

Supporting Text

A Strictly Scale-Invariant Discrete Model. In dealing with our first $q \neq 1$ *discrete* example, we start with two equal and distinguishable binary subsystems A and B ($N = 2$). The associated joint probabilities are, with all generality, indicated in Fig. 9, where κ is the

correlation* between A and B . Let us now impose (1, 2) additivity of S_q .[†] In other words, we choose $\kappa(p)$ such that $S_q(2) = 2S_q(1)$, where (for $W = 2$) $S_q(1) = 1 - p^q - (1 - p)^q/q - 1$, and (for $W = 4$) $S_q(2) = 1 - (p^2 + \kappa)^q - 2[p(1 - p) - \kappa]^q - [(1 - p)^2 + \kappa]^q/q - 1$. We focus on the solutions $\kappa_q(p)$ for $0 \leq q \leq 1$ indicated in Fig. 10.[‡]

With the convenient notation

$$\begin{aligned}\pi_{10} &\equiv r_{10} \equiv p_1^A = p \\ \pi_{11} &\equiv r_{01} \equiv p_2^A = (1 - p) \\ \pi_{20} &\equiv r_{20} \equiv p_{11}^{A+B} = p^2 + \kappa \\ \pi_{21} &\equiv r_{11} \equiv p_{12}^{A+B} = p_{21}^{A+B} = p(1 - p) - \kappa \\ \pi_{22} &\equiv r_{02} \equiv p_{22}^{A+B} = (1 - p)^2 + \kappa,\end{aligned}\tag{19}$$

we can verify

$$\begin{aligned}r_{20} + 2r_{11} + r_{02} &= 1, \\ r_{20} + r_{11} &= r_{10} = p, \\ r_{11} + r_{02} &= r_{01} = 1 - p.\end{aligned}\tag{20}$$

Let us now address the case of three equal and distinguishable binary subsystems, A , B , and C ($N = 3$). We present in Fig. 11 probabilities that are *not* the most general ones, but rather general ones for which we have *strict scale invariance*, in the sense that *all* the associated marginal probability sets exactly reproduce the above $N = 2$ case. Notice how strongly this construction reminds us of the one that occurs in the renormalization group procedures widely used in quantum field theory, the study of critical phenomena, and elsewhere (6–9).

With the convenient notation $\pi_{30} \equiv r_{30} \equiv p_{111}^{A+B+C}$; $\pi_{31} \equiv r_{21} \equiv p_{112}^{A+B+C} = p_{121}^{A+B+C} = p_{211}^{A+B+C} = p_{211}^{A+B+C}$; $\pi_{32} \equiv r_{12} \equiv p_{221}^{A+B+C} = p_{212}^{A+B+C} = p_{122}^{A+B+C}$; $\pi_{33} \equiv r_{03} \equiv p_{222}^{A+B+C}$, and so on, we can verify

$$\begin{aligned}r_{30} + 3r_{21} + 3r_{12} + r_{03} &= 1, \\ r_{30} + r_{21} &= r_{20} = p^2 + \kappa_q(p), \\ r_{21} + r_{12} &= r_{11} = p(1 - p) - \kappa_q(p), \\ r_{12} + r_{03} &= r_{02} = (1 - p)^2 + \kappa_q(p),\end{aligned}\tag{21}$$

and so on.

Let us complete this example by considering the generic case (arbitrary N). The results are presented in Fig. 12, where we have merged the Pascal triangle and the present Leibnitz-like triangle (10). For the *left* elements, we have the usual Pascal rule, i.e., every element of the N th line equals the sum of its “north-west” plus its “north-east” elements. For the *right* elements we have the property that every element of the N th line equals the sum of its “south-west” plus its “south-east” elements. In other words, for ($N = 1, 2, 3, \dots$; $n = 0, 1, 2, \dots, N$), we have that $r_{N-n,n} + r_{N-n-1,n+1} = r_{N-n-1,n}$, and also that $\sum_{n=0}^N N!/(N-n)!n! r_{N-n,n} = 1$ ($N = 0, 1, 2, \dots$). These two equations admit the following solution:

*Assuming that the states 1 and 2 of subsystems A and B correspond to the values a_1 and a_2 of the random variable, we have that the *covariance* equals $(a_1 - a_2)\kappa$, and the *correlation coefficient* equals $\kappa/[p(1 - p)]$.

[†]As previously mentioned, it is as a simple illustration that we imposed $S_q(2) = 2S_q(1)$ instead of say $S_{2-q}(2) = 2S_{2-q}(1)$. The results would then obviously be the same with $(1 - q) \leftrightarrow (q - 1)$. Consequently, we would have additivity for $1 \leq q \leq 2$, instead of $0 \leq q \leq 1$.

