MODEL STUDIES OF THE MAGNETOCARDIOGRAM

FLAVIO GRYNSZPAN *and* DAVID B. GESELOWITZ

From the University of Pennsylvania, Philadelphia, Pennsylvania 19104, and the Pennsylvania State University, University Park, Pennsylvania 16802. Dr. Grynszpan's present address is the Coordenacão des Programas de Pos-Graduacão de Engenharia, Universidade Federaldo Rio de Janeiro, Rio de Janeiro, G. B., Brasil.

ABSTRACT A general expression is developed for the quasi-static magnetic field outside an inhomogeneous nonmagnetic volume conductor containing internal electromotive forces. Multipole expansions for both the electric and magnetic fields are derived. It is shown that the external magnetic field vanishes under conditions of axial symmetry. The magnetic field for a dipole current source in a sphere is derived, and the effect of an eccentric spherical inhomogeneity is analyzed. Finally the magnetic dipole moment is calculated for a current dipole in a conducting prolate spheroid.

INTRODUCTION

Several papers have appeared concerning the theory of magnetic fields outside a volume conductor containing internal sources of electricity relevant to studies in biomagnetics (1–4). In particular Baule and McFee have pointed out a number of aspects of such fields, which may be summarized as follows. (*a*) The magnetic field should provide information different from that available from the electric field. (*b*) The external field is zero for axisymmetric configurations. (*c*) The external field is much larger for a dipole with a tangential as opposed to a radial orientation. (*d*) The effects of the boundary of the volume conductor may be quite small. (*e*) Inhomogeneities such as the more highly conducting intracavitary blood mass and more poorly conducting lung tissue would tend to enhance the magnetic field arising from a tangential dipole source. The purpose of the present paper is to develop a general theory for the magnetic field external to an inhomogeneous volume conductor, including its multipolar representation, and, using mathematical models (sphere and spheroid), to explore further the effects of inhomogeneities and boundaries.

GENERAL THEORY

Let us represent bioelectric sources by an impressed current density $\mathbf{J}^i$. Then in a region of conductivity σ the current density

$$\mathbf{J} = -\sigma \nabla V + \mathbf{J}^i. \quad (1)$$

Since the bioelectric problem of interest is a quasi-static one,

$$\nabla \times \mathbf{H} = \mathbf{J} = -\sigma \nabla V + \mathbf{J}^i. \tag{2}$$

Let

$$\mathbf{H} = \nabla \times \mathbf{A}. \tag{3}$$

Then

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\sigma \nabla V + \mathbf{J}^i, \tag{4}$$

where

$$\nabla \cdot \mathbf{A} = -\sigma V, \tag{5}$$

$$\nabla^2 \mathbf{A} = -\mathbf{J}^i. \tag{6}$$

These equations may be interpreted to indicate that the electric field is related to the divergence of $\mathbf{J}^i$ while the magnetic field is related to the curl of $\mathbf{J}^i$. It is assumed that the volume conductor is nonmagnetic.

Solutions to Eqs. 5 and 6 in an unbounded homogeneous medium are:

$$4\pi \mathbf{A}(\mathbf{r}') = \int \frac{\mathbf{J}^i}{|\mathbf{r}' - \mathbf{r}|} \, \mathrm{d}v, \tag{7}$$

$$4\pi \sigma V(\mathbf{r}') = \int \mathbf{J}^i \cdot \nabla \frac{1}{|\mathbf{r}' - \mathbf{r}|} \, \mathrm{d}v. \tag{8}$$

We have shown previously (5) (see Appendix I) that for an inhomogeneous volume conductor the currents everywhere may be determined by adding appropriate sources on the surfaces separating regions of conductivity σ' and σ''. For a bounded inhomogeneous conductor the source distribution $\mathbf{J}^i$ must be replaced by

$$\mathbf{J}^{\mathrm{inh}} \, \mathrm{d}v = \mathbf{J}^i \, \mathrm{d}v - \sum_j (\sigma' - \sigma'') \, V \, \mathrm{d}\mathbf{S}_j - \sigma V \, \mathrm{d}\mathbf{S}_0 , \tag{9}$$

where the vector element of surface $\mathrm{d}\mathbf{S}$ is directed from the primed region to the double primed region, and S_0 is the external boundary which is surrounded by an insulator (air) for which $\sigma'' = 0$. Hence, for a bounded inhomogeneous conductor

$$4\pi \mathbf{A} = \int \frac{\mathbf{I}}{|\mathbf{r}' - \mathbf{r}|} \cdot \left[\mathbf{J}^i \, \mathrm{d}v - \sum_j (\sigma' - \sigma'') V \, \mathrm{d}\mathbf{S}_j - \sigma V \, \mathrm{d}\mathbf{S}_0\right] \tag{10}$$

where $\mathbf{I}$ is a unit dyadic.

