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CONFIRMATION OF LINEAR SYSTEM THEORY PREDICTION: CHANGES IN HERRNSTEIN'S k AS A FUNCTION OF CHANGES IN REINFORCER MAGNITUDE

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Eight human subjects pressed a lever on a range of variable-interval schedules for 0.25¢ to 35.0¢ per reinforcement. Herrnstein's hyperbola described seven of the eight subjects' response-rate data well. For all subjects, the y-asymptote of the hyperbola increased with increasing reinforcer magnitude and its reciprocal was a linear function of the reciprocal of reinforcer magnitude. These results confirm predictions made by linear system theory; they contradict formal properties of Herrnstein's account and of six other mathematical accounts of single-alternative responding.

Key words: linear system theory, Herrnstein's equation, quantitative law of effect, choice, reinforcer magnitude, variable-interval schedules, lever press, humans

The linear system theory is a set of mathematical techniques that can be used to calculate the response of a system to a known input provided the system can be described at least in principle by a linear differential equation (Aseltine, 1958). McDowell and Kessel (1979) used a modified version of the linear system theory to calculate the response of organisms to reinforcement inputs provided by variable-interval (VI) schedules. Their application of the theory entailed writing the reinforcement "input" to the system and the response "output" of the system as mathematical functions of time, calculating the Laplace transformations (McDowell, Bass, & Kessel, 1983) of these functions, and taking their quotient. For a system that can be described by a linear differential equation, the ratio of the transformed output function to the transformed input function must be constant (Aseltine, 1958). The result of this application of the linear system theory was a rate equation that expresses response rate on a VI schedule as a joint function of reinforcement rate, reinforcer value, and response value. McDowell and Kessel (1979) showed that the rate equation accounted for response-rate versus reinforcement-rate data from VI schedules as accurately as Herrnstein's (1970) hyperbola.

In a later paper, McDowell (1980) compared the rate equation with Herrnstein's hyperbola and demonstrated that at ordinary rates of reinforcement and responding the hyperbola was an approximation of the rate equation. When the exponential terms in the rate equation were replaced by their series expansion approximations, the equation assumed a hyperbolic form identical to that of Herrnstein's equation. However, McDowell also demonstrated that the identity of the two equations did not extend to the structure or interpretation of their parameters. In particular, Herrnstein's (1970, 1979) derivations require the y-asymptote, k, of the hyperbola to remain invariant with respect to changes in reinforcement parameters like magnitude or immediacy (Herrnstein, 1974). McDowell (1980) showed that the y-asymptote of the hyperbolic form of the rate equation was required to vary directly with these reinforcement parameters and, specifically, that its reciprocal was required to vary as a linear function of the reciprocal of reinforcement parameters like magnitude or immediacy. Unlike the constant-k property of Herrnstein's hyperbola, the variable-k property of the hyperbolic form of the rate equation is not based on assumptions about how

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reinforcement governs behavior. As McDowell showed in detail, the variable-k property of the rate equation is a purely formal consequence of the mathematics that must hold if the original application of the linear system theory is correct. In a recent refinement of the calculation, McDowell et al. (1983) demonstrated that the rate equation exhibits this property even when the hyperbolic form is exact rather than approximate.

This difference between the rate equation and Herrnstein's hyperbola provides empirical grounds for distinguishing between the two accounts of VI responding. It also provides a rigorous test of the linear system theory because the form of the variation in k is specified. Of the infinite number of forms that variability in k may assume, only one is consistent with the theory: The reciprocal of k must vary linearly with the reciprocal of value-related parameters of reinforcement.

A k that varies with reinforcement parameters like magnitude or immediacy, in addition to being inconsistent with Herrnstein's account, is inconsistent with other existing mathematical accounts of VI responding, including those based on response-inhibiting properties of reinforcement (Catania, 1973), intertransformable "tendencies" to respond and not respond (Staddon, 1977), response threshold and time-allocation constraints (Staddon, 1977), maximizing or optimality principles (Rachlin, 1978; Staddon, 1979), and "arousing" properties of reinforcement (Killeen, 1981). These accounts will be discussed in more detail later.

