Supporting Text

A. Stability Analysis of System 2

In this *Appendix*, we study the stability of the equilibria of system **2**. If we redefine the system as $(0,0)^T$ when $V^+ = V^- = 0$, then there are at most three equilibria:

$E_0 = (0,0)$, $E_F = (\kappa_F \beta_-, 0)$,

$$E^* = \left(\frac{\gamma\kappa_F}{\gamma+\sigma}\left(\beta_- + \sigma - \frac{\tau\sigma}{\gamma+\sigma}\right), \frac{\sigma\kappa_F}{\gamma+\sigma}\left(\beta_- + \sigma - \frac{\tau\sigma}{\gamma+\sigma}\right)\right),$$

where E^* exists if $\sigma > 0$. To determine the properties of these three equilibria and the topological structures, set $x = V^- + V^+$, $y = V^+$. Recall $\sigma = \tau - \gamma + \beta_+ - \beta_-$, we have

$$\begin{cases} \dfrac{dx}{dt} = \beta_- x + (\beta_+ - \beta_-)y - \dfrac{x^2}{\kappa_F}, \\ \dfrac{dy}{dt} = \left(\beta_+ + \sigma - \dfrac{\tau y}{x} - \dfrac{x}{\kappa_F}\right)y. \end{cases} \quad \textbf{[A1]}$$

Notice that $E_0 = (0,0)$, $E_F = (\kappa_F \beta_-, 0)$ are still equilibria of system **A1,** and the third equilibrium E^* becomes

$$\overline{E}^* = (\overline{x}, \overline{y}) = \left(K_F\left(\beta_- + \sigma - \frac{\tau\sigma}{\gamma+\sigma}\right), \frac{\sigma\kappa_F}{\gamma+\sigma}\left(\beta_- + \sigma - \frac{\tau\sigma}{\gamma+\sigma}\right)\right).$$

We first discuss the stability of E_F and E^*.

Proposition A1. The semitrivial equilibrium $E_F = (\kappa_F \beta_-, 0)$ is stable if $\sigma < 0$ and unstable if $\sigma > 0$.

Proof. The variational matrix of system **A1** at E_F takes the form

$$J_K = \begin{pmatrix} -\beta_- & \beta_+ - \beta_- \\ 0 & \sigma \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = -\beta_- < 0$ and $\lambda_2 = \sigma$. Thus, E_F is stable if σ <0 and unstable if σ >0.

Proposition A2. The positive equilibrium $\overline{E}^* = (\overline{x}, \overline{y})$ is locally asymptotically stable if $\sigma > 0$.

Proof. The variational matrix of system **A1** at $\overline{E}^*$ takes the form

$$\overline{J} = \begin{pmatrix} \beta_- - \dfrac{2\overline{x}}{K_F} & \beta_+ - \beta_- \\ \dfrac{\tau \overline{y}^2}{\overline{x}^2} - \dfrac{\overline{y}}{K_F} & -\dfrac{\tau \overline{y}}{\overline{x}} \end{pmatrix}.$$

We then have

$$\det(\bar{J}) = -\frac{\tau\bar{y}}{\bar{x}}\left(\beta_- - \frac{2\bar{x}}{K_F}\right) - (\beta_+ - \beta_-)\left(\frac{\tau\bar{y}^2}{\bar{x}^2} - \frac{\bar{y}}{K_F}\right)$$

$$= -\frac{\tau\beta_-\bar{y}}{\bar{x}} + \frac{2\tau\bar{y}}{K_F} - (\beta_+ - \beta_-)\left(\frac{\tau\bar{y}^2}{\bar{x}^2} - \frac{\bar{y}}{K_F}\right)$$