For $r' > r$

$$\frac{\mathbf{I}}{|\mathbf{r}' - \mathbf{r}|} = Re \sum_{n=0}^{\infty} \sum_{m=0}^{n} \gamma_{nm} \frac{1}{n(n+1)} \mathbf{M}^1_{nm}(r, \theta, \phi) \overline{\mathbf{M}}^2_{nm}(r', \theta', \phi')$$

$$+\frac{1}{(n+1)(2n+1)}\frac{n-m+1}{n+m+1}\mathbf{N}^1_{nm}(r,\theta,\phi)\,\bar{\mathbf{G}}^2_{nm}(r',\theta',\phi')$$

$$+\frac{1}{n(2n+1)}\frac{n+m}{n-m}\mathbf{G}^1_{nm}(r,\theta,\phi)\,\bar{\mathbf{N}}^2_{nm}(r',\theta',\phi'), \quad (11)$$

where

$$\gamma_{nm} = (2-\delta^0_m)\frac{(n-m)!}{(n+m)!}. \quad (12)$$

The functions $\mathbf{M}_{nm}$, $\mathbf{N}_{nm}$, $\mathbf{G}_{nm}$ are defined in Appendix II, and the bar over the letter indicates the complex conjugate (6). Since $\mathbf{N}^2_{nm}$ has zero divergence and curl, it does not contribute to either the electric field or the magnetic field. From Eq. 5 and Appendix II, it follows that for sources in an unbounded homogeneous conductor

$$4\pi\sigma V = Re\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{a_{nm}+ib_{nm}}{(r')^{n+1}}P^m_n(\cos\theta')e^{-im\phi'}, \quad (13)$$

where

$$a_{nm}+ib_{nm} = \gamma_{nm}\int \mathbf{J}^i\cdot\mathbf{N}^1_{n-1,m}\,\mathrm{d}v, \quad (14)$$

are the multipole coefficients of the source. For the general case $\sigma = 0$ outside the volume conductor. Hence if we choose r' outside the volume conductor, then from Eqs. 9, 13, and 14

$$\gamma_{nm}\int \sigma V\mathbf{N}^1_{n-1,m}\cdot\mathrm{d}\mathbf{S}_0 = \gamma_{nm}\int\Big[\mathbf{J}^i\,\mathrm{d}v - \sum_j(\sigma'-\sigma'')V\,\mathrm{d}\mathbf{S}_j\Big]\cdot\mathbf{N}^1_{n-1,m}. \quad (15)$$

In the homogeneous case the right-hand side of Eq. 15 is just $a_{nm}+ib_{nm}$. In the general case the right-hand side can be interpreted as the multipole coefficients of an equivalent generator that would give rise to the same potential on the surface S_0 of an isomorphic homogeneous conductor.

Analogously,

$$4\pi\mathbf{A} = Re\sum\sum\gamma_{nm}\frac{\bar{\mathbf{M}}^2_{nm}(r',\theta',\phi')}{n(n+1)}\int\mathbf{J}^{inh}\cdot\mathbf{M}^1_{nm}(r,\theta,\phi)\,\mathrm{d}v$$

$$+\gamma_{n+1,m}\frac{\bar{\mathbf{G}}^2_{nm}(r',\theta',\phi')}{(2n+1)(n+1)}\int\mathbf{J}^{inh}\cdot\mathbf{N}^1_{nm}(r,\theta,\phi)\,\mathrm{d}v. \quad (16)$$

The last term in Eq. 16 vanishes as a consequence of Eq. 15. Therefore, from Eq. 3

$$4\pi\mathbf{H} = Re\sum\sum\gamma_{nm}\frac{\bar{\mathbf{N}}^2_{n+1,m}}{n+1}\int\mathbf{J}^{inh}\cdot\mathbf{M}^1_{nm}\,\mathrm{d}v. \quad (17)$$

Since there are no currents outside the volume conductor, in this region

$$\mathbf{H} = -\nabla U, \tag{18}$$

and

$$4\pi U = Re \sum_{n=1}^{\infty} \sum_{m=0}^{n} \gamma_{nm} \frac{P_n^m(\cos\theta')e^{-im\phi'}}{(r')^{n+1}(n+1)} \int \mathbf{J}^{inh} \cdot \mathbf{M}_{nm}^1 \, dv. \tag{19}$$

Eq. 19 is precisely of the form of the multipole expansion. We can therefore define the magnetic multipole coefficients as

$$\alpha_{nm} + i\beta_{nm} = \frac{\gamma_{nm}}{n+1} \int \mathbf{N}_{n-1,m}^1 \cdot \mathbf{r} \times \left[\mathbf{J}^i \, dv - \sum_j (\sigma' - \sigma'')V \, d\mathbf{S}_j - \sigma V \, d\mathbf{S}_0\right]. \tag{20}$$

The magnetic dipole moment $\mathbf{m}$ is given by

$$\mathbf{m} = \tfrac{1}{2} \int \mathbf{r} \times \left[\mathbf{J}^i \, dv - \sum_j (\sigma' - \sigma'')V \, d\mathbf{S}_j - \sigma V \, d\mathbf{S}_0\right]. \tag{21}$$

From Eq. 15 the equivalent electrical dipole moment $\mathbf{p}$ is

$$\mathbf{p} = \int \sigma V \, d\mathbf{S}_0 = \int \mathbf{J}^i \, dv - \sum_j \int (\sigma' - \sigma'')V \, d\mathbf{S}_j. \tag{22}$$

It follows from Eq. 22 that $\mathbf{p}$ and $\mathbf{m}$ are independent of the origin chosen.