De Villiers (1977) and McDowell (1980) have reviewed the data bearing on the constancy of k, and both concluded that they were equivocal. Herrnstein (1981) has also expressed doubt about the constancy of k. No experiment to date has studied k explicitly, and in only one case (Bradshaw, Szabadi, & Bevan, 1978) have more than two values of k been available for comparison. It should be noted that de Villiers' and McDowell's reviews dealt with variations in k that were produced by changes in reinforcement parameters. Other types of variation in k are permitted by Herrnstein's account. For example, changes in the force requirement on the operandum are permitted to alter the value of k, and this effect has been reported in the literature (e.g., Bradshaw, Ruddle, & Szabadi, 1981).

The purpose of the present experiment was to study the relationship between k and reinforcer magnitude. Eight human subjects in the experiment pressed a lever for varying amounts of money on a range of VI schedules.

METHOD

Subjects

Eight humans aged 24 to 40 years (seven female, one male), who were recruited by advertisement, served in the experiment. All subjects were either unemployed or employed part-time while participating. None were college students, and none were taking medication of any kind. Subjects H09, H15, and H17 had previous experience on a variety of VI schedules. The other subjects were experimentally naive.

Apparatus

The subject sat in a small room at a 54.6-cm (width) by 64.8-cm (height) console that tilted away from the subject at an angle of 23.2° from the vertical. A lever resembling a straightened bicycle handlebar extended 24.5 cm from the center of the panel and depended 20° below the horizontal. Attached to the distal end of the lever and located inside the console was a metal pan in which weights could be placed. Table 1 lists for each subject the downward force required to operate the lever successfully. In all cases successful lever operation was accompanied by a loud click. A digital counter, an amber (reinforcement) light, a small speaker, a green (session) light, and a row of five red (VI) lights were mounted on top of the console. During sessions the room could be dimly illuminated by a 7.5-W houselight and continuous white noise masked extraneous sounds. The console was controlled and data were recorded by a computer located in an adjoining room.

Procedure

All subjects worked on VI schedules of lever pressing, with money as the reinforcer. Interval values were calculated by Fleshler and Hoffman's (1962) method. Four subjects (H09, H13, H15, and H17) worked on VI 17-, 25-, 51-, 157-, and 720-s schedules at reinforcer magnitudes of 0.25, 0.40, 1, 2, and 35¢ per reinforcer. One subject (H18) worked on the same

VI schedules but at magnitudes of 1, 2, 4, 10, and 20¢ per reinforcer. The remaining three subjects (H19, H20, and H23) worked on VI 8-, 25-, 40-, and 300-s schedules at magnitudes of 1, 2, 4, and 12¢ per reinforcer. All VIs were presented in each session. The subject worked on one VI for 10 min, rested for 5 min, worked on the next VI for 10 min, rested for 5 min, and so on until all VIs were presented. The sequence of VI schedules was quasi-random within sessions, with the restriction that all VIs appear exactly once per session. A single reinforcer magnitude was in effect each session and the magnitudes were varied quasi-randomly across sessions with the restriction that all magnitudes appear once before any was repeated. Each VI schedule was correlated with one of the red stimulus lights. During work periods the houselight, session light, and the appropriate VI light were illuminated. During rest periods the subject was required to remain in the experimental room with only the session light illuminated. This procedure is similar to the VI procedure developed by Bradshaw, Szabadi, and Bevan (1976).

Table 1

Force requirement on the lever, number of sessions per day, and total number of sessions for each subject.

Subject	*Force Requirement (N)*	*Sessions per Day*	*Number of Sessions*
H09	146	3	41
H13	146	3	50
H15	144	4	45
H17	139	3	50
H18	150	3	50
H19	150	4	44
H20	150	4	24
H23	152	4	32

For subjects H09, H13, H15, and H17, reinforcement consisted of the addition of one point to the digital counter, a brief (<1 s) illumination of the amber reinforcement light, and a brief (<1 s) sounding of a 1000-Hz tone. Reinforcement duration was approximately 0.3 s and all timing stopped during reinforcer delivery. The session's exchange rate (¢/point) and examples of the dollar values of various point totals were posted on the console.

For subjects H18, H19, H20, and H23, the digital counter incremented once for every cent earned. Each increment of the counter was accompanied by the brief flash and tone described above. While the counter was incrementing, the houselight and the current VI light were extinguished and all other VI lights were illuminated. Reinforcement duration was 2 s under this procedure and was held constant across magnitudes by spacing the counter increments appropriately. All timing stopped during reinforcer delivery. The session's reinforcer magnitude was posted on the console.

