Supporting information for Kaplan and Robson (2002) *Proc. Natl. Acad. Sci. USA*, 10.1073/pnas.152502899

Appendix

Proof of Lemma 1: The result follows, because Q is (*i*) nonempty, (*ii*) bounded above, and (*iii*) closed.

(*i*) Choose any $\overline{K} > 0$. If $s(t) = F(\overline{K}, t, \alpha)/2$, for $t = 1,\ldots,T$, then $p(t) > 0$, for all $t = 0,\ldots,T$, and, whereas $y(0) < 0$, $y(t) > 0$, for $t = 1,\ldots,T$. If $r \in (-\infty, \infty)$ is sufficiently small, it follows that $\sum_{t=0}^{T} p(t)y(t)e^{-rt} \geq 0$, so that $r \in Q$.

(*ii*) The assumptions on C and F in the first two paragraphs of *The Setup* imply that there exist constants $B > 0$ and $C > 0$ such that $C(K) \geq BK + C$, for all $K \geq 0$, and a function $A(t) > 0$ such that $F(K, t, \alpha) \leq A(t)K$ for all $K \geq 0$, $\alpha \in A$, and $t = 1,\ldots,T$. Consider any $r \in Q$, so that $\sum_{t=0}^{T} p(t)y(t)e^{-rt} \geq 0$, for some $(p, y) = (p(s), y(K, s)) \in M$. Because $p(t) \leq 1$ and $y(t) = F(K, t, \alpha) - s(t) \leq A(t)K$, for $t = 1,\ldots,T$, it follows that $K\left(-B + \sum_{t=1}^{T} A(t)e^{-rt}\right) - C \geq 0.$ Hence $r < \bar{r}$ where $\bar{r}$ is the unique solution of $-B + \sum_{t=1}^{T} A(t)e^{-\bar{r}t} = 0$, and $\bar{r}$ is an upper bound for Q, as required.

(*iii*) Consider any sequence $r_n \to \bar{r}$ where $r_n \in Q$ for all n. It must be shown that $\bar{r} \in Q$. Suppose the sequences K_n and s_n generate r_n. Take N such that $n > N$ implies $r_n \geq \bar{r} - \Delta$, for any fixed $\Delta > 0$. Since $-C(K_n) + \sum_{t=1}^{T} F(K_n, t, \alpha)e^{-(\bar{r}-\Delta)t} \geq 0$, there can be no subsequence of $K_n \to \infty$, given the conditions on C and F. Hence there is a convergent subsequence of

$K_n \to \overline{K} \in [0,\infty)$, say. Now define N such that $n > N$ implies $r_n \in (\bar{r} - \Delta, \bar{r} + \Delta)$ and $K_n \in (\overline{K} - \Delta, \overline{K} + \Delta)$, for some $\Delta > 0$. Because $-B(\overline{K} - \Delta) - C + \sum_{t=1}^{T}\left(A(t)(\overline{K} + \Delta)e^{-(\bar{r}-\Delta)t} - s_n(t)e^{-(\bar{r}+\Delta)t}\right) \geq 0$, there can be no subsequence of the $s_n(t) \to \infty$, for any t = 1,…,T. Hence there is a convergent subsequence $s_n \to \bar{s} \in [0,\infty)^T$, say. It follows from continuity of the functions $p(s)$ and $y(K,s)$ that $\overline{K}$ and $\bar{s}$ generate $\bar{r}$, so that $\bar{r} \in Q$, as required.

