JOURNAL OF THE EXPERIMENTAL ANALYSIS OF BEHAVIOR 1981, 35, 413-414 NUMBER 3 (MAY)

TECHNICAL NOTE

WILKINSON'S METHOD OF ESTIMATING THE PARAMETERS OF HERRNSTEIN'S HYPERBOLA

Wetherington and Lucas (1980) have described a "nonlinear least squares" method of estimating the parameters of Herrnstein's hyperbola. Their method uses arbitrary initial estimates of k and r_e which are then adjusted by an iterative numerical procedure so that the sum of the squares of the residuals about the fitted function is a minimum. The method requires the use of a computer.

Wilkinson (1961) has described a direct method of estimating the parameters of a hyperbolic function that has two advantages over Wetherington and Lucas' method: it is easily implemented on a hand calculator, and it provides standard errors of the estimates of k and r_e. The latter advantage is especially important when the precision of the estimates is an issue as, for example, in tests of the constancy of k. Bradshaw, Szabadi, and Bevan (1976) were the first to use Wilkinson's method to estimate the parameters of Herrnstein's hyperbola.

Like the method of Wetherington and Lucas, Wilkinson's method provides least squares estimates of k and r_e. I have found that the two methods yield estimates of the parameters of the equation, and of the percentage of data variance accounted for, that are identical to at least two decimal places and usually to seven or eight. The method is applied in two steps. Initial estimates of k and r_e are obtained by a weighted least squares regression of $1/R$ on $1/r$, where R is the response rate maintained by the reinforcement rate, r. The initial estimates are then revised by fitting the bilinear equation,

$$R = b_1 f(r) + b_2 f'_{r_e}(r), \qquad (1)$$

where $f(r)$ is Herrnstein's hyperbola with k and r_e equal to their initial estimates, and $f'_{r_e}(r)$ is the first derivative of $f(r)$ with respect to r_e. The revised estimates of k and r_e are calculated from the fitting constants since $b_1 = k/k_0$ and $b_2 = b_1(r_e - r_{e_0})$, where the zero subscripts indicate initial estimates. The standard errors of the estimates also depend on b_1 and b_2. It is sometimes necessary to repeat the second step of the method with the revised estimates as new initial estimates, but I have rarely found more than three repetitions to be necessary, even when the ordinary (unweighted) linear regression of $1/R$ on $1/r$ yields a very bad fit in R and r.

An example of the hand calculation of the initial estimates of k and r_e by Wilkinson's method is given in Table 1. The first two columns of section (i) are reinforcement and response rates averaged over five sessions for a human pressing a panel for money on five variable-interval schedules (unpublished data from my laboratory). The calculations are self explanatory. Wilkinson showed that the parameters of the hyperbolic function given in section (iii) represent the least squares solution of the weighted regression problem.

Table 2 illustrates the hand calculation of the revised estimates of k and r_e and their standard errors (S.E.). The last column of section (i) is the first deriva-

Reprints may be obtained from J. J McDowell, Department of Psychology, Emory University, Atlanta, Georgia 30322.

Table 1

Hand Calculation of Initial Estimates of k and r_e

	r (rft/hr)	R (rsp/min)	$x=R^2$	$y=R^2/r$
(i)	206.39	222.55	49,528.503	239.975
	131.99	201.11	40,445.232	306.426
	68.40	199.36	39,744.410	581.059
	31.20	188.77	35,634.113	1,142.119
	3.60	114.91	13,204.308	3,667.863

(ii)	$A=\Sigma Rx$	= 35,323,913.07	$D=\Sigma xy$	= 136,502,910.6
	$B=\Sigma x^2$	= 7,112,651,286.00	$E=\Sigma y^2$	= 15,246,769.26
	$C=\Sigma Ry$	= 867,943.632	$F=AE-CD$	= 4.2009871×10^{14}
(iii)	$k_0=(BE-D^2)/F$ = 213.79 rsp/min			
	$r_{e_0}=(BC-AD)/F$ = 3.22 rft/hr			