Before the start of the experiment, all subjects signed a contract in which they agreed to participate for 150 sessions or until they were released, whichever occurred first. The contract also stated that their earnings would depend on their performance and that they would be subject to a penalty for missing sessions (forfeiture of one session's average pay per session missed) or for early withdrawal from the experiment (forfeiture of one session's average pay per session remaining in the contract). These penalties, which were designed to ensure attendance at experimental sessions and completion of the experiment, were approved by the Emory University Human Subjects Committee and meet APA guidelines regarding informed consent.

At the start of the first session all subjects were instructed as follows (cf. Bradshaw et al., 1976):

> This is a situation in which you can earn money. This green light will be on for the entire session. You earn money simply by pressing this lever. You can tell whether or not you have pressed hard enough by listening for a click from inside the machine. Now look at these red lights. When the houselight and a red light are on you can earn money. At the beginning of the session one of the red lights will come on and it will stay on for 10 min. During this time you can earn money by pressing the lever. After 10 min all lights but the green one will go off for about 5 min. During this time you are to stay in the room and rest. After the rest period another red light will come on and you will be able to earn more money by pressing the lever. Then there will be another rest period, and so on until each red light has been presented.

Subjects H09, H13, H15, and H17 received the following additional instructions:

> Sometimes when you press the lever this amber light will flash and a tone will sound. This means you will have earned one point. The total number of points you have earned is shown on this counter. Every time the amber light flashes, one point is added to the counter. Points will be worth different amounts of money in different sessions. This chart shows the exchange rate each session. At the end of the session I will take the reading from the counter and give you a receipt for the money you have earned.

Subjects H18, H19, H20, and H23 received the following additional instructions:

> Sometimes when you press the lever, the red light and houselight will go off and the other red lights will come on. While this is happening, the amber light will flash and a tone will sound a number of times. Each time the light flashes and the tone sounds, one cent will be added to the amount you have earned. The number of flashes and tones will be different in different sessions. This chart shows the number of flashes and tones. The total amount of money you earn in a session is shown on this counter. At the end of the session I will take the reading from the counter and give you a receipt for the money you have earned.

Subjects were paid at the end of the experiment, although small advances were arranged for some subjects. Questions at the first and all subsequent sessions were answered by rereading relevant portions of the instructions. To ensure that subjects did not have timepieces in the experimental room, they were told that metal jewelry might interfere with the operation of the equipment, and they were asked to leave such items with the experimenter. Either 3 or 4 daily sessions were arranged for each subject. Table 1 lists the number of sessions per day, which was constant within subjects. A break of at least 15 min intervened between sessions that occurred on the same day.

Except for H20, stability was determined by time series analysis on eight consecutive response rates in individual magnitude-by-VI conditions (alpha = .01; Tryon, 1982; von Neumann, Kent, Bellinson, & Hart, 1941; Young, 1941). In all cases, visual inspection confirmed the statistical judgment of stability. Subject H20 withdrew from the experiment before the time series criterion could be applied, but her last five sessions in each magnitude-by-VI condition appeared stable. The total number of sessions for each subject is listed in Table 1.

RESULTS

Penalties were exacted from H09, who missed seven sessions and forfeited $40.95 or about 17% of her total pay, and from H13, who missed three sessions and forfeited $17.55 or 8% of her total pay. Although H20 withdrew from the experiment prematurely, she was not penalized. None of the other subjects incurred penalties.

Cumulative records from stable sessions for the eight subjects were typical of VI performances. All subjects produced smooth linear records, with some graininess appearing in lean VI, low magnitude conditions.

Except for H20, reinforcement and response rates were averaged over the first stable eight-session block in each magnitude-by-VI condition. For H20, reinforcement and response rates were averaged over the last five-session block in each magnitude-by-VI condition. Tables 2 and 3 list each subject's average reinforcement and response rates in each stable condition. Subject H23 was inadvertently released from the experiment before her response rates had stabilized on the 2¢ magnitude, and some of the other subjects showed unstable responding in isolated magnitude-by-VI conditions. Data from unstable conditions were omitted from the analysis.

