How synchronized human networks escape local minima: supplemental materials

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S1 Analytical models

S1.1 Derivation of the effective potential

We calculate the effective potential of the system. To analyze the player dynamics, we start with the Kuramoto model for N coupled oscillators with delayed coupling. We denote each oscillator's phase as a function of time as $\varphi_n(t)$. Its dynamic is determined by:

$$\frac{\partial \varphi_n(t)}{\partial t} = \omega_n + \kappa \sin(\varphi_{n+1}(t-\Delta t) - \varphi_n(t)), \tag{S1}$$

where the frequency of the oscillator is ω_n, the coupling strength is κ, and we assume periodic boundary conditions, as $\varphi_{N+i} = \varphi_i$. We evaluate the phase difference between each adjacent oscillator as

$$\Delta\varphi_n(t) = \varphi_{n+1}(t) - \varphi_n(t) \tag{S2}$$

so:

$$\sum_{n=1}^{N} \Delta\varphi_n = 0. \tag{S3}$$

Derivative with respect to time gives:

$$\frac{\partial \Delta\varphi_n(t)}{\partial t} = \Omega_n + \kappa \sin(\varphi_{n+2}(t-\Delta t) - \varphi_{n+1}(t)) - \kappa \sin(\varphi_{n+1}(t-\Delta t) - \varphi_n(t)), \tag{S4}$$

where $\Omega_n = \omega_{n+1} - \omega_n$. We focus on the end of the ring, without loss of generality since the beginning of the ring is arbitrarily chosen. This is done to take into account the periodic boundary conditions of the ring. Therefore, we are analyzing the dynamics of $\Delta\varphi_{N-1}$ as,

$$\frac{\partial \Delta\varphi_{N-1}(t)}{\partial t} = \Omega_{N-1} + \kappa \sin(\varphi_1(t-\Delta t) - \varphi_N(t)) - \kappa \sin(\varphi_N(t-\Delta t) - \varphi_{N-1}(t)). \tag{S5}$$

When the system is close to phase locking, the phases of all the oscillators change linearly in time with about the same tempo. We assume that all the oscillators have the same frequency and are equally separated in the phase-space, so $\Delta\varphi_n = \Delta\varphi$, for all $n < N$, and $\Delta\varphi_N = -(N-1)\Delta\varphi$. This assumption is verified by the experimental results shown in Figs. 3, 4, and 5, in the manuscript, where there is an equal spacing between the phases, apart from the first and the last player. So,

$$\varphi_n(t) = \Delta\varphi(t)n + \omega t. \tag{S6}$$

So,

$$\frac{\partial \Delta\varphi_{N-1}(t)}{\partial t} = \frac{\partial \Delta\varphi(t)}{\partial t} = \Omega_n + \kappa \sin(-(N-1)\Delta\varphi - \omega\Delta t) - \kappa \sin(\Delta\varphi - \omega\Delta t), \tag{S7}$$

This equation of motion can be described as a damped dynamics in an effective potential, V, where

$$V(\Delta\varphi) = -\Omega\Delta\varphi - \frac{\kappa}{N-1}\cos((N-1)\Delta\varphi + \omega\Delta t) - \kappa\cos(\Delta\varphi - \omega\Delta t), \tag{S8}$$

and the equation of motion is:

$$\frac{\partial \Delta\varphi(t)}{\partial t} = -\frac{\partial V(\Delta\varphi)}{\partial \Delta\varphi}. \tag{S9}$$

Effective potential is an efficient way to obtain intuition about the dynamics of a system. It is easier to imagine a small ball rolling in a potential than to imagine how a solution of a differential equation changes in time. In our case, since the equation of motion has a first derivative in time, the force is proportional to the velocity and not acceleration. Thus, once the force is zero, the speed is zero, so the system has no inertia. When the system reaches a minimum in energy, namely the force is zero, the system stops. Therefore, the effective potential reveals in which states the system is stable and how it will evolve to these states.

Calculating the effective potential of our system reveals that for $\Delta t = 0$ there is a global minimum when $\Delta\varphi = 0$. So, when the delay is zeros all the oscillators have the same frequency and phase, which is the in-phase state of synchronization. However, when the delay increases beyond $\omega\Delta t > \pi/(2N)$, the $\Delta\varphi = 0$ becomes a local minimum and a new global minimum appears in $\Delta\varphi = -2\pi/N$, namely, the vortex state of synchronization. When continue to increase the delay, we find new stable states which are the higher vortex orders.