Proof of Lemma 2: Consider backward induction on t for the results in the first sentence. These results clearly hold at $t = T$. Suppose then, as the induction hypothesis, that they hold at t + 1. It follows that $V(K,t,\alpha) \geq 0$ is continuous in $K \geq 0$, because $V(K,t+1,\alpha) \geq 0$ and $F(K,t,\alpha) \geq 0$ are continuous, and $V(K,t,\alpha) = \max_{s(t)}\left\{F(K,t,\alpha) - s(t) + e^{-r}\sigma(s(t))V(K,t+1,\alpha)\right\}$. Indeed, the implicit function theorem implies the optimal $s(t)$ satisfying $\sigma'(s(t))V(K,t+1,\alpha)e^{-r} = 1$ is a continuously differentiable function of $K > 0$, for any $\alpha \in A$. Using the "envelope theorem"' it follows that $V_K(K,t,\alpha) = F_K(K,t,\alpha) + e^{-r}\sigma(s(t))V_K(K,t+1,\alpha)$, t = 1,…,T - 1. That is, although $s(t)$ is a function of K, this does not affect this expression because $s(t)$ is chosen optimally. Clearly, $V_K(K,t,\alpha) > 0$ since $F_K(K,t,\alpha) > 0$ and $V_K(K,t+1,\alpha) > 0$. Further, $V_K(K,t,\alpha)$ can be differentiated to yield $V_{KK}(K,t,\alpha)$ as a continuous function of $K > 0$, completing the induction argument.

Consider now the choice of K to maximize $\bar{p}V(K,1,\alpha)e^{-r} - C(K)$. The assumptions on F and C imply that $\bar{p}V_K(K,1,\alpha)e^{-r} - C'(K) > 0$, for all small enough $K > 0$, and that $\bar{p}V_K(K,1,\alpha)e^{-r} - C'(K) < 0$ for all large enough K. Hence there must exist an optimal $K > 0$ satisfying the first- and second-order necessary conditions as stated.

*Proof of Proposition 2: T*he dependence of variables on r is noted. For any $K > 0$, the envelope theorem implies $V_r(K,t,\alpha,r) = -e^{-r}\sigma(s(t))V(K,t+1,\alpha,r) + e^{-r}\sigma(s(t))V_r(K,t+1,\alpha,r)$,

$t = 1,\ldots,T - 1$. Since $V_r(K,T,\alpha,r) = 0$, it follows that $V_r(K,t,\alpha,r) < 0$, for $t = 1,\ldots,T - 1$. Given $\overline{p}V(K^*(r^*),1,\alpha,r^*)e^{-r^*} = C(K^*(r^*))$ and $\overline{p}V_K(K^*(r),1,\alpha,r)e^{-r} = C'(K^*(r))$, it follows that $\frac{d}{dr}(\overline{p}V(K^*(r),1,\alpha,r)e^{-r^*} - C(K^*(r))) = \overline{p}V_r(K^*(r),1,\alpha,r) < 0$, so that $\overline{p}V(K^*(r),1,\alpha,r)e^{-r^*} < C(K^*(r))$, for all $r > r^*$. That is, $L(r^*,p^*,y^*) = 0$, so that the growth rate r^* is feasible, but $\max_{p,y} L(r,p,y) < 0$, for all $r > r^*$, so no growth rate strictly greater than r^* is feasible.

Proof of Lemma 3: The optimal K^* and s^* solve the following problem $\max_{K,s(1),\ldots,s(T-1)}\left[-C(K) + \sum_{t=1}^{T}\overline{p}\left(\prod_{\tau=1}^{t-1}\sigma(s(t))\right)(F(K,t,\alpha) - s(t))e^{-r^*t}\right]$. The dynamic programming approach in *Lemma 2* can be extended to prove that such $K^* > 0$ and $s^* > 0$ are continuously differentiable functions of r^* and α. Because, in addition, $\left[-C(K^*) + \sum_{t=1}^{T}\overline{p}\left(\prod_{\tau=1}^{t-1}\sigma(s^*(\tau))\right)(F(K^*,t,\alpha) - s^*(t))e^{-r^*t}\right] = 0$, the implicit function theorem then implies that the maximum growth rate, $r^*(\alpha)$, say, is a continuously differentiable function of $\alpha \in A$. However, as another example of the envelope theorem, the derivatives of K^* and s^* play no direct role here. That is, $\frac{dr^*(\overline{\alpha})}{d\alpha} = \frac{\sum_{t=1}^{T}p^*(t)F_\alpha(K^*,1,\overline{\alpha})}{\sum_{t=1}^{T}tp^*(t)\left(F(K^*,t,\overline{\alpha}) - s^*(t)\right)} = 0$, as required, given also that $\sum_{t=1}^{T}tp^*(t)\left(F(K^*,t,\overline{\alpha}) - s^*(t)\right) = \sum_{t=1}^{T}p^*(t)V(K^*,t,\overline{\alpha}) > 0$, by *Lemma 2*.

