A THEORY OF FLUID FLOW IN COMPLIANT TUBES

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ABSTRACT Starting with the Navier-Stokes equations, a system of equations is obtained to describe quasi–one-dimensional behavior of fluid in a compliant tube. The nonlinear terms which cannot be shown to be small in the original equations are retained, and the resulting equations are nonlinear. A functional pressure-area relationship is postulated and the final set of equations are quasilinear and hyperbolic, with two independent and two dependent variables. A method of numerical solution of the set of equations is indicated, and the application to cases of interest is discussed.

INTRODUCTION

Laminar flow of an incompressible fluid in compliant tubes has received considerable attention, partly because of its relevance to the dynamics of blood flow in arteries. In principal, the problem is described exactly by the three Navier-Stokes equations of motion for the fluid, the equation of continuity for the fluid, and equations of motion for the wall.

A general solution of such a system of nonlinear partial differential equations has not been achieved. Additionally, the physiological quantities which would arise in a treatment of blood flow in mammalian arteries are not well known. For both reasons it is necessary to work in terms of approximate models, which include the important features of the system under consideration and neglect unimportant features.

A variety of models based on the Navier-Stokes equations may be found in the literature. Some of these models have been successful in predicting certain, but not all, aspects of the flow. A brief summary of some of these results will now be given.

In 1957, an extensive treatment of the problem was published by Womersley (1), who considered a segment of a uniform, infinitely long, cylindrical tube with a linearly elastic, isotropic wall. Fluid motion in the circumferential direction of the tube was neglected, and all nonlinear terms in the two remaining Navier-Stokes equations were dropped. Womersley obtained analytical traveling-wave solutions to this formally linearized system. The treatment was successful in predicting the flow produced by a known pressure gradient.

However, Womersley pointed out that this linear theory when applied to pulsatile flow in a nontapering cylindrical vessel predicted a diminution in the pressure pulse amplitude of 5 to 10% for each 10 cm of travel, in typical cases. In contrast to this theoretical result, the observed increase of the pressure pulse between the proximal and peripheral parts is one of the most striking features (2) of the arterial system. Womersley suggested that the explanation of the discrepancy between his theory of an infinitely long tube and actual systems might lie in the reflections which would occur from discontinuities such as the bifurcation of the aorta. McDonald makes the assumption that the regularly repeated heartbeat creates a steady state oscillation, i.e., that reflections from one wave are superimposed on later waves.[1]

However, the experiments of Peterson (3) and Starr (4) with single pulses showed no discrete reflections that could modify the shape of a later pulse.

Evans (5) considered the problem of flow of a viscous liquid in a tapering tube by retaining only one Navier-Stokes equation and dropping the nonlinear terms from it. For this one-dimensional linear approximation Evans gets analytical solutions in the form of a traveling wave with amplitude varying with distance. Both the pressure and velocity waves are attenuated with distance in the case of enlarging taper, and the opposite occurs with a constricting taper. The physiological observation is that the pressure pulse peak height increases and the velocity pulse decreases with distance.

Tapering tubes have been treated by Streeter, Keitzer, and Bohr (6) using a different approach. In their one-dimensional treatment the radial velocity was neglected and an empirical friction term was used in an equation of motion for the "plug" of fluid contained in a segment of the tube. This equation, together with the continuity equation and an assumed pressure-area relationship for the tube, form a quasilinear hyperbolic system of partial differential equations which were solved numerically by the method of characteristics (7).

With this method it is possible to impose a variety of boundary conditions to model tubes of finite length. Reflected waves produced at boundaries, or by changes in the tube characteristics, are automatically included in the solution.

This method was successfully used by Wiggert and Keitzer (8), who, by adjusting the frictional term, were able to predict numerically some measurements on saline solution in tapered plastic tubes.

The approach of the present paper begins with two Navier-Stokes equations for the fluid motion. Those nonlinear terms which cannot be shown to be negligible are retained. A reasonable form is assumed for the radial dependence of the axial velocity. The Navier-Stokes equations are integrated over the radial coordinate reducing the number of independent variables to two (time and axial distance) and the dependent variables to three (pressure, averaged velocity, and cross-sectional

[1] Pages 203 and 232 of reference 2.

area). A pressure-area relation is used to eliminate one of the dependent variables.

The final equation of motion for the fluid contains two parameters which depend on the assumed velocity profile. All other quantities appearing as coefficients in the equations were in principal known from physiological considerations. The relationship between this approach and those of Womersley (1), Evans (5), and Streeter et al. (6) is indicated during the development. The equations are solved numerically by the method of characteristics, so that any reflected waves occurring are automatically included in the numerical solutions.