Hyperbolas were fitted to the averaged data by the method described by McDowell (1981). Unique parameter estimates could not be obtained for H13's 0.4 or 2¢ magnitude. For the remaining 34 data sets, Table 4 lists the percentage of variance accounted for (%VAF) by the fitted hyperbolas, the estimated values of k, and the standard errors of the estimates. Where response-rate variability was small (coefficient of variation less than 0.1), the %VAF statistic was omitted from the table. In these cases the data fell along the hyperbolas' asymptotes such that visual inspection indicated good fits.

Table 2

Average reinforcement and response rates for Subjects H09 through H18 in each magnitude-by-VI condition (rft/hr = reinforcements/hour; rsp/min = responses/min).

Reinforcer magnitude (¢/reinforcer)	VI (seconds) 17 rft/hr	17 rsp/min	25 rft/hr	25 rsp/min	51 rft/hr	51 rsp/min	157 rft/hr	157 rsp/min	720 rft/hr	720 rsp/min
					H09					
0.25	not stable		not stable		49.5	20.4	16.5	15.6	6.0	10.5
0.40	159.7	34.0	113.2	28.6	60.0	29.5	25.5	22.3	not stable	
1.0	179.2	35.9	118.5	33.0	57.7	34.9	21.7	23.3	3.7	14.0
2.0	178.5	44.0	126.7	39.2	62.2	32.6	25.5	27.5	4.5	22.7
35.0	192.0	65.8	133.5	58.2	62.2	51.0	18.7	40.2	5.2	27.5
					H13					
0.25	33.7	14.4	19.5	11.7	12.0	11.7	3.0	2.6	3.0	2.0
1.0	50.2	20.3	13.5	7.3	7.5	2.2	5.2	5.9	2.2	1.4
35.0	198.7	131.6	131.2	125.1	not stable		20.2	74.8	8.2	45.4
					H15					
0.25	204.7	60.5	134.2	58.1	65.2	59.7	19.5	53.9	6.7	53.8
0.40	198.0	57.4	133.5	60.4	63.0	58.8	24.0	53.3	3.8	56.6
1.0	203.0	63.7	136.5	60.2	64.5	61.1	27.7	57.8	5.2	55.9
2.0	204.8	66.8	134.2	65.3	64.5	61.3	21.7	59.0	2.2	53.3
35.0	204.0	66.5	133.5	61.4	65.2	64.5	22.5	61.3	5.2	59.7
					H17					
0.25	188.2	32.3	129.7	33.3	59.2	31.8	18.8	26.2	3.8	7.3
0.40	185.2	35.2	130.5	35.4	61.5	33.3	not stable		5.2	4.3
1.0	198.8	37.1	129.8	37.4	63.8	34.6	23.2	27.5	3.8	5.8
2.0	201.0	36.6	132.7	36.6	60.7	34.5	not stable		4.5	6.1
35.0	199.5	37.6	132.8	38.8	64.5	37.8	23.2	30.9	6.8	9.6
					H18					
1.0	226.6	54.8	150.1	52.4	56.3	39.3	15.0	19.2	1.5	7.6
2.0	232.7	59.3	154.6	58.8	69.0	52.6	6.8	15.5	6.0	6.9
4.0	227.4	58.0	152.3	60.0	69.0	54.4	21.0	34.9	2.3	1.6
10.0	229.6	69.0	152.3	64.3	72.1	62.3	21.8	47.9	2.3	1.6
20.0	237.1	70.7	150.1	69.0	69.8	65.1	27.8	56.2	5.3	3.3

Table 4 shows that, except for H19, hyperbolas described these subjects' data well. The fitted hyperbola accounted for more than 90% of the data variance in most cases. The poor fits for H19's three largest magnitudes were due to markedly bitonic response-rate versus reinforcement-rate functions. Poor fits were also obtained for H09 at 2¢ and for H20 at 4¢ per reinforcer. Including H19's anomalous results, the hyperbola accounted for a median of 95% of the variance of the individual data sets.

The estimated values of *k* listed in Table 4 are plotted against reinforcer magnitude in Figure 1. The figure shows that *k* increased from the beginning to the end of the magnitude range for all subjects. The two flattest functions were produced by H15 and H17, but the standard errors given in Table 4 show that these subjects' *k*s were precisely determined. In particular, error bars ($k \pm 1$ standard error) on the first and last *k*s did not overlap for either subject. Among the eight subjects, only H19 (who produced the bitonic response- versus reinforcement-rate functions) showed overlapping error bars on the first and last *k*s.