S1.2 Derivation of the tempo slowing down

We model the tempo slowing down due to the delay from the Kuramoto model. We start from the equation for the relative phase, as:

$$\frac{\partial \Delta\varphi_{N-1}(t)}{\partial t} = \Omega_{N-1} + \kappa\sin(\varphi_1(t-\Delta t) - \varphi_N(t)) - \kappa\sin(\varphi_N(t-\Delta t) - \varphi_{N-1}(t)), \tag{S10}$$

All the players are starting to play in-phase and therefore as long as the delay satisfies $\Delta t < T/N$, we assume $\varphi_n(t-\Delta t) \approx \varphi_n(t) - \Delta t\,\partial\varphi_n/\partial t$. Thus,

$$\frac{\partial \Delta\varphi_{N-1}(t)}{\partial t} = \Omega_{N-1} + \kappa\sin\left(\varphi_1(t) - \varphi_N(t) - \Delta t\frac{\partial\varphi_1}{\partial t}\right) - \kappa\sin\left(\varphi_N(t) - \varphi_{N-1}(t) - \Delta t\frac{\partial\varphi_N}{\partial t}\right), \tag{S11}$$

Since we are close to phase locking, we approximate $\sin(\alpha) \approx \alpha$, and $\Delta\varphi_N = \varphi_1(t) - \varphi_N(t)$, so:

$$\frac{\partial \Delta\varphi_{N-1}(t)}{\partial t} = \Omega_{N-1} + \kappa\left(\Delta\varphi_N(t) - \Delta t\frac{\partial\varphi_1}{\partial t}\right) - \kappa\left(\Delta\varphi_{N-1}(t) - \Delta t\frac{\partial\varphi_N}{\partial t}\right). \tag{S12}$$

Considering, $\Delta\varphi_n = \Delta\varphi$ for all $n < N$ and $\Delta\varphi_N = -(N-1)\Delta\varphi$, leads to:

$$\frac{\partial \Delta\varphi(t)}{\partial t} = \frac{\Omega - N\kappa\Delta\varphi}{1-(N-1)\kappa\Delta t}, \tag{S13}$$

where Ω is the average detuning between the players. From this relation, we obtain that the phase-locking condition, $N\kappa > \Omega$, does not change when introducing a small coupling delay between the players, so, the system remains phase-locked.

However, when considering the average phase as a function of time $\varphi = \sum\varphi_n/N$, we obtain:

$$\frac{\partial\varphi(t)}{\partial t} = \omega + \frac{\kappa}{N}\sum\sin\left(\varphi_{n+1}(t) - \varphi_n(t) - \Delta t\frac{\partial\varphi_{n+1}}{\partial t}\right) = \omega + \frac{\kappa}{N}\sum\Delta\varphi_n(t) - \frac{\kappa\Delta t}{N}\sum\frac{\partial\varphi_{n+1}}{\partial t}, \tag{S14}$$

where $\omega = \sum\omega_n/N$. Since $\sum\Delta\varphi_n(t) = 0$, and $1/N\sum\partial\varphi_{n+1}/\partial t = \partial\varphi/\partial t$ this leads to:

$$\frac{\partial\varphi(t)}{\partial t} = \frac{\omega}{1+\kappa\Delta t}, \tag{S15}$$

Thus, the tempo of the coupled oscillators slows down as long as the players stay phase-locked. Assuming that $\kappa\Delta t > 1$, we obtain that $T = 2\pi\kappa\Delta t/\omega$. Therefore, this tempo slowing down ensures that $\Delta t < T/N$, indicating that our assumptions are valid.

S2 Experimental details

S2.1 Experimental setup

We set sixteen isolated electric violin players to repeatedly play a musical phrase. We collect the output from each violin and control the input to each player via noise-cancellation headphones. The players can not see or hear each other apart from what is heard in their headphones. All the players start playing the first phrase with the help of an external rhythmical beat, to verify that they all start with the same playing period and phase. The rhythmical beat is stopped after the first phrase and the only instruction to the players is to try to synchronize their rhythm to what they hear in their headphones. We established different network connectivities and introduced delayed coupling between the players while monitoring the phase, playing period, volume, and frequency of each player with a mixing system.