It is of interest to evaluate α_{nm}, β_{nm} from measurements of the external magnetic field. In principle, if $\mathbf{H}$ is known, then U can be determined, and α_{nm}, β_{nm} can then be found. From Green's theorem (see Appendix I),

$$4\pi U = \int \left[U\nabla \frac{1}{|\mathbf{r}' - \mathbf{r}|} + \frac{\mathbf{H}}{|\mathbf{r}' - \mathbf{r}|}\right] \cdot d\mathbf{S}_0. \tag{23}$$

Hence

$$\alpha_{nm} + i\beta_{nm} = \gamma_{nm} \int [r^n P_n^m(\cos\theta)e^{im\phi}\mathbf{H} + \mathbf{N}_{n-1,m}^1 U] \cdot d\mathbf{S}_0. \tag{24}$$

The magnetic dipole becomes

$$\mathbf{m} = \int U \, d\mathbf{S}_0 + \int \mathbf{r}\mathbf{H} \cdot d\mathbf{S}_0. \tag{25}$$

AXIAL SYMMETRY

Eq. 20 can be used to show that the magnetic field vanishes outside a conductor having axial symmetry for a radial current source. Choose the origin at a point on the axis. Then $\mathbf{J}^i = \mathbf{r}_1 J^i$, and $d\mathbf{S}_j = \mathbf{r}_1 dS_r + \boldsymbol{\theta}_1 dS_\theta$. Therefore $\alpha_{nm} + i\beta_{nm}$ is proportional to

$$\int_0^{2\pi} ime^{im\phi} V \, dS_\theta = 0.$$

In the case of a sphere with centric inhomogeneities, including shells, the terms involving $d\mathbf{S}_j$ and $d\mathbf{S}_0$ vanish. Hence the external field is identical with that of a homogeneous sphere. Since the external field vanishes for a radial dipole because of axial symmetry, only tangential dipoles will contribute. Note that the field will remain unchanged if the radius of the sphere is changed, if a spherical hole is cut in the center, if the center is made more conducting, etc.

DIPOLE IN SPHERE

From the above argument, only the tangential component of $\mathbf{J}^i$ will contribute to the external magnetic field surrounding a homogeneous sphere. Let us determine the field for a dipole located at a distance a from the center of the sphere and oriented in the x direction.

$$\mathbf{J}^i = \mathbf{i} p_x \delta(\theta - 0)\delta(r - a)\delta(\phi - 0), \tag{26}$$

$$\mathbf{i} = \mathbf{r}_1 \sin\theta\cos\phi + \boldsymbol{\theta}_1 \cos\theta\cos\phi - \boldsymbol{\phi}_1 \sin\phi. \tag{27}$$

From Eq. 20

$$\alpha_{nm} + i\beta_{nm} = \frac{\gamma_{nm}}{n+1} \int \nabla[r^n P_n^m(\cos\theta)e^{im\phi}] \cdot a\boldsymbol{\phi}_1 p_x \delta(\theta - 0)\delta(\phi - 0)\delta(r - a)\, dv$$

$$= \lim_{\theta\to 0} \frac{\gamma_{nm} p_x a^n im}{n+1} \frac{P_n^m(\cos\theta)}{\sin\theta} = \lim_{\theta\to 0} \frac{\gamma_{nm} p_x a^n im}{n+1} \sin^{m-1}\theta \frac{d^m P_n}{d\theta^m}. \tag{28}$$

It follows from Eq. 28 that all coefficients vanish except β_{n1}, and that

$$\beta_{n1} = \frac{p_x a^n}{n+1}. \tag{29}$$

Note that β_{n1} is independent of the radius of the sphere. The magnetic scalar potential can be put in the following closed form as shown in Appendix III (7).

$$4\pi U = \frac{p_x \sin\phi'}{a\sin\theta'} \left[\frac{a\cos\theta' - r'}{(r'^2 - 2ar'\cos\theta' + a^2)^{1/2}} + 1\right], \tag{30}$$

and a closed form solution for $\mathbf{H}$ can be obtained by taking the gradient. Fig. 1