[‡]The $(1 - q) \leftrightarrow (q - 1)$ “duality” appears naturally in nonextensive statistical mechanics (see, for instance, refs. 3 and 4). J. Marsh and S. Earl (see ref. 5) noticed and kindly communicated to us that, for the present κ -model, there were also $\kappa > 0$ solutions, and also that the additivity of the $q \neq 1$ entropy $S_q(N)$ was limited to values of N that only achieved infinity for $p = 1$.

$$\begin{aligned}
r_{N,0} &= p^N + \kappa_q(p) \frac{[N(1-p) + (p^N - 1)]}{(1-p)^2}, \\
r_{N-1,1} &= p^{N-1}(1-p) - \kappa_q(p) \frac{1-p^{N-1}}{1-p}, \\
r_{N-n,n} &= p^{N-n}(1-p)^n \left[1 + \frac{\kappa_q(p)}{(1-p)^2} \right] (2 \leq n \leq N).
\end{aligned} \tag{22}$$

Summarizing, as long as $r_{N,0} \geq 0$, this interesting structure takes automatically into account (i) the standard constraints of the theory of probabilities (nonnegativity and normalization of probabilities), and (ii) the scale-invariant structure which guarantees that *all the possible sets of marginal probabilities derived from the joint probabilities of N subsystems reproduce the corresponding sets of joint probabilities of $N - 1$ subsystems*. Consistently S_q is strictly additive for all $N \leq N_{max}$, where N_{max} depends on (p, q) . In this way, the correlation $\kappa_q(p)$ that we introduced between two subsystems will itself be preserved for all $N \leq N_{max}$.

Let us now address the following question: how *deformed*, and in what manner, is the occupation of the phase space (N -dimensional “hypercube”, in the same sense that the $N = 2$ phase space may be seen as a “square”, and the $N = 3$ one as a “cube”) in the presence of the scale-invariant correlation $\kappa_q(p)$ determined once and for all (see Fig. 10)? The most natural comparison is with the case of independence (which corresponds to $\kappa = 0$, hence to $q = 1$). It is then convenient to define the *relative discrepancy* $\eta_{N-n,n} \equiv \{r_{N-n,n}/$

$[p^{N-n} (1-p)^n] - 1$ (naturally, other definitions for *discrepancy* can be used as well, but the present one is particularly simple). Since $n = 0, 1, 2, \dots, N$, we may expect in principle to have $N + 1$ *different* discrepancies. *It is not so!* Quite remarkably there are only *three* different ones, namely $\eta_{N,0}$, $\eta_{N-1,1}$, and all the others, which therefore coincide with $\eta_{0,N}$. They are given by

$$\begin{aligned}\eta_{N,0} &= \frac{\kappa_q(p)}{(1-p)^2} \left[1 + \frac{N(1-p) - 1}{p^N} \right] \leq 0, \\ \eta_{N-1,1} &= \frac{\kappa_q(p)}{(1-p)^2} \left(1 - \frac{1}{p^{N-1}} \right) \geq 0, \text{ and} \\ \eta_{N-n,n} &= \frac{\kappa_q(p)}{(1-p)^2} \leq 0 \quad (2 \leq n \leq N),\end{aligned}\tag{23}$$

where the inequalities hold for $0 \leq q < 1$, for which $\kappa_q(p) \leq 0$. Of course, the equalities in Eq. 23 correspond to $q = 1$ (i.e., $\kappa = 0$) (see Fig. 13). We see that, for arbitrary $N \geq 2$, only three different types of vertices emerge in the N -dimensional hypercube. These can be characterized by the $(1, 1, \dots, 1)$ corner, the N sites along each cartesian axis emerging from this corner, and all the others. As N increases, the middle type predominates more and more, with increasingly uneven occupation of phase space.

The present example corresponds to $\pi_{N,0} = r_{N,0}$ as given in Eq. 22. It is important to notice in this case that, for fixed (p, q) such that $p < 1$ and $q < 1$, there is a maximal value of N , noted $N_{max}(p, q)$, for which the analytical expression for $r_{N,0}$ in Eq. 22 is nonnegative. For $N > N_{max}$, we are obliged to consider $r_{N,0} = 0$, which, through application of the Leibnitz rule, leads to violations of the nonnegativity of *all* $r_{N-n,n}$. When this happens, of course the additivity of the entropy, i.e., $S_q(N) = NS_q(1)$, does not hold any more. Unless we have the trivial situation $q = 1$ (for which entropic additivity holds for all $0 \leq p \leq 1$), the thermodynamic limit $N \rightarrow \infty$ imposes $p = 1$ for $0 \leq q < 1$. Indeed $N_{max}(1, q) \rightarrow \infty \forall q \in [0, 1]$. For all other values of $p < 1$ and $q < 1$, $N_{max}(p, q)$ is finite.