$$= -\frac{\tau\sigma\beta_-}{\gamma+\sigma} + \frac{2\tau\sigma}{\gamma+\sigma}\left(\beta_- + \sigma - \frac{\tau\sigma}{\gamma+\sigma}\right) - (\beta_+ - \beta_-)\left[\frac{\tau\sigma^2}{(\gamma+\sigma)^2} - \frac{\sigma}{\gamma+\sigma}\left(\beta_- + \sigma - \frac{\tau\sigma}{\gamma+\sigma}\right)\right]$$

$$= \frac{\tau\sigma}{\gamma+\sigma}\left(\beta_- + 2\sigma - \frac{2\tau\sigma}{\gamma+\sigma}\right) + (\beta_+ - \beta_-)\frac{\sigma}{\gamma+\sigma}\left(\beta_- + \sigma - \frac{2\tau\sigma}{\gamma+\sigma}\right)$$

$$= \frac{\tau\sigma}{\gamma+\sigma}\left(\beta_- + 2\sigma - \frac{2\tau\sigma}{\gamma+\sigma}\right) + (\gamma + \sigma - \tau)\frac{\sigma}{\gamma+\sigma}\left(\beta_- + \sigma - \frac{2\tau\sigma}{\gamma+\sigma}\right)$$

$$= \frac{\tau\sigma}{\gamma+\sigma}\left(\beta_- + 2\sigma - \frac{2\tau\sigma}{\gamma+\sigma}\right) + \sigma\left(\beta_- + \sigma - \frac{2\tau\sigma}{\gamma+\sigma}\right) - \frac{\tau\sigma}{\gamma+\sigma}\left(\beta_- + \sigma - \frac{2\tau\sigma}{\gamma+\sigma}\right)$$

$$= \frac{\tau\sigma^2}{\gamma+\sigma} + \sigma\left(\beta_- + \sigma - \frac{2\tau\sigma}{\gamma+\sigma}\right)$$

$$= \sigma\left(\beta_- + \sigma - \frac{\tau\sigma}{\gamma+\sigma}\right) > 0$$

and

$$tr(\bar{J}) = -\frac{\tau\bar{y}}{\bar{x}} - \frac{2\bar{x}}{K_F} + \beta_+ = -\frac{\tau\sigma}{\gamma+\sigma} - 2\left(\beta_- + \sigma - \frac{\tau\sigma}{\gamma+\sigma}\right) + \beta_-$$

$$= -\left(\beta_- + \sigma - \frac{\tau\sigma}{\gamma+\sigma}\right) - \sigma < 0.$$

Therefore, $\bar{E}^*$ is locally asymptotically stable.

$E_0 = (0,0)$ is a degenerate equilibrium of system **A1**. To determine its property, rescale the age variable by $da = xdt$, system **A1** becomes

$$\begin{cases} \dfrac{dx}{dt} = (\beta_+ - \beta_-)xy + \beta_- x^2 - \dfrac{x^3}{\kappa_F} = X_2(x,y) + \phi(x,y), \\ \dfrac{dy}{dt} = (\beta_+ + \sigma)xy - \tau\ y^2 - \dfrac{x^2 y}{\kappa_F} = Y_2(x,y) + \psi(x,y), \end{cases} \qquad \textbf{[A2]}$$

where $X_2(x,y)$ and $Y_2(x,y)$ are homogeneous polynomials in *x* and *y* of degree 2, $\phi(x,y)$ and $\psi(x,y)$ are polynomials in *x* and *y* of higher degrees.

Let $x = r\cos\theta, y = r\sin\theta, 0 \le \theta \le \pi/2.$ Define

$$G(\theta) = \cos\theta \cdot Y_2(\cos\theta, \sin\theta) - \sin\theta \cdot X_2(\cos\theta, \sin\theta).$$

The characteristic equation is as follows:

$$G(\theta) = r^2 \cos\theta \sin\theta \left[(\sigma + \beta_+ - \beta_-)\cos\theta - \frac{\sigma + \beta_+ - \beta_-}{\gamma + \sigma} \sin\theta \right] = 0. \qquad \textbf{[A3]}$$