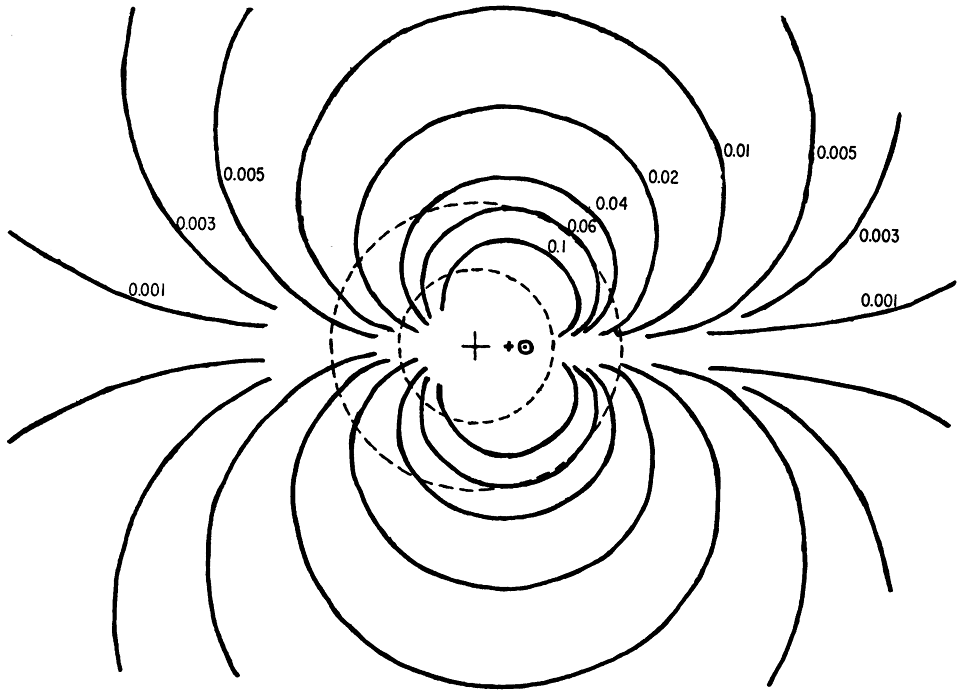

FIGURE 1 Magnetic scalar potential of current dipole in conducting sphere. Current dipole p indicated by $\odot$ is perpendicular to plane of drawing at distance a from center. Solid lines show isopotential contours *outside* any concentric sphere enclosing dipole. Broken lines show two such spheres. Potentials are negative in the upper half plane and positive below. A *magnetic* dipole of magnitude $pa/2$ located at + and pointing down will give virtually identical isopotential contours.

shows isopotential contours of U plotted in the plane $\phi' = 90°$ for $p/4\pi a$ equal to unity. If only the dipole term of the expansion is used,

$$4\pi U_D = \tfrac{1}{2}(p_x a/r'^2) \sin \phi' \sin \theta'. \quad (31)$$

The optimum location for this dipole is at a distance $\beta_{21}/\beta_{11} = 2a/3$ from the origin (8). If the magnetic dipole is located at this point the isopotential contours are virtually identical with the actual magnetic potential U (see Fig. 1). A plot of the magnetic field **H** is shown in Fig. 2.

It was shown that a centric inhomogeneity would not affect the external field. The effect of an eccentric inhomogeneity will now be analyzed. Consider a sphere of radius R with an internal sphere of radius b whose center is eccentric by a distance e along the z axis. The coordinate system x, y, z has its origin at the center of the external sphere and the coordinate system x', y', z' has its origin at the center of the internal sphere (see Fig. 3).

Consider for a moment the primed coordinate system and place a current source I at (0, 0, c) and a sink of the same magnitude at (0, 0, 0). Then if the conductivity

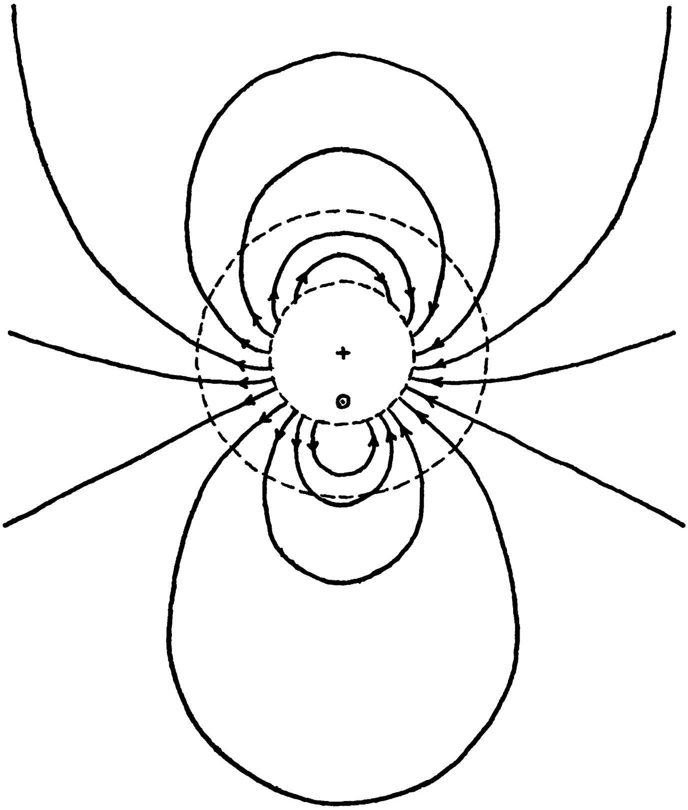

FIGURE 2 Magnetic field of current dipole in conducting sphere. See caption to Fig. 1.