The right-hand panel of Figure 1 shows the median *k* plotted against reinforcer magnitude. From the low to the high end of the magnitude range, the median *k* increased 34 responses/min, or about 115%. This translates into an average rate of change in *k* of about +1 response/min per one-cent increase in reinforcer magnitude, although it is evident

Table 3

Average reinforcement and response rates for Subjects H19 through H23 in each magnitude-by-VI condition (rft/hr = reinforcements/hour; rsp/min = responses/min).

Reinforcer magnitude (¢/reinforcer)	VI (seconds) 8 rft/hr	8 rsp/min	25 rft/hr	25 rsp/min	40 rft/hr	40 rsp/min	300 rft/hr	300 rsp/min
H19								
1.0	416.3	59.5	124.4	71.4	81.7	54.1	6.0	32.3
2.0	417.9	59.3	121.4	79.8	82.4	61.6	5.2	41.7
4.0	422.0	60.9	130.0	93.0	82.4	57.2	8.2	49.3
12.0	420.1	63.5	113.9	89.6	81.6	59.2	6.7	64.0
H20								
1.0	408.2	49.3	122.2	45.2	82.7	42.3	6.0	39.1
2.0	399.6	50.6	118.7	46.3	80.3	44.0	7.2	41.1
4.0	414.2	55.9	123.5	48.0	82.7	44.3	7.2	40.9
12.0	410.4	55.0	122.2	53.6	82.7	50.2	6.0	41.6
H23								
1.0	431.9	183.4	129.9	163.4	89.6	161.5	12.0	112.6
4.0	429.6	175.4	125.0	164.7	89.6	165.0	11.2	102.0
12.0	not stable		127.8	194.3	89.8	176.6	12.0	119.8

from Figure 1 that k increased far more rapidly at the low than at the high end of the magnitude range.

To test the predicted form of the variation in k, the reciprocal of k was plotted against the reciprocal of reinforcer magnitude for each subject. These plots are shown in Figure 2. In this figure's double-reciprocal coordinates, the ks and magnitudes increase as the data points approach the origins of the coordinate axes. Lines drawn through the points were fitted by the method of least squares. Although the data in Figure 2 were subject not only to the usual measurement error but also to the error associated with the estimation of k, the figure shows that straight lines described the individ-