Finally, the application of the theory to a cylindrical tube, driven sinusoidally at the proximal end and almost closed at the distal end, is outlined. This is intended as a model of a catheter with a pressure-transducer at the distal end. Numerical results for this case will be compared to experimental results in the following paper (11).

DERIVATION OF THE EQUATION OF MOTION OF THE FLUID

If, in a cylindrical coordinate system, motion in the circumferential direction is neglected, the Navier-Stokes equations of motion for an incompressible Newtonian fluid become

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right) \quad (1)$$

and

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial r} = \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2}\right) \quad (2)$$

where r is the radial and z the axial direction. The constants ρ and ν are the density and kinematic viscosity of the fluid; the rest of the notation is standard (9).

In considering these equations, Womersley (1) remarked that for the case of interest v_r is much less than v_z, and used this to justify neglect of all nonlinear terms:

$$v_r \frac{\partial v_z}{\partial r}, v_z \frac{\partial v_z}{\partial z}, v_r \frac{\partial v_r}{\partial r}, v_z \frac{\partial v_r}{\partial z}.$$

The terms

$$\frac{\partial^2 v_z}{\partial z^2} \text{ and } \frac{\partial^2 v_r}{\partial z^2}$$

were also neglected. Evans (5) additionally ignored equation 2. Although $v_r/v_z \ll 1$, it does not follow that $v_z\ (\partial v_r/\partial z)$ etc. are small. By neglecting these terms, Womersley obtained an excellent prediction of flow from a known pressure gradient. His model, however, predicted attenuation of the pressure wave along the tube, contrary to physiological observations.

It is convenient first to make the equations nondimensional. If U_o and V_o are typical velocities in the axial (z) and radial (r) directions, then a parameter may be defined by $\epsilon = V_o/U_o$. The following treatment is valid when ϵ is small.

For laminar flow in flexible tubes, the maximum value of v_r is the radial velocity of the wall, which will be small unless the tube is very flexible. Thus ϵ is small for blood flow in mammalian arteries, as Womersley pointed out.

Such flow may be termed quasi–one-dimensional. The velocities U_o and V_o define a characteristic length λ in the z direction over which deviations from the axis become equal to R_o (the tube radius): $\lambda = R_oU_o/V_o$ or $R_o/\lambda = \epsilon$. Nondimensional quantities, designated by primes, may be defined as follows:

$$r = R_0 r' \quad z = \lambda z'$$

$$v_z = U_0 v_z' \quad v_r = V_0 v_r' \quad t = \frac{\lambda}{U_0} t' \quad \text{and} \quad p = \rho U_0^2 p'.$$

In terms of primed quantities, equations 1 and 2 reduce to

$$\frac{\partial v_z'}{\partial t'} + v_z' \frac{\partial v_z'}{\partial z'} + v_r' \frac{\partial v_z'}{\partial r'} + \frac{\partial p'}{\partial z'} = \beta\left(\frac{\partial^2 v_z'}{\partial r'^2} + \frac{1}{r'}\frac{\partial v_z'}{\partial r'} + \epsilon^2 \frac{\partial^2 v_z'}{\partial z'^2}\right) \tag{3}$$

and

$$-\frac{\partial p'}{\partial r'} = \epsilon^2\left\{\frac{\partial v_r'}{\partial t'} + v_z' \frac{\partial v_r'}{\partial z'} + v_r' \frac{\partial v_r'}{\partial r'} - \beta\left[\frac{\partial^2 v_r'}{\partial r'^2} + \frac{1}{r'}\frac{\partial v_r'}{\partial r'} - \frac{v_r'}{r'^2} + \epsilon^2 \frac{\partial^2 v_r'}{\partial z'^2}\right]\right\} \tag{4}$$

where $\beta = (\lambda/R_o)(\nu/U_oR_o)$. Provided $\epsilon \ll 1$, the last term in equation 3 is negligible and equation 4 implies that p' is not a function of r'.

The equation of continuity in nondimensional form is

$$\frac{\partial v_z'}{\partial z'} + \frac{1}{r'}\frac{\partial}{\partial r'}(r'v_r') = 0. \tag{5}$$

Now equations 3 and 5 may be rewritten

$$\frac{\partial}{\partial t'}(r'v_z') + \frac{\partial}{\partial z'}(r'v_z'^2) + \frac{\partial}{\partial r'}(r'v_r'v_z') + \frac{\partial}{\partial z'}(r'p') = \beta\left(\frac{\partial}{\partial r'}\left(r'\frac{\partial v_z'}{\partial r'}\right)\right) \tag{6}$$

and

$$\frac{\partial}{\partial z'}(v_z'r') + \frac{\partial}{\partial r'}(r'v_r') = 0. \tag{7}$$

Equations 6 and 7 may be integrated over r' from $r' = 0$ to $r' = R'$.