of the inner sphere is σ_1 and of the outer sphere σ_2, where $\sigma_1 = k\sigma_2$,

$$4\pi\sigma_2 V = \begin{cases} I\sum_{n=1}^{\infty} \dfrac{2n+1}{kn+n+1}[B_n + c^{-(n+1)}]r'^n P_n(\cos\theta') \\ \qquad\qquad -\dfrac{1}{kr'} + \dfrac{1-k}{kb}, & 0 \le r' \le b \\ I\sum_{n=1}^{\infty} [B_n + c^{-(n+1)}]\left[r'^n + \dfrac{n(1-k)b^{2n+1}}{(kn+n+1)r'^{n+1}}\right] \\ \qquad\qquad \cdot P_n(\cos\theta') - \dfrac{1}{r'}, & b \le r' < c \\ I\sum_{n=1}^{\infty}\left(B_n r'^n + \dfrac{D_n}{r'^{n+1}}\right) P_n(\cos\theta'), & r' > c \end{cases} \tag{32}$$

where

$$D_n = \frac{n(1-k)b^{2n+1}}{kn+n+1}[B_n + c^{-(n+1)}] + c^n. \tag{33}$$

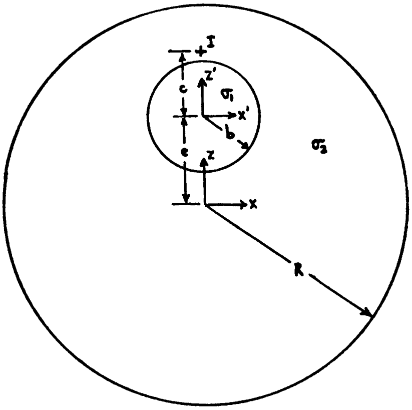

FIGURE 3 Geometry for sphere with eccentric spherical inhomogeneity.

V satisfies the boundary conditions of continuity of potential and of normal current density at $r' = b$ (9).

Let us now translate the origin from 0' to 0 so that the new coordinate system is x, y, z. One could then formally introduce the boundary condition that the normal component of V vanishes on the sphere $r = R$, determine D_n, and then determine B_n from Eq. 33. This solution has been worked out by Grynszpan (7). If, however, b/R is not very large then the potential on $r' = b$ is, to a very good approximation, unaffected by the insulating boundary at $r = R$. To simplify matters, let us use this approximation. If no boundary is present, then B_n must be 0 for V to be finite as $r' \rightarrow \infty$.

Therefore

$$4\pi\sigma_2 V(b) = I \sum \frac{2n+1}{kn+n+1} \left(\frac{b}{c}\right)^n \frac{1}{c} P_n(\cos\theta') - \frac{I}{b}. \qquad (34)$$

Following Geselowitz and Ishiwatari (9), we find for a dipole oriented in the x direction

$$V = \lim_{I\epsilon c \to p_x} \{\epsilon \sin\theta' \cos\phi' \cdot \partial V/\partial(\cos\theta')\}, \qquad (35)$$

$$4\pi\sigma_2 V = \frac{p_x}{b^2} \sum \frac{2n+1}{kn+n+1} \left(\frac{b}{c}\right)^{n+2} P_n^1(\cos\theta') \cos\phi'. \qquad (36)$$

The magnetic dipole moment can be found from Eq. 21. Note that

$$\mathbf{r} = \mathbf{r}' + ek', \qquad (37)$$

$$\mathbf{dS} = b^2 \sin\theta'(\mathbf{k}' \cos\theta' + \mathbf{i}' \sin\theta' \cos\phi' + \mathbf{j}' \sin\theta' \sin\phi')d\theta'\, d\phi', \qquad (38)$$

$$\mathbf{m} = \mathbf{j}\left[\frac{1}{2} p_x(c+e) - \frac{1}{2} p_x e \left(\frac{b}{c}\right)^3 \frac{k-1}{k+2}\right]. \qquad (39)$$

The first term on the right is the dipole moment for the homogeneous sphere. The second is the correction arising from the eccentric inhomogeneity, which tends to diminish the magnetic dipole moment for k greater than 1.

Let us use the results of the homogeneous spherical model to estimate the magnitude of the electric and magnetic fields. Consider first a centric dipole p in a sphere of radius $R = 10$ cm, and conductivity $\sigma = 0.2$ mho/m. The potential is given by $V = 4p \cos\theta/3\pi\sigma R^2$. Therefore the potential difference between an electrode at $\theta = 0°$ and one at $\theta = 90°$ is approximately $200p$. If p is of the order of 1 mA-cm then V is of the order of 2 mV.

To estimate the magnetic field, let R, σ, and p be the same, but let the dipole have an eccentricity $a = 2$ cm. Then for $\theta = 90°$,

$$U = \frac{p}{4\pi a}\left[1 - \frac{r}{(r^2 + a^2)^{1/2}}\right], \tag{40}$$

$$H(R) = -\frac{\partial U}{\partial r} = \frac{p}{4\pi}\frac{a}{(R^2 + a^2)^{3/2}}, \tag{41}$$

which is approximately $2/4\pi \times 10^{-4}$ A/m or 2×10^{-7} Oe.

DIPOLE IN SPHEROID

Consider a current dipole located at $(\xi_0, \eta_0, 0)$ in a homogeneous prolate spheroid $\xi = \xi_1$, whose foci are at $z = \pm c$. Let the dipole lie in the $\phi = 0$ plane and be oriented in the normal direction. From the derivation of Yeh and Martinek (10), the potential on the surface is

$$V = \sum_{n=1}^{\infty}\sum_{m=0}^{n} A_{nm} \cos m\phi P_n^m(\eta)\left[Q_n^m(\xi_1) - \frac{(\partial/\partial\xi)Q_n^m(\xi_1)}{(\partial/\partial\xi)P_n^m(\xi_1)} P_n^m(\xi_1)\right], \tag{42}$$

