Supporting Text

In these notes we outline the main steps of the derivation of the results. Let us firstly focus on the limit $b_0 = d_0 = 0$ when the process reduces to a Galton - Watson branching process. Without loss of generality, we can set $d_1 = 1$ and $b_1 = \alpha$ in what follows. Introducing the characteristic function

$$G(x,t) = \sum_{n=0}^{\infty} P_n(t) x^n, \tag{1}$$

the birth and death equation can be transformed in a first-order p.d.e. for $G(x,t)$

$$\partial_t G(x,t) = (\alpha x^2 + 1 - (\alpha+1)x)\partial_x G(x,t). \tag{2}$$

This equation can be integrated using, for example, the characteristic method (see ref. 1). Taking as initial condition $G(x,0) = x$, which corresponds to Eq. **4** in the main text, the complete solution is

$$G(x,t) = \frac{(1-x) - (1-\alpha x)e^{(1-\alpha)t}}{\alpha(1-x) - (1-\alpha x)e^{(1-\alpha)t}}, \tag{3}$$

from which we obtain

$$P(0,t) = G(0,t) = \frac{1 - e^{(1-\alpha)t}}{\alpha - e^{(1-\alpha)t}}, \tag{4}$$

and, taking the time derivative of this, we derive Eq. **7** of the main text. It is also easy to see that in the scaling limit, i.e. for $t^* = 1/(1-\alpha) \gg 1$ and t/t^* fixed, $p(t)$ can be cast in the scaling form **8**.

In order to deal with the general case, we make a Laplace transform with respect to time of the generating function and define

$$\tilde{G}(x,s) = \int_0^{\infty} dt e^{-st} G(x,t) = \int_0^{\infty} dt e^{-st} \sum_{n=0}^{\infty} P_n(t) x^n. \tag{5}$$

Then the equation of the dynamics becomes

$$\begin{aligned} & \left[\alpha x^2 + 1 - (\alpha+1)x\right] \partial_x \tilde{G}(x,s) + \\ + \quad & \left[b_0 x + \tfrac{d_0}{x} - b_0 - d_0 - s\right] [\tilde{G}(x,s) - g_0(s)] = s g_0(s) - x, \end{aligned} \tag{6}$$

where we defined $g_0(s) = \tilde{G}(0,s)$, which is the Laplace transform of $P_0(t)$, the function we wish to compute. Defining $F(x,s) = \tilde{G}(x,s) - g_0(s)$ and using the fact that $g_0(s)$ does not depend on x, we obtain the following equation for $F(x,s)$:

$$\partial_x F(x,s) + p(x,s)F(x,s) = q(x,s), \tag{7}$$

where

$$\begin{aligned} p(x,s) &= \left[\frac{d_0}{x} - \frac{b_0 - d_0\alpha}{1-\alpha x} - \frac{s}{(1-\alpha x)(1-x)}\right] \\ q(x,s) &= \frac{s g_0(s) - x}{(1-\alpha x)(1-x)}. \end{aligned} \tag{8}$$

Eq. **7** should be solved with the boundary conditions

$$F(1,s) = \frac{1}{s} - g_0(s) \tag{9}$$

$$F(0,s) = 0. \tag{10}$$

Due to the presence of singularities at $x=0$ and $x=1$, some care must be taken when imposing these conditions on the general solution of Eq. **7**. Our strategy is that of solving Eq. **7** with a modified initial condition (Eq. **9**) at $x = 1-\epsilon$

$$F(1-\epsilon, s) = \frac{1}{s} - g_0(s). \tag{11}$$

Then we will impose condition **10** on the resulting expression, which leaves us with an equation for $g_0(s)$. Finally, we shall restore the boundary condition **9** by taking the limit $\epsilon \to 0$. Such an ϵ-"regularization" procedure allows us to circumvent the problem of dealing with the singularities at $x=1$ of Eq. **7**. Notice that, as long as $\alpha = b_1 \leq 1 = d_1$, one has $\lim_{t\to\infty} P_0(t) = 1$, i.e. the probability of being asymptotically extinct approaches 1.

The generic form of the solution of Eq. **7** with boundary condition **11** is

$$F(x,s) = e^{\int_x^{1-\epsilon} dx' p(x',s)} \left[\frac{1}{s} - g_0(s)\right] - \int_x^{1-\epsilon} dx' q(x',s) e^{\int_x^{x'} dx' p(x',s)}. \tag{12}$$

The resulting expression is rather complex and it will be considered later on. We shall first specialize to the particular case $b_0 = d_0 = r$ and $\alpha = 1$ discussed in the main text, which describes the crossover between the two power law regimes, and then the sub-critical case $\alpha < 1$.

