a Gaussian process.

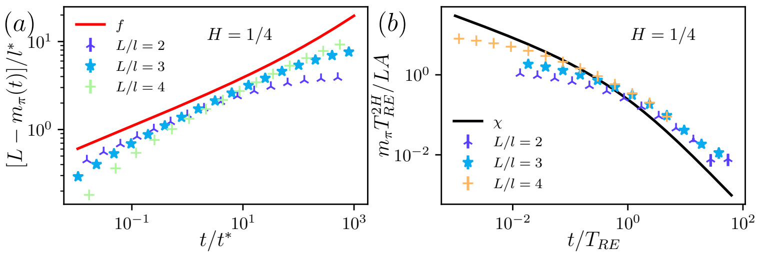

FIGURE S2: Additional checks for scaling behavior of $m_\pi(t)$ for $H = 1/4$. (a) Check of the short time scaling (S36) $m_\pi(t) = L - l^* f(t/t^*)$ in the limit $L \to \infty$. Here f is calculated by numerically solving (S37). Note that the larger discrepancy between f and the data at short times comes from the finiteness of the time step Δt compared to t^* (since $t^* \propto 1/L^4$ here). The fact that one needs to generate trajectories that are longer than $\langle T\rangle \propto e^{L^2/2l^2}$ prevents us from using smaller time steps for large L. (b) Check of the long time scaling regime given by Eq. (13) in the main text. Here the initial position is drawn from an equilibrium distribution, corresponding to our predictions for $x_0 = 0$.

H	L/l	dt/τ_d	Figures
3/8	0	5.96×10^{-6}	Fig 2(a)
3/8	1	5.96×10^{-6}	Figs. 2(a), 3(a), 4(b), 4(d), S1(b)
3/8	2	5.96×10^{-6}	Figs. 2(a), 3(a), 4(a), 4(c), 4(d), S1(a)
3/8	3	7.45×10^{-6}	Figs. 2(a), 3(a), 4(c), 4(d), S1(b)
3/8	3	1.49×10^{-5}	Figs. 4(a), S1(a)
3/8	4	1.86×10^{-5}	Figs. 3(a), 4(d)
3/8	4	7.45×10^{-6}	4(c)
1/4	0	1.86×10^{-7}	Figs. 2(b)
1/4	1	1.86×10^{-7}	Figs. 2(b), 3(b), S1
1/4	2	7.45×10^{-7}	Figs. 2(b), 3(b), 4(a), 4(b), S1, S2
1/4	3	3.72×10^{-6}	Figs. 2(b), 3(b), 4(a), S1, S2
1/4	4	1.30×10^{-5}	Figs. 3(b), S2

TABLE S1: Value of the time steps used in the simulations.

E. Exactness of the theory at first order for weakly non-Markovian processes

Let us consider the case of weakly non-Markovian processes, for which the covariance and mean of the process $x(t)$ are given by Eqs. (S13) and (S14), with

$$\phi(t) = e^{-\lambda t} + \varepsilon \phi_1(t), \tag{S54}$$

with $\lambda > 0$, ε is a small parameter, and $\phi_1(t)$ is an arbitrary function. For simplicity, and without loss of generality, we set $\lambda = 1$ and $l = 1$. We start with the generalization of Eq. (S16) for an arbitrary number of positions and times x_i, t_i :

$$p_\pi(L,t;x_1,t+t_1;x_2,t+t_2;...;x_N,t+t_N) = \\ \int_0^\infty d\tau F(\tau) p(L,t+\tau;x_1,t+t_1+\tau;x_2,t+t_2+\tau;...;x_N,t+t_N+\tau|\text{FPT}=\tau). \tag{S55}$$

Following the approach of Section B, this equation leads to

$$\int_0^\infty dt[p_\pi(0,t;x_1,t+t_1;x_2,t+t_2;...) - p(0,t;x_1,t+t_1;x_2,t+t_2;...)] = \langle T\rangle p_s(0;x_1,t_1;x_2,t_2;...). \tag{S56}$$