We can see that either $G(\theta) = 0$ has a finite number of real roots $\theta_k (k = 1,2,...,n)$ or $G(\theta) \equiv 0.$ By the results in Zhang *et al.* (1), no orbit of system **A2** can tend to (0,0) spirally. If $G(\theta) \equiv 0$, then it is singular. If $G(\theta)$ is not identically zero, there are at most 2(2 + 1) = 5 directions $\theta = \theta_i$ along which an orbit of system **A2** may approach the origin; these directions $\theta = \theta_i$ are given by solutions of the characteristic Eq. **A3**. If the orbit of system **A1** tends to the origin as a sequence, $\{t_n\}$ tends to $+\infty$ or $-\infty$ along a direction $\theta = \theta_i$, then the direction is called a *characteristic direction*. The orbits of system **A2,** which approach the origin along characteristic directions, divide a neighborhood of the origin into a finite number of open regions, called sectors. For an analytic system, there are three types of sectors: hyperbolic, parabolic, and elliptic [see Perko (2)].

Solving for θ in Eq. **A3** for $0 \le \theta \le \pi/2$, we obtain three simple roots:

$$\theta_1 = 0, \theta_2 = \pi/2, \theta_3 = \arctan\frac{\sigma + \beta_+ - \beta_-}{\gamma + \sigma}.$$

Proposition A3. There exist $\varepsilon_1 > 0$ and $r_1 > 0$ such that there is a unique orbit of system **A2** in $\{(r,\theta) : 0 < r < r_1, 0 \le |\theta - \theta_3| < \varepsilon_1\}$ that tends to the trivial equilibrium $E_0 = (0,0)$ along $\theta = \theta_3$ as $t \to -\infty$.

Proof. Making the Briot-Bouquet transformation

$$x = x, y = zx, ds = xdt,$$

we transform system **A2** into the following system:

$$\begin{cases} \dfrac{dx}{ds} = \beta_- x + (\beta_+ - \beta_-)xz - \dfrac{x^2}{\kappa_F}, \\ \dfrac{dz}{ds} = \sigma\ z - (\gamma + \sigma)z^2. \end{cases} \quad [\mathbf{A4}]$$

The Briot-Bouquet transformation maps the first, second, third, and fourth quadrants in the (*x*, *y*) plane into the first, third, second, and fourth quadrants in the (*x*, *z*) plane, respectively. The inverse Briot-Bouquet transformation maps the *z* axis in the (*x*, *z*) plane to the point (0,0) in the (*x*, *y*) plane, and maps the orbits in the left of the *z* axis in the (*x*, *z*) plane to orbits in the left of the *y* axis in the (*x*, *y*) plane with reversed directions. Thus, we consider only equilibria of system **A4** in the *z* axis. There are two equilibria: (0,0) and $(0, \sigma/(\gamma + \sigma))$. Linear analysis shows that (0,0) is an unstable node, and $(0, \sigma/(\gamma + \sigma))$ is an unstable saddle. Thus, there is a unique separatrix in the interior of the first quadrant of system **A4**, which tends to $(0, \sigma/(\gamma + \sigma))$ as $t \to -\infty$.

By the inverse Briot-Bouquet transformation, there exist $\varepsilon_1 > 0$ and $r_1 > 0$, such that there is a unique orbit of system **A2** in $\{(r,\theta): 0 < r < r_1, 0 \leq |\theta - \theta_3| < \varepsilon_1\}$ that tends to (0,0) along $\theta = \theta_3$ as $t \to -\infty$.

Proposition A4. System **A2** does not have limit cycles.

Proof. Denote the functions on the right-hand side in system **A2** by $f(x, y)$ and $g(x, y)$. Choose a Dulac function $D(x,y) = \frac{1}{xy}$ and notice that $x, y > 0$. We have

$$\frac{\partial(D(x,y)f(x,y))}{\partial y} + \frac{\partial(D(x,y)g(x,y))}{\partial x} = -\left(\frac{\beta_{-}x}{y^2} + \frac{x^2}{K_F y^2} + \frac{\tau y}{x^2} + \frac{1}{K_F}\right) < 0.$$

This implies that system **A2** does not have a limit cycle.