Table 2

Hand Calculation of Revised Estimates and Standard Errors

	r (rft/hr)	R (rsp/min)	$r+r_{e_0}$	$f=k_0r/(r+r_{e_0})$	$f'=-k_0r/(r+r_{e_0})^2$
(i)	206.39	222.55	209.61	210.51	−1.0043
	131.99	201.11	135.21	208.70	−1.5435
	68.40	199.36	71.62	204.18	−2.8509
	31.20	188.77	34.42	193.79	−5.6302
	3.60	114.91	6.82	112.85	−16.5471
(ii)	$A=\Sigma f^2$ = 179,849.309			$E=\Sigma Rf'$ = −4,066.516	
	$B=\Sigma ff'$ = −4,074.057			$F=\Sigma R^2$ = 178,556.565	
	$C=\Sigma Rf$ = 179,075.314			$G=AD-B^2$ = 40,418,606.90	
	$D=\Sigma f'^2$ = 317.024				
(iii)	$b_1=(CD-BE)/G=$ .9947			$s=\sqrt{(F-b_1C-b_2E)/(n-2)}=9.1176$	
	$b_2=(AE-BC)/G=-.0445$				
(iv)	$k=b_1k_0$ = 212.66 rsp/min			S.E.$(k)=k_0s\sqrt{D/G}$ = 5.46 rsp/min	
	$r_e=r_{e_0}+b_2/b_1$ = 3.18 rft/hr			S.E.$(r_e)=(s/b_1)\sqrt{A/G}$ = .61 rft/hr	

tive of Herrnstein's hyperbola with respect to r_e. The last three columns use the initial estimates of k and r_e from Table 1. The assignment of letters to sums in section (ii) is unique to this table. Wilkinson showed that b_1 and b_2 as given in section (iii) represent the least squares solution of the bilinear regression problem (Equation 1). The standard deviation, s, in section (iii) is based on n-2 degrees of freedom, where n is the number of data pairs. Section (iv) gives the revised estimates of k and r_e and their standard errors.

Tables 1 and 2 follow Wilkinson's example of the hand calculation of the parameters of a hyperbola. Although he does not discuss the percentage of data variance accounted for (%VAF), it may be calculated by the method used by McDowell and Kessel (1979) (and others) from

$$\%\text{VAF} = \frac{\sigma_y^2 - \sigma^2_{resid}}{\sigma_y^2} \times 100\,, \qquad (2)$$

where σ^2_{resid} is the residual mean square. Table 3 illustrates the hand calculation of the percentage of variance accounted for by the hyperbola. The values of k and r_e used in section (i) are the revised estimates from Table 2. As before, the assignment of letters to sums in section (ii) is unique to this table. Notice that D is equal to twice the indicated sum.

Wilkinson's method can, of course, be written for execution by computer. A program in North Star BASIC, which is convertible to other versions of BASIC with minor modifications, is available from J. J McDowell.

J. J McDowell
Emory University

Table 3

Hand calculation of percentage of variance accounted for (%VAF).

	r (rft/hr)	R (rsp/min)	$r+r_e$	$f=kr/(r+r_e)$
(i)	206.39	222.55	209.57	209.43
	131.99	201.11	135.17	207.66
	68.40	199.36	71.58	203.21
	31.20	188.77	34.38	192.99
	3.60	114.91	6.78	112.92
(ii)	$A=\Sigma R$ = 926.700		$C=\Sigma f^2$ = 178,273.971	
	$B=\Sigma R^2$ = 178,556.565		$D=2\Sigma Rf$ = 356,578.908	
(iii)	$\%\text{VAF}=\{1-[(B-D+C)/(B-A^2/n)]\}\times 100 = 96.30\%$			

REFERENCES

Bradshaw, C. M., Szabadi, E., & Bevan, P. Behavior of humans in variable-interval schedules of reinforcement. *Journal of the Experimental Analysis of Behavior,* 1976, **26**, 135-141.

McDowell, J. J, & Kessel, R. A multivariate rate equation for variable-interval performance. *Journal of the Experimental Analysis of Behavior,* 1979, **31**, 267-283.

Wetherington, C. L., & Lucas, T. R. A note on fitting Herrnstein's equation. *Journal of the Experimental Analysis of Behavior,* 1980, **34**, 199-206.

Wilkinson, G. N. Statistical estimations in enzyme kinetics. *Biochemical Journal,* 1961, **80**, 324-332.