where p_ξ and p_η are the ξ and η components of the dipole and

$$A_{mn} = \frac{2n+1}{c}\gamma_{nm}\left[\frac{(n-m)!}{(n+m)!}\right]^2 \cdot \left\{\frac{p_\xi}{h_1} P_n^m(\eta_0)\frac{\partial}{\partial\xi}P_n^m(\xi_0) + \frac{p_\eta}{h_2}P_n^m(\xi_0)\frac{\partial}{\partial\eta}P_n^m(\eta_0)\right\}, \tag{43}$$

$$h_1 = c\left\{\frac{\xi_0^2 - \eta_0^2}{\xi_0^2 - 1}\right\}^{1/2}, \tag{44 a}$$

$$h_2 = c\left\{\frac{\xi_0^2 - \eta_0^2}{1 - \eta_0^2}\right\}^{1/2}; \tag{44 b}$$

We define a radial dipole as one oriented outwards along a line connecting it to the center of the spheroid. For such a dipole

$$\mathbf{p} = (pc/a)\{[(\xi_0^2 - 1)(1 - \eta_0^2)]^{1/2}\mathbf{i} + \eta_0\xi_0\mathbf{k}\}, \tag{45}$$

where a is the distance of the dipole from the center and p is the dipole moment.

$$a = c(\xi_0^2 + \eta_0^2 - 1)^{1/2}. \tag{46}$$

Conversion to spheroidal coordinates is accomplished using the relations

$$\mathbf{i} = (\xi_0 c/h_2)\boldsymbol{\xi}_1 - (\eta_0 c/h_1)\boldsymbol{\eta}_1, \tag{47}$$

$$\mathbf{k} = (\eta_0 c/h_1)\boldsymbol{\xi}_1 + (\xi_0 c/h_2)\boldsymbol{\eta}_1,$$

with the result

$$\mathbf{p} = \frac{pc}{a(\xi_0^2 - \eta_0^2)^{1/2}}[\xi_0(\xi_0^2 - 1)^{1/2}\boldsymbol{\xi}_1 + \eta_0(1 - \eta_0^2)^{1/2}\mathbf{n}_1]. \tag{48}$$

From Eq. 21

$$\mathbf{m} = -1/2\int \sigma V(\mathbf{r} \times \mathrm{d}\mathbf{S}), \tag{49}$$

$$\mathbf{r} \times \mathrm{d}\mathbf{S} = c^3\eta(\xi_1^2 - 1)^{1/2}(1 - \eta^2)^{1/2}(\cos\phi\mathbf{j} - \sin\phi\mathbf{i})\,\mathrm{d}\eta\,\mathrm{d}\phi. \tag{50}$$

Hence from Eqs. 42 and 49 and the orthogonality relations for the Legendre functions only the term A_{21} is nonzero, and

$$\mathbf{m} = -\mathbf{j}\frac{6pc^2}{a}\xi_0\eta_0(\xi_0^2 - 1)^{1/2}(1 - \eta_0^2)^{1/2}(\xi_1^2 - 1)^{1/2} \cdot \left[Q_2^1(\xi_1) - \frac{(\partial/\partial\xi)Q_2^1(\xi_1)}{(\partial/\partial\xi)P_2^1(\xi_1)}P_2^1(\xi_1)\right],$$

$$= \mathbf{j}\frac{12pc^2/a}{2\xi_1^2 - 1}\xi_0\eta_0[(\xi_0^2 - 1)(1 - \eta_0^2)]^{1/2}. \tag{51}$$

Therefore for ξ_1 much greater than 1, which gives a spheroid close to a sphere,

$$\mathbf{m} = \mathbf{j}\frac{6pa\eta_0\xi_0}{\xi_1^2}\frac{[(\xi_0^2 - 1)(1 - \eta_0^2)]^{1/2}}{\xi_0^2 + \eta_0^2 - 1}. \tag{52}$$

The ratio of the minor axis to the major axis of the spheroid is $(\xi_1^2 - 1)^{1/2}/\xi_1$. For $\xi_1 = 10$ this ratio is 0.995. Let $\xi_0 = 2$ and $\eta_0 = 0.5$. Then $m = 0.048pa$. In the sphere the radial current dipole would have zero magnetic dipole moment. If it were rotated to be tangential to the surface of the sphere, keeping p and a constant, the magnetic dipole moment would be a maximum of $0.5pa$. Hence a very slight change in the geometry of a sphere (0.5 %) results in the appearance of a magnetic dipole moment which is 10 % of this maximum. Another way of looking at this result is that the change to a spheroid is equivalent to the rotation of the current dipole source through an angle whose sine is 0.1, or 6°.

DISCUSSION

The results presented by Baule and McFee are derived largely from lead field theory, which is based on the reciprocity theorem (1). Many of the results obtained in the present paper can also be derived from lead field theory. According to this theory, the voltage V in a lead is given by

$$V = \int \mathbf{J}^i \cdot \mathbf{E}_L \, dv,$$

where $\mathbf{E}_L$ is the electric field in the volume conductor when the terminals of the lead are energized with a unit current i. In the present case the "lead" is a small single turn coil, V is proportional to dH/dt, and E_L is proportional to di/dt.