S2.2 Choosing the musical phrase

When choosing the musical phrase for the violin players, we consider these points:

- **Cyclic**. It is important for the phrase not to have a clear beginning or end, but rather, have a cyclic characteristic to it.
- **Same octave**. For post-processing with Fourier transform and not confusing with other notes due to over-tones, it is easier when the entire phrase is at the same octave.
- **Repeating notes.** To help the players identify quickly where their neighbor is playing, it is preferable to have a minimum number of repeating notes.
- **4th finger**. We need our players to play for a long time and with a minimum number of mistakes, therefore, we prefer finding a musical phrase that does not require using the 4th finger.

S2.3 Analyzing the results

We record the output from each player and analyze it off-line. We use a fast Fourier transform algorithm on a moving temporal window to retrieve the spectrogram of the player. If the temporal window is too long we will not have enough temporal resolution and if the temporal window is too short, we will not have enough spectral resolution. We are using a temporal window of $0.15s$. Next, we identify the different notes according to their frequency and map the time each player played a specific note. Next, we identified which player is coupled to which, and compensated for the delay between them to check if they are phase-locked or not. All the experimental results and the code to analyze them are available online: 10.6084/m9.figshare.22822103

S3 Experimental results

We study the synchronization dynamics of human networks with closed-ring topology and unidirectional time-delayed coupling. Such a network has a complex potential landscape with well-defined local and global minima that we can tune and control in real-time. We prepare the system in a global minimum (fully synchronized) state and then adiabatically transform the potential landscape by tuning the coupling delay time such that this state becomes a local minimum. We then study in detail how the system escapes this local minimum into the new fully synchronized global minimum by measuring the amplitude, tempo, and phase of each node and identifying which node is following which. In this study, we set the players' network as a unidirectional ring, so that each player hears only a single neighbor with periodic boundary conditions.

S3.1 Spreading of the phase

The first dynamic we present is when the players are spreading their phases to escape from the in-phase local minimum into a vortex-phase state global minimum. This dynamic is enabled by the players' ability to reduce their coupling until they find a stable state. The phase results as a function of time for networks with $N = 4, 5, 7$, and 8 players are shown in Supplementary Fig. 1. As shown, in all network sizes, we observe a phase locking in the beginning, where all the players have the same phase at the same time, so all the curves coincide, or at least parallel to each other. Then, at least one curve leaves all the other curves. Then, all the different curves spread, until they spread over 2π.

S3.2 Slowing down of the tempo and oscillations death

The second dynamic we observe is the slowing down of the players' tempo as an alternative strategy to maintain a stable in-phase synchronization state in the presence of delayed coupling. The measured phase results as a function of time for networks with $N = 3$ to 8 coupled players are shown in Supplementary Fig. 2. From these results, we evaluate the players' tempo by calculating the average derivative of the phase.

When the tempo slows too much, the players get stuck in a state of oscillation death, namely, all the players are playing the same note indefinitely, thereby maintaining a degenerate form of synchronization. We present oscillation death for $N = 3$, 4, and 5 coupled violin players, in Supplementary Fig. 2(a), (b), and (c), respectively.