Proof of Theorem 1: Note that $r^*(\overline{\alpha}) = 0$ and $\frac{dr^*(\overline{\alpha})}{d\alpha} = 0$ throughout. Consider first:

Lemma A: (*i*) $V_\alpha(K,t,\overline{\alpha}) > 0$, for all $K > 0$, and $t = 2,\ldots,T$. (*ii*) $V_{K\alpha}(K,1,\overline{\alpha}) > 0$, for all $K > 0$.

Proof of Lemma A: (*i*) By the envelope theorem,

$V_\alpha(K,t,\overline{\alpha}) = F_\alpha(K,t,\overline{\alpha}) + \sigma(s(t))V_\alpha(K,t+1,\overline{\alpha})$, for all $K>0$, $t=1,...,T-1$. Recall that $F_\alpha(K,t,\overline{\alpha})<0$, for all $t<\bar{t}$, but $F_\alpha(K,t,\overline{\alpha})>0$, for all $t\geq\bar{t}$. Hence backwards recursion from *T* implies that $V_\alpha(K,t,\overline{\alpha})>0$, for $t=\bar{t},...,T$. Moreover, if $V_\alpha(K,t,\overline{\alpha})\leq 0$, for some $t<\bar{t}$, then $V_\alpha(K,t-1,\overline{\alpha})<0$. But, since $V_\alpha(K,1,\overline{\alpha}) = \sum_{t=1}^{T} p^*(t)F_\alpha(K,t,\overline{\alpha})/\overline{p} = 0$, it must then be that $V_\alpha(K,t,\overline{\alpha})>0$, for any $K>0$, and $t=2,...,T$.

(*ii*) Differentiating $\sigma'(s(t))V(K,t+1,\alpha)e^{-r^*}=1$ with respect to α, at $\alpha=\overline{\alpha}$, holding $K>0$ constant, yields $\dfrac{\partial s(t)}{\partial\alpha} = -\dfrac{\sigma'(s(t))V_\alpha(K,t+1,\overline{\alpha})}{\sigma''(s(t))V(K,t+1,\overline{\alpha})}>0$, for $t=1,\ldots,T-1$. The envelope theorem implies that $V_K(K,1,\alpha) = \sum_{t=1}^{T} e^{-r^*(t-1)}\sigma(s(1))...\sigma(s(t-1))F_K(K,t,\alpha)>0$, for all $K\geq 0$. Since $F_{K\alpha}(K,t,\alpha)\geq 0$, differentiation of $V_K(K,1,\alpha)$ with respect to α, at $\alpha=\overline{\alpha}$, with $K>0$ constant, then yields $V_{K\alpha}(K,1,\overline{\alpha})>0$.

Now *Lemmas 2* and *A* imply the results of *Theorem 1*:

(*i*) Because $\overline{p}V_{KK}(K^*,1,\alpha)-C''(K^*)<0$, it follows from differentiating $\overline{p}V_K(K^*,1,\alpha)e^{-r^*}=C'(K^*)$ with respect to α, at $\alpha=\overline{\alpha}$, that

$$\frac{dK^*}{d\alpha} = \frac{\overline{p}V_{K\alpha}(K^*,1,\overline{\alpha})}{C''(K^*)-\overline{p}V_{KK}(K^*,1,\overline{\alpha})}>0.$$

(*ii*) Differentiating $\sigma'(s^*(t))V(K^*,t+1,\alpha)e^{-r^*}=1$ with respect to α, at $\alpha=\overline{\alpha}$, where *K* can vary, finally yields $\dfrac{ds^*(t)}{d\alpha} = -\dfrac{\sigma'(s^*(t))\left[V_\alpha(K^*,t+1,\overline{\alpha})+V_K(K^*,t+1,\overline{\alpha})\dfrac{dK^*}{d\alpha}\right]}{\sigma''(s^*(t))V(K^*,t+1,\overline{\alpha})}>0$, for $t=1,\ldots,T-1$.