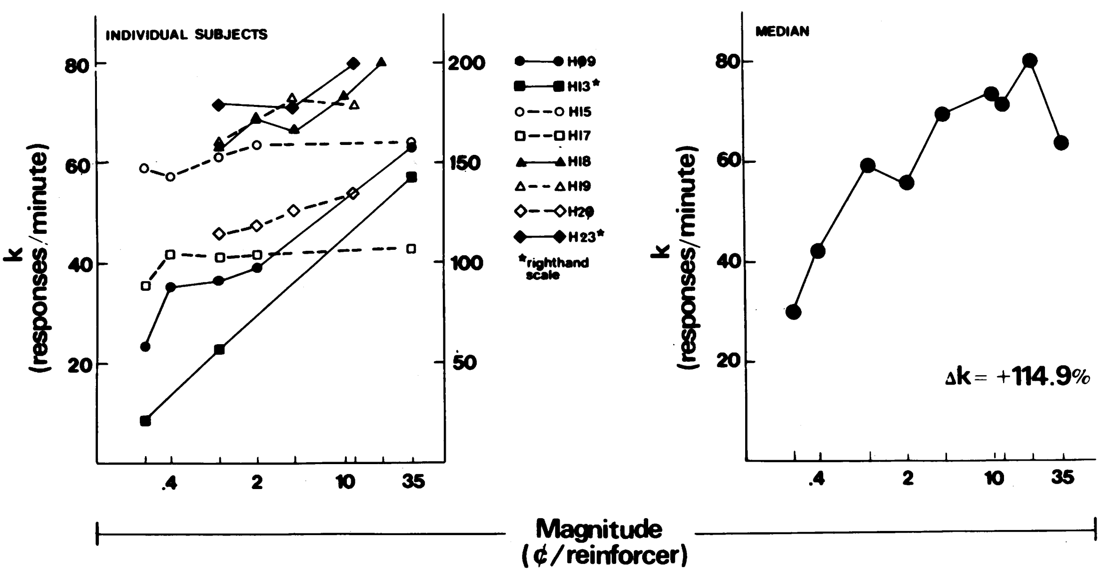

Fig. 1. In the left panel, the ks from Table 4 are plotted against reinforcer magnitude for each subject. The right-hand ordinate scale of this panel applies to H13 and H23; the left-hand ordinate scale applies to the remaining subjects. In the right panel, the median k across subjects is plotted against reinforcer magnitude. The abscissae of both panels are scaled logarithmically.

Table 4

Percentage of variance accounted for, *k*, and the standard error (S.E.) of *k* for fits of hyperbolas to individual-subject response and reinforcement rates at each reinforcer magnitude (rsp/min = responses/min). The values in Table 4 were calculated from unrounded reinforcement and response rates. If hyperbolas are fitted to the rounded data in Tables 2 and 3, values slightly different from those in Table 4 may result.

Reinforcer magnitude (¢/reinforcer)	*Variance accounted for (%)*	*k (rsp/min)*	*S.E. (rsp/min)*
	H09		
0.25	99.6	23.4	0.7
0.40	84.0	35.2	2.8
1.0	92.2	36.6	2.4
2.0	68.2	39.2	3.7
35.0	92.0	63.1	3.8
	H13		
0.25	93.6	21.8	5.6
1.0	94.8	56.6	36.0
35.0	100.0	142.9	0.9
	H15		
0.25	gvf[a]	59.2	1.3
0.40	gvf	57.7	1.6
1.0	gvf	61.4	1.2
2.0	gvf	63.8	1.6
35.0	gvf	64.0	1.3
	H17		
0.25	96.2	35.8	2.0
0.40	96.5	42.1	5.0
1.0	98.0	41.3	2.0
2.0	98.0	41.8	3.0
35.0	92.2	43.0	3.6
	H18		
1.0	98.6	63.0	3.8
2.0	98.0	68.8	4.8
4.0	98.1	66.4	4.1
10.0	97.0	73.4	5.1
20.0	90.9	80.0	10.0
	H19		
1.0	81.3	64.4	5.9
2.0	63.5	68.4	7.4
4.0	30.1	72.3	13.0
12.0	5.8	71.2	10.2
	H20		
1.0	gvf[a]	46.1	2.0
2.0	gvf	47.5	1.9
4.0	50.9	50.3	3.5
12.0	91.9	53.7	1.3
	H23		
1.0	96.0	178.6	5.4
4.0	99.7	178.3	1.6
12.0	97.2	200.2	9.1
Median	94.8		

[a]Good visual fit (little variance to account for).

ual-subject data well. There was a slight indication of downward concavity for H18, but this was not confirmed by any other subject's results. Coefficients of determination for the eight subjects, in numerical order, were 0.9, 1.0, 0.7, 0.7, 0.7, 0.9, 0.9, and 0.4. The unusually small coefficient of determination for H23 was due to his small regression slope. As is evident from H23's plot, the coefficient of determination is not always a suitable indicator of goodness of fit when the regression slope is small.

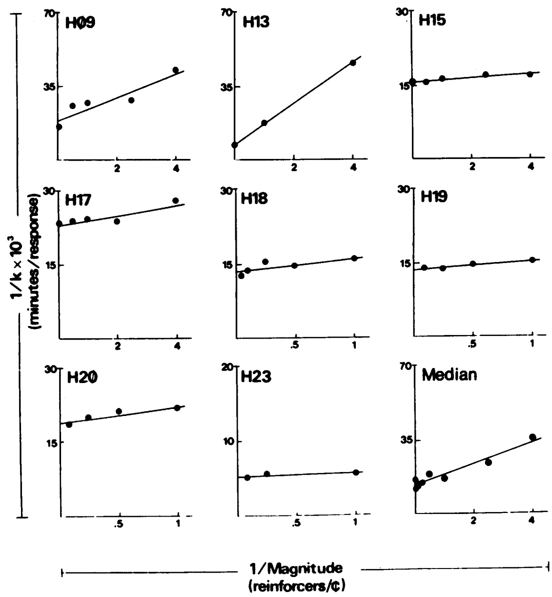

Fig. 2. The reciprocals of the *k*s from Table 4, multiplied by 1000, are plotted against the reciprocals of reinforcer magnitude for each subject. The median reciprocal of *k*, multiplied by 1000, is plotted against reinforcer magnitude in the bottom right panel of the figure.