S3.3 Amplitude death

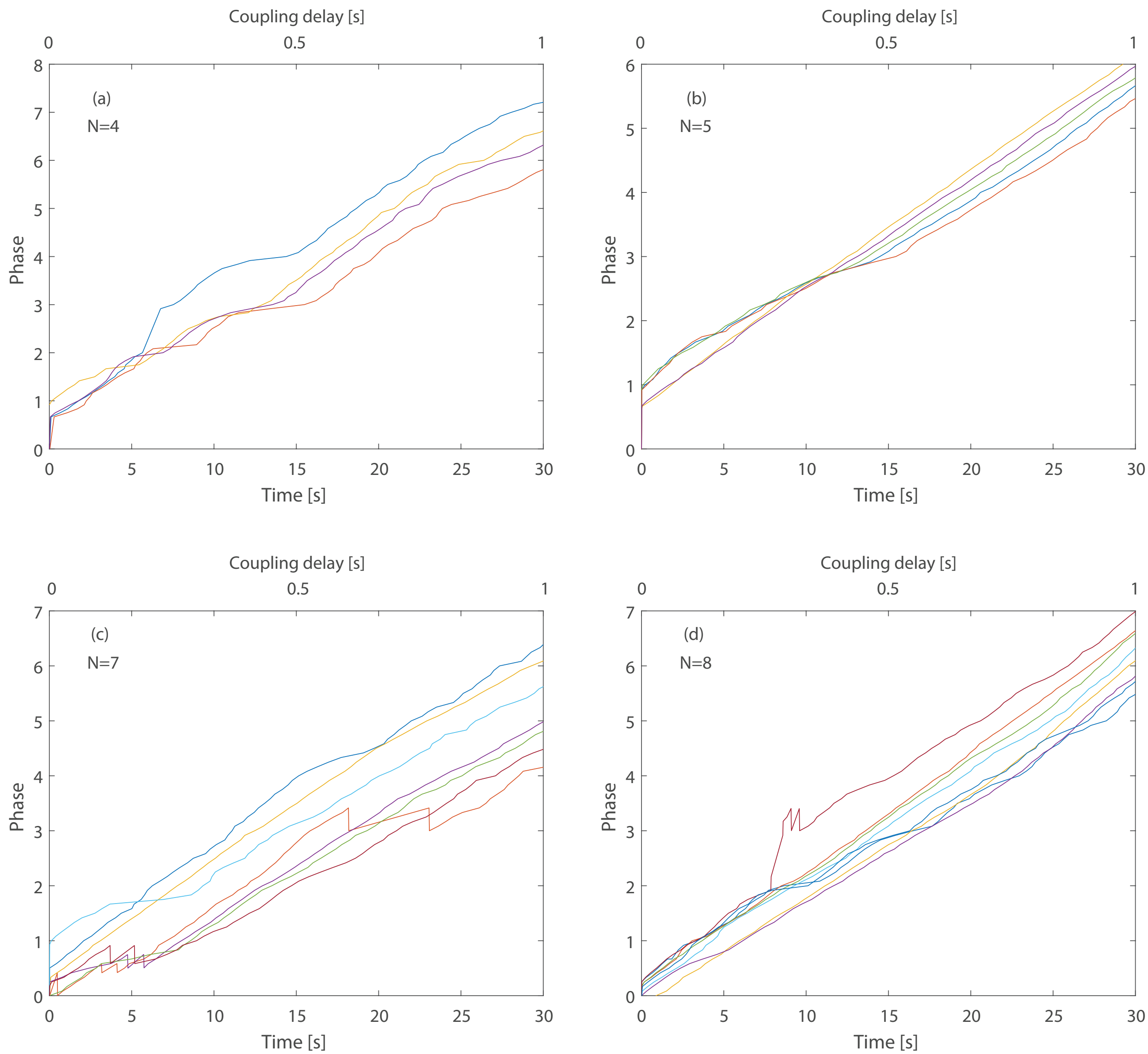
The third dynamic we observed was amplitude death. Here, one of the players stops playing breaking the periodic boundary conditions and changing the topology into a unidirectional open-chain network. In such a network, there is no global frustration, and the players find a solution at any value of delay. Once the players reach the vortex solution, the players that stopped playing continue playing at the correct phase. We present amplitude death for $N = 3$, 5, and 7 players in Supplementary Fig. 3(a), (b), and (c).

