Supplementary Material – Activities and Sensitivities in Boolean Network Models

Ilya Shmulevich[1] and Stuart A. Kauffman[2,1]
[1]Cancer Genomics Laboratory
University of Texas M.D. Anderson Cancer Center
Houston, TX 77030
[2]Department of Cell Biology and Physiology
University of New Mexico Health Sciences Center
Albuquerque, NM 87131

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Abstract

In this appendix, we give a proof of the fact that the expected activity vector of a random canalizing function with one canalizing variable is equal to $E[\boldsymbol{\alpha}^f] = \left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{4}\right)$.

Let the symbols $\vee$ and $\wedge$ denote the Boolean disjunction and conjunction, respectively. Also, recall that $\wedge$ takes precedence over $\vee$ so that writing $a \vee b \wedge c$ is the same as $a \vee (b \wedge c)$. Let $f(x_1, \ldots, x_K)$ be a random canalizing function of the form

$$f(x_1, \ldots, x_K) = x_1 \vee g(x_2, \ldots, x_K),$$

where g is chosen randomly from the set of all $2^{2^{K-1}}$ Boolean functions. Without loss of generality, we are supposing that the first variable, x_1, is a canalizing variable. Furthermore, the discussion for other types of canalizing functions (e.g., $f(x_1, \ldots, x_K) = x_1 \wedge g(x_2, \ldots, x_K)$) would be nearly identical. Our first aim is to characterize the activities of each of the variables, which are also random variables themselves by virtue of f being random. It is clear that the activity of variables $x_2, \ldots, x_K$ should behave identically in the probabilistic sense if $g(x_2, \ldots, x_K)$ is a random unbiased function. Consequently, it will suffice to examine the activity of variable x_2, with the other variables behaving identically.

Let us first compute α_1^f – the activity of x_1 in f. Firstly, we have

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= (0 \vee g(x_2, \ldots, x_K)) \oplus (1 \vee g(x_2, \ldots, x_K)) \\ &= g(x_2, \ldots, x_K) \oplus 1 \\ &= g'(x_2, \ldots, x_K).\end{aligned}$$

Now, since g is a random unbiased function (i.e. $p = 1/2$), the expected activity of the canalizing variable x_1 is equal to

$$\begin{aligned}E\left[\alpha_1^f\right] &= E[2^{-(K-1)} \cdot \sum_{\mathbf{x}\in\{0,1\}^{K-1}} g'(x_2, \ldots, x_K)] \\ &= 2^{-(K-1)} \cdot \sum_{\mathbf{x}\in\{0,1\}^{K-1}} E[g'(x_2, \ldots, x_K)] \\ &= 2^{-(K-1)} \cdot \sum_{\mathbf{x}\in\{0,1\}^{K-1}} \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

Now let us consider the expected activity of variable x_2. We have

$$\begin{aligned}\frac{\partial f}{\partial x_2} &= \left(x_1 \vee g\left(\mathbf{x}^{(2,0)}\right)\right) \oplus \left(x_1 \vee g\left(\mathbf{x}^{(2,1)}\right)\right) \\ &= \left(x_1 \vee g\left(\mathbf{x}^{(2,0)}\right)\right) \wedge \left(x_1 \vee g\left(\mathbf{x}^{(2,1)}\right)\right)' \\ &\quad \vee \left(x_1 \vee g\left(\mathbf{x}^{(2,0)}\right)\right)' \wedge \left(x_1 \vee g\left(\mathbf{x}^{(2,1)}\right)\right) \\ &= \left(x_1 \vee g\left(\mathbf{x}^{(2,0)}\right)\right) \wedge \left(x_1' \wedge g'\left(\mathbf{x}^{(2,1)}\right)\right) \\ &\quad \vee \left(x_1' \wedge g'\left(\mathbf{x}^{(2,0)}\right)\right) \wedge \left(x_1 \vee g\left(\mathbf{x}^{(2,1)}\right)\right) \\ &= x_1' \wedge g\left(\mathbf{x}^{(2,0)}\right) \wedge g'\left(\mathbf{x}^{(2,1)}\right) \vee x_1' \wedge g'\left(\mathbf{x}^{(2,0)}\right) \wedge g\left(\mathbf{x}^{(2,1)}\right) \\ &= x_1' \wedge \left(g\left(\mathbf{x}^{(2,0)}\right) \oplus g\left(\mathbf{x}^{(2,1)}\right)\right) \\ &= x_1' \wedge \frac{\partial g}{\partial x_2},\end{aligned}$$

where in the second equality we used the fact that $a \oplus b = a \wedge b' \vee a' \wedge b$, in the third equality we used de Morgan's identity: $(a \vee b)' = a' \wedge b'$, in the fifth equality we again used the definition of $\oplus$, and in the last equality, we used the definition of partial derivative. The expected activity of variable x_2 is now equal to

$$E\left[\alpha_2^f\right] = E[2^{-(K-1)} \cdot \sum_{\mathbf{x} \in \{0,1\}^{K-1}} x_1' \wedge \frac{\partial g}{\partial x_2}].$$

Note that $\frac{\partial g(x_2,\ldots,x_K)}{\partial x_2}$ is a Boolean function of $K-2$ variables and the above summation is taken over all $\mathbf{x} = (x_1, x_3, \ldots, x_K)$. Let us break up this summation into parts, corresponding to $x_1 = 0$ and $x_1 = 1$. We have

$$\begin{aligned} E\left[\alpha_2^f\right] &= 2^{-(K-1)} \cdot \left[\sum_{\mathbf{x}^{(1,0)} \in \{0,1\}^{K-1}} E[1 \wedge \frac{\partial g}{\partial x_2}] + \sum_{\mathbf{x}^{(1,1)} \in \{0,1\}^{K-1}} E[0 \wedge \frac{\partial g}{\partial x_2}]\right] \\ &= 2^{-(K-1)} \cdot \sum_{\mathbf{x}^{(1,0)} \in \{0,1\}^{K-1}} E[\frac{\partial g}{\partial x_2}]. \end{aligned}$$

Since g is a random unbiased function, so is $\frac{\partial g}{\partial x_2}$. This essentially means that the probability that a random function g differs on $\mathbf{x}^{(j,0)}$ and $\mathbf{x}^{(j,1)}$ is equal to 1/2. Thus,

$$E\left[\alpha_2^f\right] = 2^{-(K-1)} \cdot \sum_{\mathbf{x}^{(1,0)} \in \{0,1\}^{K-1}} \frac{1}{2}$$

and since there are exactly 2^{K-2} different vectors $\mathbf{x}^{(1,0)} = (0, x_3, \ldots, x_K)$,

$$E\left[\alpha_2^f\right] = 2^{-(K-1)} \cdot \frac{1}{2} \cdot 2^{K-2} = \frac{1}{4}.$$

Thus, the expected activity of all non-canalizing variables is equal to 1/4. The expected activity vector is then equal to $E[\boldsymbol{\alpha}^f] = \left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{4}\right)$ and the expected sensitivity is equal to $E\left[s\left(f\right)\right] = \frac{1}{2} + \frac{1}{4} \cdot (K-1) = (K+1)/4$.