The median of the individual 1/*k*s is plotted against the reciprocal of reinforcer magnitude in the lower right panel of Figure 2. Evidently, the least squares regression line drawn through the points provided an excellent description of the median data. The coefficient of determination for this linear fit was 0.9. No downward concavity was evident in the median data, nor were any other consistent departures from linearity apparent.

DISCUSSION

The results summarized in Figure 1 confirm the linear system theory prediction that *k* should vary directly with reinforcer magnitude. The more stringent formal requirement that the variation be linear in double-reciprocal coordinates is verified by the data shown in Figure 2. These results were robust with respect to procedural variations, including the use of two VI series, three magnitude ranges, and two methods of reinforcer delivery.

The original derivation of Herrnstein's (1970) hyperbola was based on the assumption that single-alternative responding entails choice between responding and not responding (i.e., doing other things). Herrnstein (1974) and McDowell (1980) showed how this assumption translates into the formal requirement that k remain invariant across changes in parameters of reinforcement like magnitude. Because the results summarized in Figure 1 violate the constant-k requirement, they suggest that single-alternative responding does not entail choice and, consequently, that Herrnstein's original derivation of the hyperbola is untenable. Herrnstein (1979) recently proposed a "molecular" derivation of his hyperbola that is closely related to the original derivation. He showed that the hyperbola is the solution of a differential equation that expresses changes in responding as a function of changes in reinforcement. Like its predecessor, this derivation incorporates the matching (or choice) principle, requires a constant k, and, evidently, is untenable.

The data shown in Figure 1 also support Williams' (1983) recent criticism of mathematical accounts of multiple-schedule performance that are based on Herrnstein's hyperbola (e.g., Herrnstein, 1970; McLean & White, 1983). As Williams noted, these accounts typically require a constant k.

The few extant data that show a roughly constant k (McDowell, 1980) are not necessarily inconsistent with the linear system theory, or with the results of the present experiment. McDowell (1980) showed that the theory requires the rate of change of k to depend on the aversiveness of the response, such that the increase in k over a given magnitude range should be larger for a more aversive than for a less aversive response. In other words, k may appear to be roughly constant for a relatively effortless response, especially if the range of magnitudes sampled is small or if only a few values of k are compared. McDowell argued that many of the data supporting the constancy of k were obtained under circumstances that should favor a rough constancy in k. By constrast, the force requirements on the operandum in the present experiment (Table 1) represented at least 25% of the typical subject's body weight, and exceeded those used in other studies of human lever pressing (e.g., Bradshaw et al., 1976; McDowell & Sulzen, 1981) by two orders of magnitude. The range of reinforcer magnitudes sampled in this experiment was also large (two orders of magnitude). Although this explanation of the few contradictory data is plausible from the standpoint of the linear system theory, the argument would be more compelling if it were shown that an apparently constant k could be made to vary by increasing response aversiveness.

Since 1970, many efforts have been made to derive a function relating response and reinforcement rates from assumptions that are unrelated to choice. Not including McDowell and Kessel's (1979) application of the linear system theory, at least six distinct derivations have been published in the last decade. The results of the present experiment are not easily reconciled with any of the six accounts, each of which is summarized below.

Catania's (1973) derivation, which is the earliest, is based on the assumption that reinforcement both supports and inhibits responding. The asymptote of Catania's hyperbola is required to vary with units of measurement but, apparently, not with reinforcement parameters like magnitude.