Continuous Model

Let us now address our last example, namely a *continuous* model. It is known that classical mechanics violates the *3rd principle of thermodynamics*, whereas quantum mechanics conforms to it. Indeed, in the latter we typically have $\lim_{T \rightarrow 0} \lim_{N \rightarrow \infty} S(N, T)/N = 0$ (T being the absolute temperature), whereas in the former such a limit is typically negative, and can even diverge to $-\infty$. Consistently, the present continuous model is going to have, as we shall see, difficulties of the same type. This, however, does not affect its scaling properties with N , which constitutes the central scope of the present paper. We shall therefore dedicate some effort to explore such continuous cases. We consider the following probability distribution.

$$p(x) = \frac{2}{\sqrt{\pi}(2+a)} e^{-x^2}(1+ax^2)(a \geq 0). \quad [24]$$

We can verify that $\int_{-\infty}^{\infty} dx p(x) = 1$. This distribution is illustrated in Fig. 14.

The entropy corresponding to one subsystem (i.e., $N = 1$) is given by

$$\begin{aligned}
 S_q(1) &= \frac{1 - \int_{-\infty}^{\infty} dx [p(x)]^q}{q - 1} \\
 &= \frac{1 - \left[\frac{2}{\sqrt{\pi}(2+a)} \right]^q \int_{-\infty}^{\infty} dx e^{-q(x^2+y^2)} (1+ax^2)^q}{q - 1} \\
 &= \frac{1 - \frac{1}{\sqrt{q}} \left[\frac{2}{\sqrt{\pi}(2+a)} \right]^q I(a, q)}{q - 1} \quad [25]
 \end{aligned}$$

with (11)

$$\begin{aligned}
 I(a, q) &\equiv \int_{-\infty}^{\infty} dz e^{-z^2} \left(1 + \frac{a}{q} z^2 \right)^q \\
 &= \frac{\sqrt{\pi q} \Gamma(-\frac{1}{2} - q) {}_1F_1\left(\frac{1}{2}, \frac{3}{2} + q, \frac{q}{a}\right)}{\sqrt{a} \Gamma(-q)} \\
 &\quad + \left(\frac{a}{q}\right)^q \Gamma\left(\frac{1}{2} + q\right) {}_1F_1\left(-q, \frac{1}{2} - q, \frac{q}{a}\right), \quad [26]
 \end{aligned}$$

Γ and ${}_1F_1$ being respectively the Riemann's Γ and the hypergeometric functions. The a -dependence of S_q for typical values of q is depicted in Fig. 15. As expected for continuous distributions, negative values for S_q do emerge.

Let us now compose two such subsystems. If they are independent ($q = 1$) we have

$$P_1(x, y) = p(x)p(y) = \frac{4}{\pi(2+a)^2} e^{-(x^2+y^2)} [1 + a(x^2 + y^2) + a^2x^2y^2] \quad [27]$$

Of course, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy P_1(x, y) = 1$. For the general case, we propose the following simple generalization of $p(x)p(y)$:

$$P_q(x, y) = \frac{4}{\pi(4 + 4A + B)} e^{-(x^2+y^2)} [1 + A(x^2 + y^2) + Bx^2y^2], \quad [28]$$

which satisfies $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy P_q(x, y) = 1$. Of course, for $q = 1$, we expect $(A, B) = (a, a^2)$. Let us now calculate the marginal probability, i.e.,

$$\int_{-\infty}^{\infty} dy P_q(x, y) = \frac{2(2+A)e^{-x^2}}{\sqrt{\pi}(4+4A+B)} \left[1 + \frac{2A+B}{2+A} x^2 \right]. \quad [29]$$

We want this marginal probability to *recover* the original $p(x)$, so we impose $(2A+B)/(2+A) = a$, which implies $B = aA + 2(a-A)$ and $\int_{-\infty}^{\infty} dy P_q(x, y) = p(x)$. It follows that

$$P_q(x, y) = \frac{4}{\pi[4 + 2(a+A) + aA]} e^{-(x^2+y^2)} \cdot \{1 + A(x^2 + y^2) + [aA + 2(a-A)]x^2y^2\}. \quad [30]$$