$$\frac{\partial}{\partial t'}\left[\int_0^{R'} r'v_z'\,dr'\right] - [r'v_z']_{R'}\frac{\partial R'}{\partial t'} + \frac{\partial}{\partial z'}\left[\int_0^{R'} r'v_z'^2\,dr'\right]$$

$$- [r'v_z'^2]_{R'}\frac{\partial R'}{\partial z'} + [r'v_r'v_z']_{R'} + \int_0^{R'} r'\frac{\partial p'}{\partial z'}\,dr' = \beta\left[r'\frac{\partial v_z'}{\partial r'}\right]_{R'} \tag{8}$$

and

$$\frac{\partial}{\partial z'}\left[\int_0^{R'} v_z' r' \, dr'\right] - [v_z' r']_{R'} \frac{\partial R'}{\partial z'} + [r' v_r']_{R'} = 0. \tag{9}$$

Since the wall is a stream surface,

$$[v_r']_{R'} = \left[\frac{\partial r'}{\partial t'}\right]_{R'} + \left[v_z' \frac{\partial r'}{\partial z'}\right]_{R'}$$

or

$$[r' v_r' v_z']_{R'} = [r' v_z']_{R'} \frac{\partial R'}{\partial t'} + [r' v_z'^2]_{R'} \frac{\partial R'}{\partial z'}.$$

If a mean velocity in the z direction is defined by

$$U' = \frac{1}{R'^2} \int_0^{R'} 2r' v_z' \, dr' \tag{10}$$

and a parameter α by

$$\alpha = \frac{1}{R'^2 U'^2} \int_0^{R'} 2r' v_z'^2 \, dr' \tag{11}$$

then equations 8 and 9 may be rewritten

$$\frac{\partial}{\partial t'}(R'^2 U') + \frac{\partial}{\partial z'}(\alpha R'^2 U'^2) + R'^2 \frac{\partial p'}{\partial z'} = 2\beta R' \left[\frac{\partial v_z'}{\partial r'}\right]_{R'} \tag{12}$$

and

$$\frac{\partial}{\partial z'}(R'^2 U') + 2R' \frac{\partial R'}{\partial t'} = 0. \tag{13}$$

In terms of dimensional quantities these equations are

$$\frac{\partial U}{\partial t} + \frac{U}{A}(1 - \alpha)\frac{\partial A}{\partial t} + \alpha U \frac{\partial U}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{2\nu}{R}\left[\frac{\partial v_z}{\partial r}\right]_R \tag{14}$$

and

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z}(UA) = 0 \tag{15}$$

where A is the cross-sectional area of the tube. Note that the integration over r has introduced new nonlinearities in the dependent variables U and A, but has eliminated the velocity component in the r direction. The mean velocity is the quantity measured in most experiments.

It is now necessary to know the variation of v_z with r (the "velocity profile") in order to evaluate α and $(\partial v_z/\partial r)_R$. This variation cannot be properly determined without carrying out a full two-dimensional nonlinear treatment of the problem. In this paper the Polhausen approach will be followed: profiles will be assumed which satisfy the boundary conditions $(\partial v_z/\partial r)_{r=0} = 0$ and $(v_z)_{r=R} = 0$ and the sensitivity of the results to the assumed profile will be investigated. A particularly simple assumption is the parabolic form

$$v_z = 2U(1 - r^2/R^2)$$

in which case $\alpha = 4/3$ and $(\partial v_z/\partial r)_R = -4\ U/R$.
The right-hand side of equation 14 may then be written as $-\gamma U/A$, where $\gamma = 8\ \pi\nu = 0.251\ \text{cm}^2/\text{sec}$ for water at room temperature.
It should be noted that the fact that γ is a constant is a property of the particularly simple profile assumed.

The relation of the present work to that of Streeter et al. (6) can now be seen. If the profile $v_z(r)$ = constant were assumed, although it would violate the no-slip condition at the wall, then $\alpha = 1$ and $\gamma = 0$. Since such a choice would remove friction effects, an empirical friction term such as $-FU^n/2D$ would have to be added. The equation 14 would become

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{FU^n}{2D} = 0$$

which is that used by Streeter et al. In the present work the friction term comes from the basic Navier-Stokes equations.

CONSIDERATION OF THE SYSTEM OF EQUATIONS

Quasi–one-dimensional flow of a fluid through a distensible tube will normally be described by a system of three equations: the equation of motion for the fluid, the continuity equation for the fluid, and the equation of motion of the wall. The third equation depends on the properties of the wall, which are not well known. If the approximation is used that the wall is thin and linearly elastic with Poisson's Ratio 1/2, the equation of motion for a wall tethered in the z direction is:

$$pR + \frac{4}{6}\,Eh_0\left[\left(\frac{\partial R}{\partial z}\right)^2 + (R - R_0)\frac{\partial^2 R}{\partial z^2} - 2\,\frac{(R - R_0)}{R_0}\right] = \rho_w h_0 R_0\,\frac{\partial^2 R}{\partial t^2} \qquad (16)$$

where E is Young's Modulus for the wall of thickness h and density ρ_w.