S3.4 Experimental results of a two-dimension lattice

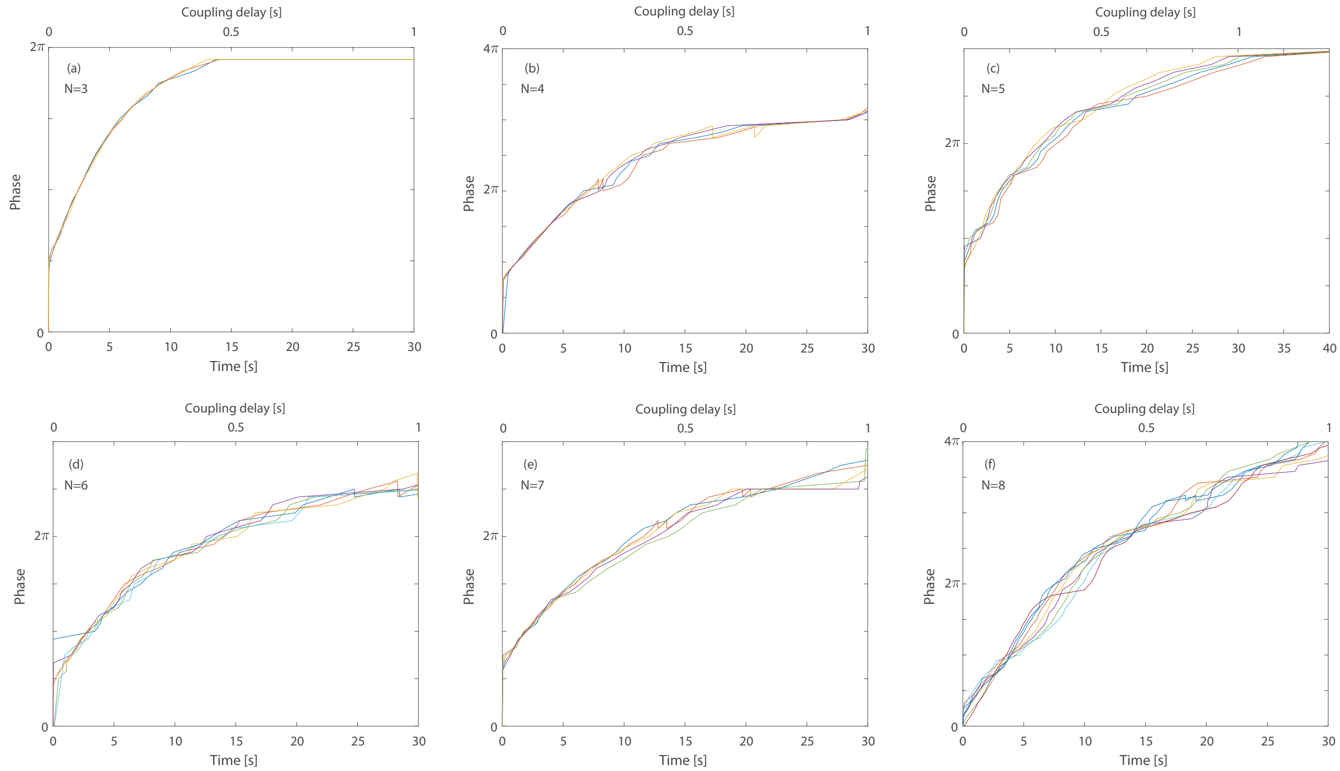
To generalize our findings into networks with higher complexity, we measured the dynamics of the players when situated in two-dimensional networks with periodic boundary conditions. We start with a square two-dimension lattice shown in Fig. S4(a). We measure the phase as a function of time, shown in Supplementary Fig. 4(b), and evaluate the average tempo, shown in Supplementary Fig. 4(c). As evident, we observe the slowing down of the tempo similar to the one shown in the one-dimensional rings. Next, we analyze four players from the lattice that are situated on a ring and observe the same spreading of phase observed in the one-dimensional ring. This is presented with two sets of four players in different orientations, shown in Supplementary Fig. 4(d) and (e). These results verify that our findings are general to other networks, with unidirectional motifs. We analyze the effective connectivity of the network as a function of time and show it visually in https://youtu.be/h9C0XQgrLPI.