For $b_0 = d_0 = r$ and $\alpha = 1$, the coefficients take the simpler form

$$\begin{aligned} p(x,s) &= \tfrac{r}{x} - \tfrac{s}{(1-x)^2} \\ q(x,s) &= \tfrac{s g_0(s) - x}{(1-x)^2}. \end{aligned} \tag{13}$$

Up to the leading order in ϵ, the solution is

$$F(x,s) = \frac{e^{-\frac{s}{\epsilon}}(g_0(s) - \frac{1}{s}) - \int_x^{1-\epsilon} dt \frac{sg_0(s)-t}{(1-t)^2} t^r e^{-\frac{s}{1-t}}}{x^r e^{-\frac{s}{1-x}}}. \tag{14}$$

Since the denominator diverges when $x \to 0$, in order to have $F(0,s) = 0$, we have to impose that the numerator should be equal to zero. After taking the limit $\epsilon \to 0$, this yields an equation for $g_0(s)$, which reads

$$\int_1^x dt \frac{sg_0(s) - t}{(1-t)^2} t^r e^{-\frac{s}{1-t}} = 0. \tag{15}$$

Finally, upon making the substitution $\frac{1}{1-t} = y$ and rearranging terms, we arrive at our main result, Eq. **11** of the main text with $N(s,r)$ given by

$$N(s,r) = \int_1^\infty \frac{dy}{y} e^{-sy} (1 - \frac{1}{y})^r. \tag{16}$$

For r fixed and $s \ll 1$, the integral in $N(s,r)$ is dominated by the region $y \sim 1/s$ and hence $N(s,r) \sim -\log s$; the application of the Tauberian theorem (see ref. 2) finally demonstrate the t^{-2} asymptotic behavior of the lifetimes. In order to derive Eq. **12** of the paper, in the limit $s \ll 1$ with rs fixed, we make the change of variables $x = \sqrt{\frac{s}{r}} y$ in Eq. **16** , exponentiate the term $(1 - 1/y)^r$ in the integral and make a power expansion

$$N(s,r) = \int_{\sqrt{\frac{s}{r}}}^\infty \frac{dx}{x} e^{-\sqrt{rs}\left(x + \frac{1}{x} - \sqrt{\frac{s}{r}}\frac{1}{x^2} + \ldots\right)}, \tag{17}$$

which, neglecting corrections of order $\sqrt{s/r}$ leads to Eq. **12** of the main text. When $rs \gg 1$, i.e. for $t \ll r \gg 1$, we can use the asymptotic expansion for the modified Bessel function, K_0 (see Eq.**12** of the main text) or, more directly, we can estimate the integral with the saddle point method: the maximum of the argument of the exponential occurs at $x^* = 1$ and, expanding it to second order around $x^* = 1$, we find

$$N(s,r) \approx e^{-2\sqrt{rs}} \int_{\sqrt{\frac{s}{r}}}^\infty dx e^{-\sqrt{rs}(x-1)^2} \approx e^{-2\sqrt{rs}} (rs)^{-\frac{1}{4}}. \tag{18}$$

Hence

$$sg_0(s) - 1 = \frac{1}{\partial_s \log N(s,r)} = -\frac{1}{\sqrt{\frac{r}{s}} + \frac{1}{4s}}, \tag{19}$$

which means that for $s \to 0$, $sg_0(s) - 1 \sim -\sqrt{s}$ corresponding, according to the Tauberian theorem, to the random walk behavior $P_0(t) \sim 1/\sqrt{t}$. The fact that the scaling variable in the derivation above is rs, implies that the crossover time should be proportional to r. Indeed using Eqs.**11** and **12** of the main text and the inverse Laplace transform one derives the scaling form

$$p(t) = \frac{1}{t^2} f\left(\frac{t}{r}\right), \tag{20}$$

where the function $f(x) \sim \sqrt{x}$ for small value of the argument (i.e. when $x \ll 1$) and approaches a constant when x becomes large.

Finally, let us discuss the sub-critical case $b_1 < d_1$. Using exactly the same strategy as for the critical case, we find that the condition $F(0,s) = 0$ leaves us with the following equation:

$$\int_0^1 dt t^{d_0} \, (1-\alpha t)^{b_0/\alpha - d_0 - s/(1-\alpha) - 1} (1-t)^{1/(1-\alpha)-1} (s g_0(s) - t) = 0. \tag{21}$$

Now, we substitute $y = 1 - t$ and solve for $g_0(s)$

$$s g_0(s) - 1 = -\frac{\int_0^1 dy \; (1-y)^{d_0} [1 - \alpha(1-y)]^{b_0/\alpha - d_0 - s/(1-\alpha) - 1} \; y^{s/(1-\alpha)}}{\int_0^1 dy \; (1-y)^{d_0} [1 - \alpha(1-y)]^{b_0/\alpha - d_0 - s/(1-\alpha) - 1} \; y^{s/(1-\alpha) - 1}}. \tag{22}$$

The integral on the numerator is finite when $s \to 0$, whereas that on the denominator has a leading singularity of order $(1-\alpha)/s$. This implies that $s g_0(s) \simeq -A/[1 + st^*]$, with A constant and $t^* \sim 1/(1-\alpha)$, which is exactly the Laplace transform of a distribution of the form

$$p(t) \simeq e^{-t/t^*}.$$

This confirms both the asymptotic exponential decay of $p(t)$ and the scaling of the cutoff time $t^* \sim 1/(1-\alpha)$.

References

1. Polyanin, A.D., Zaitsev, V.F. & Moussiaux, A. (2002) *Handbook of First Order Partial Differential Equations* (Taylor & Francis, London).

2. Feller, W. (1966) *An Introduction to Probability Theory and Its Applications* (Wiley, New York), vol. II.