The nondimensional form of equation 16 is:

$$p'R'(\rho\,U_0^2R_0) - \frac{4}{3}\,Eh_0(R' - 1)$$
$$= \epsilon^2\left\{\rho_w h_0\,U_0\,\frac{\partial^2 R'}{\partial t'^2} + \frac{4}{6}\,Eh_0\left[\left(\frac{\partial R'}{\partial z'}\right)^2 + (R' - 1)\,\frac{\partial^2 R'}{\partial z'^2}\right]\right\}$$

Since the right-hand side of the previous equation is negligible if $\epsilon \ll 1$, in terms of dimensional quantities

$$p = \frac{4}{3}\,Eh_0(R - R_0)/RR_0 \simeq \frac{4}{3}\,E\left(\frac{h_0}{R_0}\right)\left(\frac{1}{2}\,\frac{\Delta A}{A_0}\right)$$

or

$$A = f(p) \tag{17}$$

The applicability of this solution is indicated by the results of Patel, Greenfield, and Fry (10) who show that the variations in the radius of different blood vessels follow variations in the pressure remarkably closely and without significant phase lag.

If the special solution (equation 17) to the wall equation is used, equations 14 and 15 may be written:

$$\frac{\partial U}{\partial t} + \alpha U \frac{\partial U}{\partial z} + (1 - \alpha) U \frac{f'(p)}{f(p)} \frac{\partial p}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial z} = -\frac{\gamma U}{f(p)} \tag{18}$$

and

$$f'(p) \frac{\partial p}{\partial t} + f(p) \frac{\partial U}{\partial z} + U f'(p) \frac{\partial p}{\partial z} = 0. \tag{19}$$

Note that the dependent variable A does not appear in equations 18 and 19, which are a pair of quasi-linear hyperbolic partial differential equations in the two dependent variables p and U. These equations may be solved numerically using the method of characteristics (7).

The characteristic curve in the $z - t$ plane (Fig. 1) has the property that along

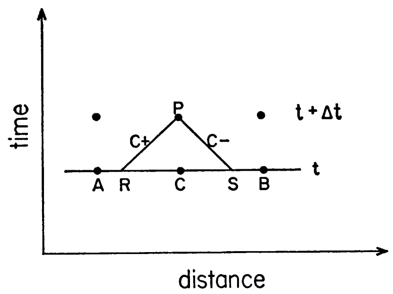

FIGURE 1 The time-distance plane. If the solutions are known at points A, B, and C (time t) they can be found at point P (time $t + \Delta t$) by integrating along the characteristic curves $C+$ and $C-$.

it the partial differential equations 18 and 19 are equivalent to an ordinary differential equation of the form

$$\frac{dU}{dt} + g_2 \frac{dp}{dt} = g_1. \tag{20}$$

For a hyperbolic system there are two such curves passing through every point in the $z - t$ plane. For the point p, at time $(t + \Delta t)$ the curves are indicated by $C+$ and $C-$. Then for the curve $C+$, equation 20 may be approximated by the finite-difference equation

$$(U_p - U_R) + g^{+}_{2_c}(p_P - p_R) = g^{+}_{1_c}\, \Delta t \tag{21}$$

and for $C-$:

$$(U_P - U_S) + \bar{g_{2}}_{*}(p_P - p_S) = \bar{g_{1}}_{*}\, \Delta t \quad (22)$$

where the subscripts indicate the (z, t) points at which the quantities are evaluated. Now if U and p are known at time t for the points R, C, and S, the equations 21 and 22 may be solved simultaneously for the quantities U_P and p_P, which are values at time $(t + \Delta t)$. Thus the solutions may be propagated forward in time.

Consider now the proximal boundary. If either p or U is specified as a function of time, then equation 22 is sufficient to find the other. Similarly equation 21 may be used if either p or U is specified at the distal boundary. A simple example is a closed tube, for which $U = 0$ for all time at the distal end.

More generally an equation specifying a pressure-flow relationship may be given at either end, and solved simultaneously with equations 21 or 22 to give p and U separately. Streeter et al. (6) used a "terminal bed condition" at the distal end. Another example is a pressure transducer placed at the distal end. If for applied pressure p the volume displacement of the transducer is $k_t p$, then

$$U_{\text{distal}} = (k_t/A)(\partial p/\partial t)_{\text{distal}}. \quad (23)$$

If the fluid and tube are assumed everywhere at rest at time zero, the problem is properly specified and may be solved numerically. Numerical results will be given in the following paper (11).

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