Combining Propositions **A1-A4** and using Poincaré-Bendixson's Theorem [see Perko (2)], we have the following theorem about the dynamical behavior of solutions to systems **A1** and thus system **2**.

Theorem A5. *For the original system **2**, the topological structure of the trivial equilibrium E_0 in the interior of the first quadrant consists of a parabolic sector and a hyperbolic sector; the semitrivial equilibrium E_F is stable if σ <0 and unstable if σ >0; the positive equilibrium E^* is a global attractor if σ >0.*

B. Steady States for System 3

To find the steady states of system **3**, consider

$$\lambda - \nu \bar{S} - \eta \left[\Phi_{V_F}(\bar{i}_N) + \Phi_{V^-}(\bar{i}_R) + \Phi_{V^+}(\bar{i}_R) \right] \bar{S} = 0,$$
$$\frac{d}{da}\bar{i}_N(a) = -\mu_N(a)\bar{i}_N(a),$$
$$\frac{d}{da}\bar{i}_R(a) = -\mu_R(a)\bar{i}_R(a).$$
[**B1**]

From the first equation in **B1**, we have

$$\bar{S} = \frac{\lambda}{\nu + \eta \overline{\Phi}}, \ \overline{\Phi} = \left[\Phi_{V_F}(\bar{i}_N) + \Phi_{V^-}(\bar{i}_R) + \Phi_{V^+}(\bar{i}_R) \right].$$ [**B2**]

From the second and third equations in **B1**, we have

$$\bar{i}_N(a) = \exp\left[-\int_0^\infty \mu_N(\hat{a}) d\hat{a} \right] \bar{i}_N(0) = \exp\left[-\int_0^\infty \mu_N(\hat{a}) d\hat{a} \right] \eta \left[\Phi_{V_F}(\bar{i}_N) + \Phi_{V^-}(\bar{i}_R) \right]$$
$$\bar{i}_R(a) = \exp\left[-\int_0^\infty \mu_R(\hat{a}) d\hat{a} \right] \bar{i}_R(0) = \exp\left[-\int_0^\infty \mu_R(\hat{a}) d\hat{a} \right] \eta \left[\Phi_{V^+}(\bar{i}_R) \right]$$

Using the definitions of T_F, T_{V^-} and T_{V^+}, we have

$$\bar{i}_N(0) = \eta \left[\int_0^\infty (V_F(a)\bar{i}_N(0) + V^-(a)\bar{i}_R(0)) da \right] \exp\left[-\int_0^\infty \mu_N(\hat{a}) d\hat{a} \right] \bar{S}$$
$$= \eta \left[\bar{i}_N(0) T_F + \bar{i}_R(0) T_{V^-} \right] \bar{S}$$
[**B3**]

and

$$\bar{i}_R(0) = \eta \left[\int_0^\infty V^+(a)\bar{i}_R(0) da \right] \exp\left[-\int_0^\infty \mu_R(\hat{a}) d\hat{a} \right] \bar{S} = \eta \left[\bar{i}_R(0) T_{V^+} \right] \bar{S}$$ [**B4**]

From **B2**, we have $\overline{\Phi} = (\lambda - \nu \bar{S})/(\eta \bar{S})$. **B3** and **B4** imply that $\overline{i_N}(0) + \overline{i_R}(0) = \eta \overline{\Phi} \overline{S}$. We then have the following claim:

Claim B1. If $\overline{i_N}(0) + \overline{i_R}(0) \neq 0$, then $\overline{i_N}(0) + \overline{i_R}(0) = \lambda - \mu \overline{S}$.