S4 Numerical simulations

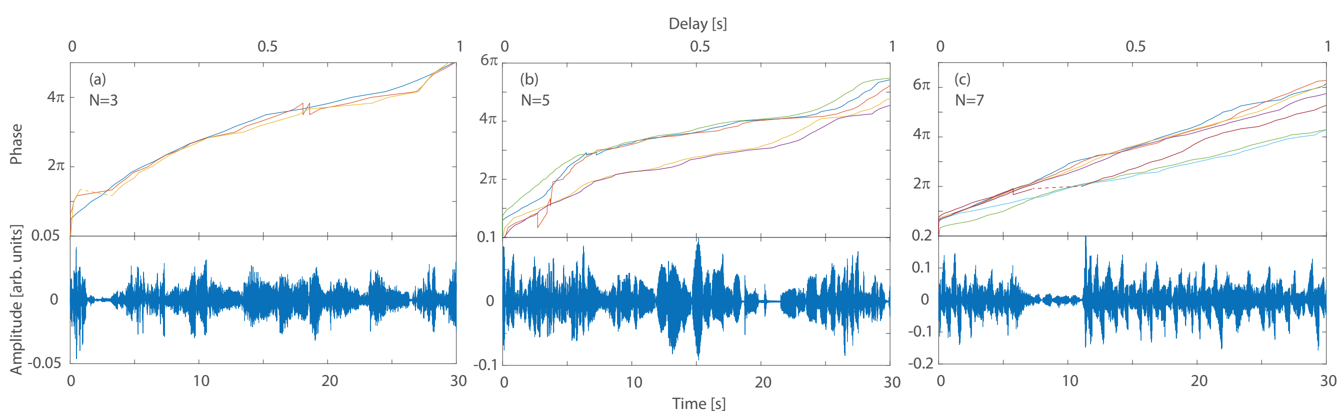
To study our models in large networks, we performed numerical simulations of random networks with unidirectional and bidirectional coupling. We assume a linear increasing delay between the nodes as a function of time. Two representative results of the different phases of the nodes as a function of time are shown in Supplementary Fig. 5. We repeated this calculation numerous times for $N = 10$, 100, and 1000 players, and observed the slowing down of the tempo. We show representative results of the phase of the players as a function of time with the average tempo in Supplementary Fig. 5(a)-(c). Next, we change the coupling strength as a function of time according to $\kappa(t) = \kappa(t = 0)\cos^2(\Delta t N\pi/T)$ as done in Section 3 in the main text. We repeated the calculations for different networks and observed the spreading of the phases, representative results are shown in Supplementary Fig. 5(d)-(e). Here, the players are divided into several clusters with a phase shift between them according to the different unidirectional rings in the random networks. These results indicate that the dynamics of the motifs are general dynamics that appear in large complex networks.



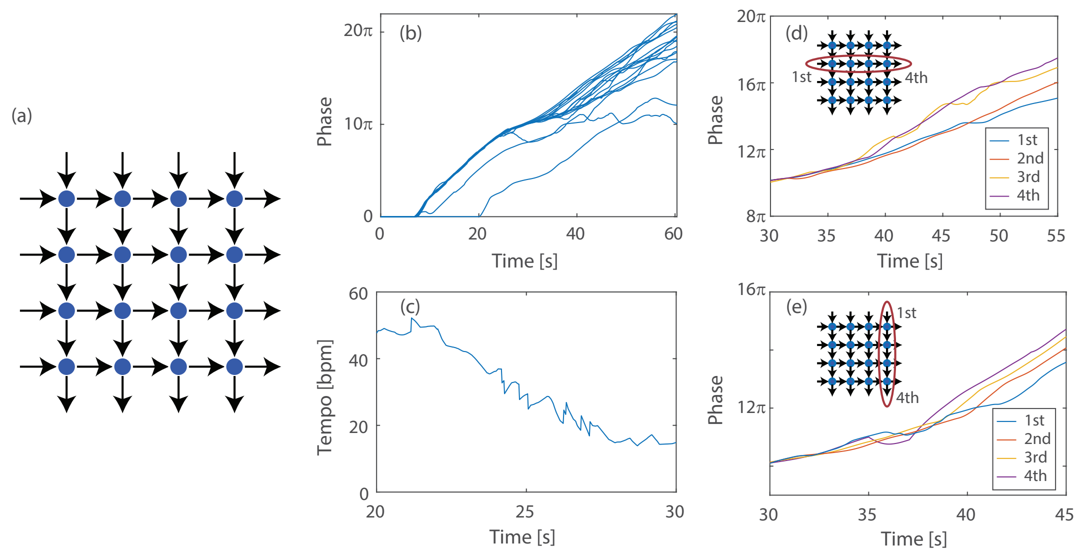
Supplementary Figure 1. Coupled violin players situated on a ring with unidirectional coupling showing how the phase between the players is spread when at least one of the players reduces its coupling compared to the others.