Staddon (1977) proposed two distinct derivations of a hyperbolic function. In the first, "tendencies" to respond and not respond are assumed to be intertransformable and, at equilibrium, equal. From this assumption, Staddon derived an equation that is identical to Herrnstein's hyperbola both in form and in the interpretation of its parameters. This means that the asymptote of the hyperbola is required to remain invariant across changes in reinforcement parameters. Staddon's second derivation is based on an assumed exponential distribution of interreinforcement intervals, a threshold reinforcement probability below which responding cannot occur, and a time-allocation constraint. The asymptote of the resulting hyperbola is the reciprocal of response duration, which represents the organism's physically maximal response rate. Response duration includes an hypothesized refractory period during which additional responding cannot occur. Although Staddon did not specify the variables that might affect the refractory period, it seems reasonable to suppose that they would be properties of the response. For example, it may be reasonable to suppose that the refractory period for a key peck is shorter than that for a treadle press. However, with a constant response form (as was used in the present experiment), it appears that the asymptote of Staddon's hyperbola should invariably estimate the organism's physically maximal response rate (for that response form), regardless of changes in reinforcement parameters.

The fourth derivation, also provided by Staddon (1979), is based on the assumption that organisms allocate time to instrumental responding such that the distribution of time spent responding, consuming reinforcers, and doing other things is as close as possible to the distribution that would result if the three types of activity were freely available. This is a kind of optimality analysis. Incorporating a hyperbolic feedback function, Staddon obtained an equation relating single-alternative responding to obtained reinforcement. Unlike the equations discussed thus far, Staddon's equation is a cubic polynomial. For most parameter values the function is bitonic in the first quadrant and consequently has no y-asymptote for admissible (i.e., positive) response and reinforcement rates. With the possible exception of H19's data, this cubic polynomial would not describe well the present response- versus reinforcement-rate data.

A conceptually similar derivation was provided by Rachlin (1978). It is based on the assumption that organisms distribute their time among responding, consuming reinforcers, and doing other things in ways that maximize value, which is a function of the time spent engaging in each of the three types of activity. Incorporating a feedback function that, unlike Staddon's, is a power function, Rachlin obtained an equation for single-alternative VI responding. This equation has no analytic solution for time spent responding or for response rate. Although it is not a hyperbola in the analytic sense, the equation may assume an approximately hyperbolic form for certain parameter values. The y-asymptote of the function, when it exists, cannot vary with reinforcer magnitude because the value of reinforcement is assumed to depend only on the amount of time spent consuming reinforcers. In the present experiment, within-subject reinforcement time remained constant across changes in reinforcer magnitude.

The sixth derivation is based on Killeen's "arousal" theory (Killeen, Hanson, & Osborne, 1978), which assumes that reinforcers generate arousal that decays with time, but that may cumulate when reinforcers are presented repeatedly. Using an exponentially weighted moving average of the arousal generated by repeated reinforcements, and a ceiling on response rates, Killeen (1981) derived a hyperbolic VI response function. The asymptote of the function depends on an arbitrary constant of proportionality (which determines how arousal is "mapped onto" response rate) and on the organism's physically maximal response rate. Like the other hyperbolas reviewed here, the y-asymptote of Killeen's hyperbola is not required to vary with reinforcer magnitude.

Although the present results violate formal properties of Herrnstein's derivations and of the six nonchoice derivations, there is no doubt that at least some of these accounts could be modified so as to allow appropriate variations in the y-asymptote of the hyperbola. In Herrnstein's later derivation, for example, the constant-k requirement is a definition, or a "constraint on the measurement of behavior" (Herrnstein, 1979, p. 487) rather than a formal consequence of the mathematics. Although response rate has been the principal dependent variable in operant conditioning since its inception, Herrnstein's later derivation might be preserved if time-rate measurement were abandoned in favor of time-allocation measurement. As another example, Rachlin's (1978) account could be modified by arguing that the exponent, x, on the value function for contingent responding (i.e., consuming reinforcers) is affected by reinforcement parameters like magnitude. This exponent is a free parameter in the present version of Rachlin's theory. Staddon's (1977) second derivation could be modified by arguing that the refractory period of a response varies inversely with reinforcer magnitude. As a final example, Killeen's (1981) derivation would accommodate the present data if the constant of proportionality relating response rate to arousal were allowed to increase with increasing reinforcer magnitude. No doubt most, if not all, of the remaining derivations could be defended by arguing that they were never intended to account for changes in reinforcement parameters other than rate, or by introducing post-hoc modifications of one sort or another.

The linear system theory's a priori prediction of both the variation in k and its form is distinctive. Rather than following from data-based assumptions about how reinforcement governs behavior, the prediction is a purely formal consequence of the mathematics (McDowell, 1980). The empirical confirmation of this prediction evidently provides unique and substantial support for the linear system approach.

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