We may write formally a continuous version of this equation, for all paths $[y(\tau)]$ with $y(0) = L$:

$$\langle T\rangle P_\text{s}([y(\tau)]) - \int_0^\infty dt \left\{\Pi([y(\tau)],t) - P([y(\tau)],t)\right\} = 0, \tag{S57}$$

where $P_\text{s}([y(\tau)])$ is the stationary probability to follow the path $[y(\tau)]$, $\Pi([y(\tau)],t)$ is the probability to follow the path $[y]$ in the future t of the FPT (ie, the probability density that $x(\text{FPT}+\tau+t) = y(\tau)$ for all $\tau > 0$), and $P([y(\tau)],t)$ is the probability density that $x(t+\tau) = y(\tau)$ for all $\tau > 0$. Using Bayes' formula, we can write (S57) as

$$\int_0^\infty dt \left\{\Pi([y(\tau)],t|y(0)=L)p_\pi(L,t) - P([y(\tau)],t|y(0)=L)p(L,t)\right\} - \langle T\rangle p_s(L) P_\text{s}([y(\tau)]|y(0)=L) = 0, \tag{S58}$$

which is valid for all paths $[y(\tau)]$ (if $y(0) \neq L$ the above equation is simply $0 = 0$). Let us define a functional $\mathcal{F}([k])$ as the value of the above expression when multiplied by $e^{\int_0^\infty d\tau k(\tau) y(\tau)}$ and integrated over all paths y. In principle $\mathcal{F}([k])$ should vanish for all functions $k(\tau)$. Let us evaluate $\mathcal{F}$ for a distribution of paths Π that satisfies our hypotheses, i.e. by assuming that the process in the future of the first passage time is Gaussian with mean $m_\pi(t)$ and with the stationary covariance approximation. Using formulas for the moment generating function of Gaussian processes, we find

$$\begin{aligned}\mathcal{F}([k(\tau)]) &= \langle T\rangle p_s(L) e^{\int_0^\infty d\tau k(\tau) m_s^*(\tau)} e^{\frac{1}{2}\int_0^\infty d\tau \int_0^\infty d\tau' k(\tau)k(\tau')\sigma(\tau,\tau')} \\ &- \int_0^\infty dt \left[p_\pi(0,t) e^{\int_0^\infty d\tau k(\tau) m_\pi^*(t+\tau|L,t)} - p(0,t) e^{\int_0^\infty d\tau k(\tau) m_0^*(t+\tau|L,t)} \right] e^{\int_0^\infty d\tau \int_0^\infty d\tau' \frac{k(\tau)k(\tau')}{2}\sigma(t+\tau,t+\tau'|t)},\end{aligned} \tag{S59}$$

where we remind that $\langle T\rangle$ is evaluated with Eq. (S33), and

$$\sigma(t+\tau, t+\tau'|t) = \sigma(t+\tau, t+\tau') - \frac{\sigma(t+\tau,t)\sigma(t+\tau',t)}{\sigma(t,t)}. \tag{S60}$$

If one could find a function $m_\pi(t)$ so that $\mathcal{F}([k(\tau)])$ vanishes for all $k(\tau)$, it would mean that the theory is exact. It does not seem to be the case in general. However, when $\varepsilon \to 0$, assuming that

$$m_\pi(t) = Le^{-\lambda t} + \varepsilon \mu_1(t), \tag{S61}$$

we can evaluate (S59) as

$$\mathcal{F}([k(\tau)]) = -\varepsilon \int_0^\infty d\tau k(\tau) Q_1(\tau) \times e^{\int_0^\infty du \int_0^\infty du' k(u)k(u')\frac{1}{2}\sigma(u,u')} + \mathcal{O}(\varepsilon^2), \tag{S62}$$

where

$$\begin{aligned}Q_1(\tau) = \int_0^\infty \frac{dt}{\sqrt{2\pi(1-e^{-2t})}} \Bigg\{ & e^{-\frac{[L(1-e^{-t})]^2}{2(1-e^{-2t})}} \left[\mu_1(t+\tau) - \mu_1(t)e^{-\tau} - Le^{-t}S_1(t,\tau) - L\phi_1(\tau) \right] \\ & - e^{-\frac{(L-x_0e^{-t})^2}{2(1-e^{-2t})}} \left[m_1^*(t,\tau) - L\phi_1(\tau) \right] \Bigg\},\end{aligned} \tag{S63}$$

where we have defined S_1 and m_1^* such that

$$\frac{\sigma(t+\tau,t)}{\sigma(t,t)} = e^{-\tau} + \varepsilon S_1(t,\tau) + \mathcal{O}(\varepsilon^2), \qquad m_0^*(t+\tau|L,t) = Le^{-\tau} + \varepsilon m_1^*(t,\tau) + \mathcal{O}(\varepsilon^2). \tag{S64}$$

Note that, to obtain (S62), it is important to remark that

$$\sigma(t+\tau, t+\tau'|t) = \sigma(\tau,\tau') + \mathcal{O}(\varepsilon^2). \tag{S65}$$

We observe that the equality $Q_1(\tau) = 0$ for all τ can be realized by a proper choice of μ_1, so that $\mathcal{F}([k(\tau)])$ vanishes at order ε for all functions $k(\tau)$. This suggests that our theory is exact at order ε.

Supplementary References

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