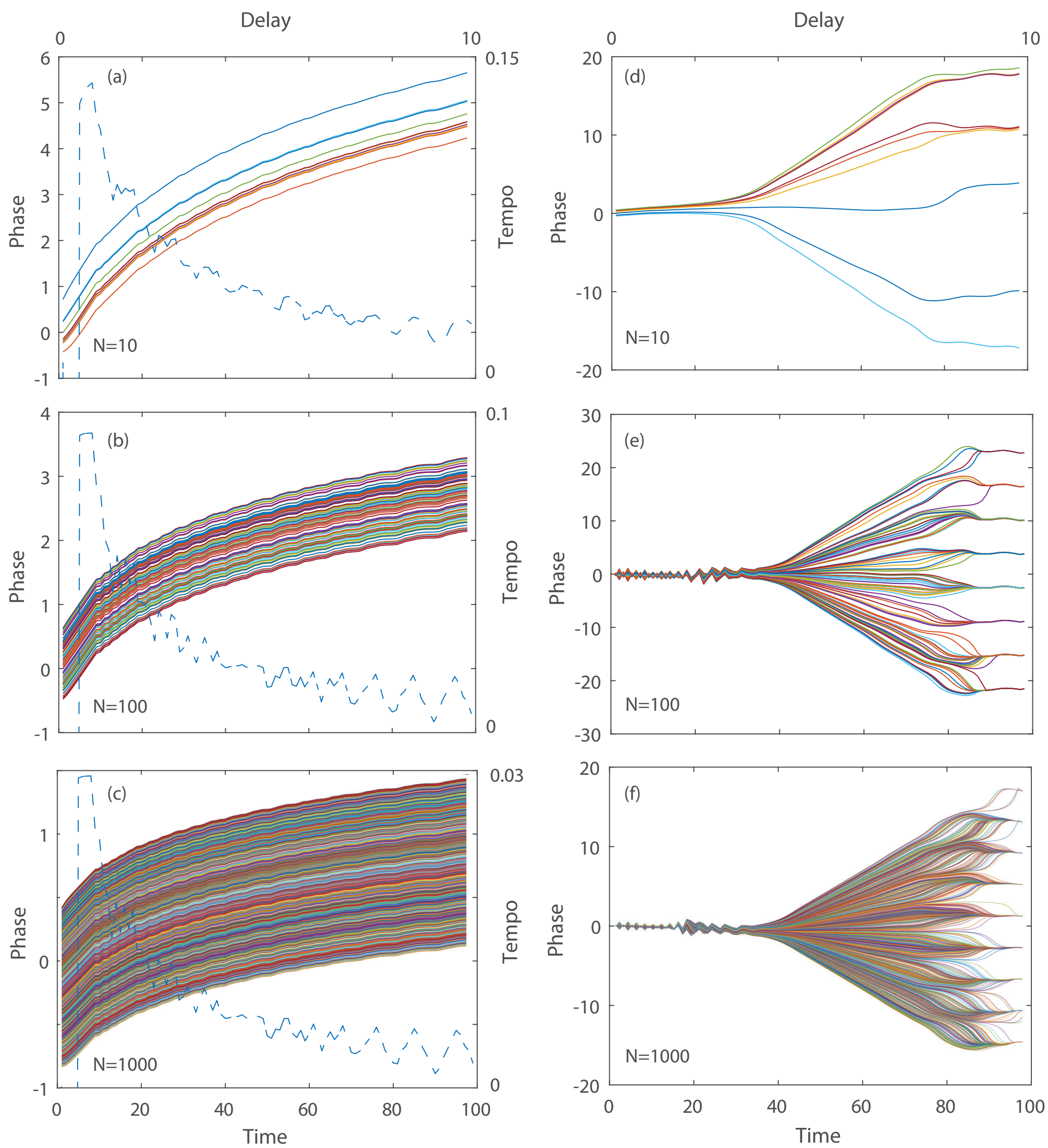
Supplementary Figure 2. Coupled violin players situated on a ring with unidirectional coupling showing how the tempo slows down until reaching a state of oscillation death where all the players are playing the same note indefinitely.



Supplementary Figure 3. Coupled violin players situated on a ring with unidirectional coupling showing how one of the players stopped playing due to amplitude death.



Supplementary Figure 4. Coupled violin players in a square lattice with periodic boundary conditions. (a) Schematics of the lattice showing the 16 violin players and their coupled neighbors. (b) The phase of all the players as a function of time as we increase the delay between the players. (c) The average tempo of the players as a function of time, showing the slowing down of the playing tempo due to the delay. (d) and (e) Representative phase as a function of time of four players emphasized by the ellipse. These results show the spreading of the phase similar to the one shown in a one-dimensional ring.



Supplementary Figure 5. Representative numerical calculations of the phase dynamics in random networks with N=10, 100, and 1000 players. We observe the two main dynamics, namely, the slowing down of the tempo in (a)-(c) and the spreading of the phase in (d)-(e). Solid curves denote the phase of each player as a function of time, and the dashed curves in (a)-(c) denote the average tempo of the players.