Assume that the geometry of the volume conductor possesses symmetry about an axis perpendicular to the coil and passing through its center. Then at a point in the volume conductor lying at a distance a from the axis and in a plane d below the coil,

$$\mathbf{E}_L = \boldsymbol{\phi}_1 E_L(a, d),$$

where $E_L(a, d)$ is independent of the geometry and inhomogeneities. This result follows from the fact that $E_L(a, d)$ is proportional to the rate of change of flux through the circle of radius a in this plane, and that the flux created by the reciprocally energized coil is independent of the geometry for a nonmagnetic volume conductor in the quasi-static case under consideration. The fact that the magnetic field outside an inhomogeneous sphere is not affected by centric inhomogeneities follows immediately, since for such a sphere all axes through the center are axes of symmetry.

A special case involving axial symmetry is given by Eq. 41 where R can be replaced by d. From the above argument this result is perfectly general for axisymmetric configurations. Baule and McFee previously derived the identical equation for a dipole in a slab (1), and presented a similar estimate of the magnitude of the magnetic field.

Baule and McFee investigated the effects of a finite slab and showed that the introduction of side boundaries has a small effect on the magnetic field. Furthermore, Eq. 41 shows that the magnetic field outside the slab is not affected by the thickness of the slab or the location of the surfaces in relation to the source. This result is, of course, implicit in their analysis, although Baule and McFee do not point it out explicitly. By the same token the magnetic field outside the sphere is independent of the radius of the sphere. It does, however, depend on the fact that there is a boundary present.

The fact that gross changes in the boundary which preserve the symmetry will not change the external magnetic field is a rather interesting result. If on the other hand, a rather small and subtle change which does destroy the symmetry is introduced, the magnetic field can be substantially altered. At least this conclusion can be drawn from the spheroid model studied here. On the other hand it is difficult to extrapolate this result to the torso without considering more realistic geometries. From Eqs. 17 and 9 it is possible to calculate the effect of the external boundary of the volume conductor. Since this calculation involves the surface electrocardiogram and geometry, it would be extremely difficult to accomplish with accuracy in a practical case.

Baule and McFee state that the magnetic field will increase when the source is in a more highly conducting medium, or when the heart is directly below the detector (1). Our results, in an arrangement with the same symmetry, show that the magnetic field will not depend on the relative conductivities. This discrepancy is explained by the fact that we have considered current dipoles while Baule and McFee have used voltage dipoles. $\mathbf{J}^i$ is the current dipole moment per unit volume. If we let $\mathbf{E}^i$ be the voltage dipole moment per unit volume, then $\mathbf{E}^i = \mathbf{J}^i/\sigma$, and (11)

$$V = \int \mathbf{J}^i \cdot \mathbf{E}_L \, dv = \int \sigma \mathbf{E}^i \cdot \mathbf{E}_L \, dv. \tag{53}$$

Since $\mathbf{E}_L$ is independent of conductivity for axisymmetric configurations, V and hence H will be constant for J^i constant, but will be proportional to σ for E^i constant.

Our results for the sphere show that the external field is unaffected by centric inhomogeneities, but that for an eccentric inhomogeneity the external field will be somewhat diminished for a tangential dipole near a more highly conducting region. The basic pattern of the external magnetic field is not strongly affected by the inhomogeneities. Baule and McFee, on the other hand, argue qualitatively that the more highly conducting blood mass and the more poorly conducting lung tissue will have a substantial effect on the pattern of the magnetic field and will tend to enhance the effect of tangential sources. The discrepancy apparently arises from the fact that they have considered a predominately planar distribution of current, while we have considered a volume distribution, particularly in a spherical geometry.

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APPENDIX I

We start with Green's theorem

$$\sum_j \int [\sigma'(\psi'\nabla V' - V'\nabla\psi') - \sigma''(\psi''\nabla V'' - V''\nabla\psi'')]\cdot dS_j$$
$$= \sum_i \int (\psi\nabla\cdot\sigma\nabla V - V\nabla\cdot\sigma\nabla\psi)\, dv_i.$$

The convention for primes and double primes follows that given in the text. The volume v_i is a region in which the conductivity σ is constant. S_j is a surface separating regions with conductivities σ' and σ''. In general the surface bounding v_i will consist of several such surfaces S_j.

Let

$$\psi = 1/\rho = |\mathbf{r}' - \mathbf{r}|^{-1}.$$

where $\mathbf{r}'$ designates a fixed observation point and $\mathbf{r}$ designates the variable coordinates of the terms in the two integrands. The normal component of current density and the potential must be continuous at all boundaries. Hence

$$[-\sigma'\nabla V' + \mathbf{j}^{i'}]\cdot d\mathbf{S}_j = [-\sigma''\nabla V'' + \mathbf{J}^{i''}]\cdot d\mathbf{S}_j,$$

$$V'(S_j) = V''(S_j).$$

Therefore, with the use of Eq. 2

$$\sum_j \int [(1/\rho)(\mathbf{J}^{i'} - \mathbf{J}^{i''}) - (\sigma' - \sigma'')V\nabla(1/\rho)]\cdot d\mathbf{S}_j$$
$$= \sum_i \int [(1/\rho)\nabla\cdot\mathbf{J}^i - V\sigma\nabla^2(1/\rho)]\, dv_i.$$