Now **B4** implies that

$$\overline{S} = \frac{1}{\eta T_{V^+}} \text{ if } \overline{i_R}(0) \neq 0. \qquad \textbf{[B5]}$$

Substituting **B5** into **B3**, we have $\bar{i}_N(0) = \left[\bar{i}_N(0)\, T_F + \bar{i}_R(0)\, T_{V^-}\right] / T_{V^+}$, which implies that

$$\frac{\overline{i_R}(0)}{\overline{i_N}(0)} = \frac{T_{V^+} - T_F}{T_{V^-}} \text{ if } \overline{i_N}(0) \neq 0. \qquad \textbf{[B6]}$$

We now prove the second claim.

Claim B2. A necessary condition for $\overline{i_N}(0) > 0$ is $T_{V^+} > T_F$.

Because

$$\bar{i}_N(0) = \bar{i}_R(0)\left(\frac{T_{V^-}}{T_{V^+} - T_F}\right) = (\lambda - \nu \overline{S} - \bar{i}_N(0))\left(\frac{T_{V^-}}{T_{V^+} - T_F}\right),$$

we have

$$\bar{i}_N(0) = \left(\lambda - \frac{\nu}{\eta T_F}\right)\left(\frac{T_{V^-}}{T_{V^+} + T_{V^-} - T_F}\right), \qquad \textbf{[B7]}$$

and

$$\bar{i}_R(0) = \left(\lambda - \frac{\nu}{\eta T_F}\right)\left(\frac{T_{V^+} - T_F}{T_{V^+} + T_{V^-} - T_F}\right). \qquad \text{[B8]}$$

Claim B3. A necessary condition for $\overline{i_N}(0) > 0$ and $\overline{i_R}(0) > 0$ is $\lambda > \nu/(\eta T_F)$.

For the existence of steady states, we have three cases.

(a) If $R_0 = (\lambda\eta/\nu)\max(T_F, T_{V+}) < 1$, then $\overline{i_N}(0) = 0$. **B3** thus implies $\overline{i_R}(0) = 0$. The equilibrium is given by

$$\bar{S} = \lambda/\mu, \overline{I_N} = 0, \overline{I_R} = 0.$$

(b) If $R_0 = (\lambda\eta/\nu)\max(T_F, T_{V+}) = (\lambda\eta/\nu)T_F > 1$, then $\overline{i_N}(0) > 0$ and $\overline{i_R}(0) = 0$. **B3** and **B4** give the steady-state values

$$\bar{S} = 1/(\eta T_{V+}), \bar{I}_N = \left(\lambda - \frac{\nu}{\eta T_F}\right)\exp\left[-\int_0^\infty \mu_N(\hat{a})d\hat{a}\right], \bar{I}_R = 0.$$

(c) If ($R_0 = (\lambda\eta/\nu)\max(T_F, T_{V+}) = (\lambda\eta/\nu)T_{V+} > 1$, then $\overline{i_N}(0) > 0$ and $\overline{i_R}(0) > 0$. **B5**, **B7,** and **B8** yield the following steady-state values:

$$\bar{S} = 1/(\eta T_{V+}), \bar{I}_N = \left(\lambda - \frac{\nu}{\eta T_F}\right)\left(\frac{T_{V^-}}{T_{V^+} + T_{V^-} - T_F}\right)\exp\left[-\int_0^\infty \mu_N(\hat{a})d\hat{a}\right],$$
$$\bar{I}_R = \left(\lambda - \frac{\nu}{\eta T_F}\right)\left(\frac{T_{V^+} - T_F}{T_{V^+} + T_{V^-} - T_F}\right)\exp\left[-\int_0^\infty \mu_R(\hat{a})d\hat{a}\right].$$

1. Zhang, Z., Ding, T., Huang, W. & Dong, Z. (1991) *Qualitative Theory of Differential Equations* (Transl. Math. Monogr. 101, Am. Math. Soc., Providence, RI).

2. Perko, L. (1996) *Differential Equations and Dynamical Systems* (Springer, New York).