Finally, to have A as a function of (q, a) , we impose, as for the binary case,

$$S_q(2) = 2S_q(1), \quad [31]$$

where $S_q(1)$ is given by Eq. 25 and

$$\begin{aligned} S_q(2) &= \frac{1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy [P_q(x, y)]^q}{q-1} \\ &= \frac{1 - \left[\frac{4}{\pi[4 + 2(a+A) + aA]} \right]^q \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy e^{-q(x^2+y^2)} \cdot \{1 + A(x^2 + y^2) + [aA + 2(a-A)]x^2y^2\}^q}{q-1} \\ &= \frac{1 - \frac{1}{q} \left[\frac{4}{\pi[4 + 2(a+A) + aA]} \right]^q J(a, A, q)}{q-1} \quad [32] \end{aligned}$$

with (11)

$$\begin{aligned}
J(a, A, q) &\equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du dv e^{-(u^2+v^2)} \\
&\cdot \left[1 + \frac{A}{q}(u^2 + v^2) + \frac{aA + 2(a - A)}{q^2} u^2 v^2 \right]^q \\
&= \frac{1}{\Gamma(-q)} \int_{-\infty}^{\infty} dz \sqrt{\frac{1 + (A/q)z^2}{(A/q) + [(aA + 2(a - A))/q^2]z^2}} e^{-z^2} \\
&\cdot (1 + (A/q)z^2)^q \\
&\times \left[\sqrt{\pi} \Gamma\left(\frac{1}{2} - q\right) \right]_1 \\
&\cdot F_1\left(\frac{1}{2}, \frac{3}{2} + q, \frac{1 + (A/q)z^2}{(A/q) + [(aA + 2(a - A))/q^2]z^2}\right) \\
&+ \left(\frac{(A/q) + [(aA + 2(a - A))/q^2]z^2}{1 + (A/q)z^2}\right)^{\frac{1}{2}+q} \\
&\Gamma(-q) \Gamma\left(\frac{1}{2} + q\right) \\
&\times {}_1F_1\left(-q, \frac{1}{2} - q, \frac{1 + (A/q)z^2}{(A/q) + (aA + 2(a - A))/q^2} z^2\right).
\end{aligned} \tag{33}$$

See in Fig. 16 the a -dependence of A for typical values of q . Finally, the *relative discrepancy*

$$\eta(x, y) \equiv \frac{P_q(x, y)}{P_1(x, y)} - 1 \tag{34}$$

is illustrated in Fig. 17 for a typical set (a, q) . For higher values of N we follow here a procedure similar to the one in our discrete example *SSF* of Fig. 3. Let us address the $N = 3$ case. For the case of independence, we have

$$\begin{aligned}
P_1(x, y, z) &= p(x)p(y)p(z) \propto e^{-(x^2+y^2+z^2)} \\
&\cdot [1 + a(x^2 + y^2 + z^2) + a^2(x^2y^2 + y^2z^2 + z^2x^2) + a^3x^2y^2z^2]. \tag{35}
\end{aligned}$$

We consistently assume

$$\begin{aligned}
P_q(x, y, z) &= \frac{8}{\pi^{3/2}(8 + 12A_3 + 6B_3 + C_3)} e^{-(x^2+y^2+z^2)} \\
&\cdot [1 + A_3(x^2 + y^2 + z^2) + B_3(x^2y^2 + y^2z^2 + z^2x^2) \\
&+ C_3x^2y^2z^2], \tag{36}
\end{aligned}$$

which satisfies $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy dz P_q(x, y, z) = 1$. Clearly, for $q = 1$, $(A_3, B_3, C_3) = (a, a^2, a^3)$. For the general case, we impose that $\int_{-\infty}^{\infty} dz P_q(x, y, z) = P_q(x, y)$, i.e., the $N = 2$ distribution as given by Eq. 30. This imposition implies

$$\frac{2A_3 + B_3}{2 + A_3} = A_2 \equiv A,$$

$$\frac{2B_3 + C_3}{2 + A_3} = B_2 \equiv B, \quad [37]$$

$$\frac{2 + A_3}{8 + 12A_3 + 6B_3 + C_3} = \frac{1}{4 + 4A_2 + B_2},$$

hence

$$A_3 = \frac{4A_2 - 2B_2 + C_3}{4 - 2A_2 + B_2},$$
$$B_3 = \frac{4B_2 + (A_2 - 2)C_3}{4 - 2A_2 + B_2}. \quad [38]$$

The coefficient $C_3 > 0$ must satisfy that $C_3 = a^3$ for $q = 1$. If $S_q(3) = 3S_q(1)$ is automatically satisfied, we have some freedom for choosing C_3 . Natural choices could be $C_3 = a^3$ and $C_3 = A_3B_3$ (which automatically satisfies $C_3 = a^3$ for $q = 1$). If, however, $S_q(3) \neq 3S_q(1)$, we can impose the equality and determine a better approximation for q . The new value is expected to be only slightly different from the one that we already determined by imposing entropic additivity for $N = 2$. The procedure can in principle be iteratively repeated for increasing N . Although such a study has its own interest, it lies outside the scope of this article.

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