The two terms on the right are evaluated as follows

$$\int V\sigma\nabla^2(1/\rho)\,dv = -4\pi\sigma(\mathbf{r}')V(\mathbf{r}')$$

$$\sum_i \int (1/\rho)\nabla\cdot\mathbf{J}^i\,dv_i = \sum_i \int \nabla\cdot(\mathbf{J}^i/\rho) - \mathbf{J}^i\cdot\nabla(1/\rho)\,dv_i$$

$$= \sum_j \int (\mathbf{J}^{i'} - \mathbf{J}^{i''})(1/\rho)\cdot d\mathbf{S}_j - \sum_i \int \mathbf{J}^i\cdot\nabla(1/\rho)\,dv_i.$$

Hence

$$4\pi\sigma V = \int \mathbf{J}^i\cdot\nabla(1/\rho)\,dv - \sum_j \int (\sigma' - \sigma'')V\nabla(1/\rho)\cdot d\mathbf{S}_j.$$

Green's theorem can be written alternatively as follows

$$\int [(1/\rho)\nabla U - U\nabla(1/\rho)]\cdot d\mathbf{S} = \int [(1/\rho)\nabla^2 U - U\nabla^2(1/\rho)]\,dv.$$

If we let the volume integral be over all space outside the volume conductor and let U be the magnetic scalar potential then $\nabla^2 U = 0$, $d\mathbf{S} = -d\mathbf{S}_0$, $\mathbf{H} = -\nabla U$, and

$$4\pi U = \int [U\nabla(1/\rho) + (H/\rho)]\cdot d\mathbf{S}_0.$$

APPENDIX II

The following list shows the definition and properties of the sets **M**, **N**, and **G**. With

$$X_n^m(\theta, \phi) = P_n^m(\cos\theta)e^{im\phi},$$

$$\mathbf{M}_{nm}^1(r, \theta, \phi) = \nabla \times [\mathbf{r}\, r^n X_n^m(\theta, \phi)] = \nabla [r^n X_n^m(\theta, \phi)] \times \mathbf{r},$$

$$\mathbf{M}_{nm}^2(r, \theta, \phi) = \nabla \times [\mathbf{r}(1/r^{n+1})X_n^m(\theta, \phi)] = \nabla [(1/r^{n+1})X_n^m(\theta, \phi)] \times \mathbf{r},$$

$$\mathbf{N}_{nm}^1(r, \theta, \phi) = \nabla [r^{n+1}X_{n+1}^m(\theta, \phi)],$$

$$\mathbf{N}_{nm}^2(r, \theta, \phi) = \nabla [(1/r^n)X_{n-1}^m(\theta, \phi)],$$

$$\nabla \times N_{nm}^1 = \nabla \times N_{nm}^2 = \nabla\cdot\mathbf{N}_{nm}^1 = \nabla\cdot\mathbf{N}_{nm}^2 = \nabla\cdot\mathbf{M}_{nm}^1 = \nabla\cdot\mathbf{M}_{nm}^2 = 0,$$

$$\nabla\cdot\mathbf{G}_{nm}^2 = -(n+1)(2n+1)(1/r^{n+2})X_{n+1}^m(\theta, \phi),$$

$$\nabla \times \mathbf{M}_{nm}^2 = -n\,\mathbf{N}^2_{n+1,m},$$

$$\nabla \times \mathbf{G}_{nm}^2 = (2n+1)\,\mathbf{M}^2_{n+1,m}.$$

APPENDIX III

From Eqs. 19, 20, and 29

$$4\pi U = Re \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{\alpha_{nm} + i\beta_{nm}}{(r')^{n+1}} P_n^m (\cos \theta')e^{-im\phi'},$$

$$4\pi U = \frac{p_x \sin \phi'}{r'} \sum_{n=1}^{\infty} \left(\frac{a}{r'}\right)^n \frac{P_n^1 (\cos \theta')}{n + 1}. \qquad (A\ 1)$$

The reciprocal of the distance between points at distances a and r' from the origin can be expanded in Legendre polynominals as follows, with $\mu = \cos \theta'$,

$$[(r')^2 - 2a\mu r' + a^2]^{-1/2} = (1/r') \sum_{n=0}^{\infty} (a/r')^n P_n (\mu). \qquad (A\ 2)$$

If both sides of Eq. A 2 are differentiated with respect to μ

$$ar'[(r')^2 - 2a\mu r' + a^2]^{-3/2} = \frac{1}{(1 - \mu^2)^{1/2} r'} \sum_{n=0}^{\infty} \left(\frac{a}{r'}\right)^n P_n^1(\mu). \qquad (A\ 3)$$

Now integrate both sides of Eq. A 3 with respect to a between the limits of 0 and a. Then

$$\frac{a/r'}{(1 - \mu^2)^{1/2}} \sum_{n=1}^{\infty} \left(\frac{a}{r'}\right)^n \frac{P_n^1(\mu)}{n + 1} = \frac{a}{(1 - \mu^2)} \left\{\frac{\mu - r'/a}{[(r')^2 - 2a\mu r' + a^2]^{1/2}} + \frac{1}{a}\right\}. \qquad (A\ 4)$$

Substitution of Eq. A 4 into Eq. A 1 gives Eq. 30.