# 1 Definitions

We have considered 3 structures: The cortex (abbreviated $cx$), the cerebellum (abv. $cb$) and the remaining area (abv. $ra$). The scaling laws obtained apply to each separately, and in each case the relevant variables are denoted by the superscripts $^{cx}$, $^{cb}$ or $^{ra}$. The calculations below apply to all cases, and for simplicity we omit the superscripts. We shall use the following definitions:

$M$ Total mass of the structure
$M_n$ Total mass of neurons in the structure
$M_{nn}$ Total mass of non-neuronal cells in the structure
$m_n$ Average mass of neurons in the structure
$m_{nn}$ Average mass of non-neuronal cells in the structure
$N$ Total number of cells in the structure
$N_n$ Total number of neurons in the structure
$N_{nn}$ Total number of non-neuronal cells in the structure

## 2 The power laws for $m_n$ and $m_{nn}$

It has been shown experimentally that the total mass of each structure $M$ can be expressed as a power law of both $N_n$ and $N_n$

$$M = k_n N_n^{\alpha_n} \ , \tag{1a}$$
$$M = k_{nn} N_{nn}{}^{\alpha_{nn}} \ . \tag{1b}$$

Combining both expressions above one can relate the number of neurons to the number of non-neuronal cells

$$N_{nn} = \left(\frac{k_n}{k_{nn}}\right)^{\frac{1}{\alpha_{nn}}} N_n^{\frac{\alpha_n}{\alpha_{nn}}} \ . \tag{2}$$

The total mass $M$ can be decomposed into its neuronal and non-neuronal components

$$M = M_n + M_{nn} \ ,$$
$$M_n = N_n m_n \ , \tag{3a}$$
$$M_{nn} = N_{nn} m_{nn} \ . \tag{3b}$$

We shall assume that $m_n$ e $m_{nn}$ for different brain sizes also conform to power laws of $N_n$ and $N_{nn}$ respectively

$$m_n = h_n N_n^{\beta_n} \ , \tag{4a}$$
$$m_{nn} = h_{nn} N_{nn}^{\beta_{nn}} \ , \tag{4b}$$

where $h_n$, $h_{nn}$, $\beta_n$ and $\beta_{nn}$ are to be determined. By combining (3) and (4) we then get

$$M_n = h_n N_n^{\beta_n + 1} \ , \tag{5a}$$
$$M_{nn} = h_{nn} N_{nn}^{\beta_{nn}+1} \ , \tag{5b}$$
$$M = h_n N_n^{\beta_n+1} + h_{nn} N_{nn}^{\beta_{nn}+1} \ . \tag{5c}$$

To obtain the values of $\beta_n$ and $\beta_{nn}$ we express $M$ and $N_{nn}$ as functions of $\alpha_n$, $\alpha_{nn}$ and $N_n$, by substituting (1a) and (2) into (5c).

$$k_n N_n^{\alpha_n} = h_n N_n^{\beta_n+1} + h_{nn} \left(\frac{k_n}{k_{nn}}\right)^{\frac{\beta_{nn}+1}{\alpha_{nn}}} N_n^{\frac{\alpha_n}{\alpha_{nn}}(\beta_{nn}+1)} \ . \tag{6}$$

For non-zero $h_n$, $h_{nn}$, $k_n$ and $k_{nn}$, the equation above will not hold for all values of $N_n$ unless the exponents are equal, i.e.

$$\beta_n + 1 = \alpha_n \ ,$$
$$\frac{\alpha_n}{\alpha_{nn}}(\beta_{nn} + 1) = \alpha_n \ .$$

And thus the unknown exponents $\beta_n$ and $\beta_{nn}$ can be expressed in terms of $\alpha_n$and $\alpha_{nn}$

$$\beta_n = \alpha_n - 1 \ , \tag{7a}$$
$$\beta_{nn} = \alpha_{nn} - 1 \ . \tag{7b}$$

Note that we cannot fully determine $m_n$ and $m_{nn}$ without another independent measurement, since $h_n$ and $h_{nn}$ are unknown. But by substituting (7) into (6), cancelling the common $N_n^{\alpha_n}$ terms and dividing by $k_n$ we can relate the coefficients $h_n$, $h_{nn}$, $k_n$ and $k_{nn}$

$$\frac{h_n}{k_n} + \frac{h_{nn}}{k_{nn}} = 1 \ . \tag{8}$$

## 3 The ratio $\frac{M_n}{M_{nn}}$

Let $r$ be the ration between the total mass of neurons and the total mass of non-neuronal cells in the structure, $r = M_n/M_{nn}$. We use (3) to write

$$r = \frac{M_n}{M_{nn}} \ ,$$
$$r = \frac{h_n N_n^{\beta_n+1}}{h_{nn} N_{nn}^{\beta_{nn}+1}} \ . \tag{9}$$

Substituting the formulas (2) e (7) in the expression above we get

$$r = \frac{h_n N_n^{\alpha_n}}{h_{nn}\left[\left(\frac{k_n}{k_{nn}}\right)^{\frac{1}{\alpha_{nn}}} N_n^{\frac{\alpha_n}{\alpha_{nn}}}\right]^{\alpha_{nn}}} \ ,$$
$$r = \frac{h_n k_{nn}}{h_{nn} k_n} \ . \tag{10}$$

This ratio does not depend on the mass of the brain or the number of cells and is therefore constant for all brains as long as the power laws (1) and (4) remain valid.

# 4 The power law for $M(N)$

Since we know as an experimental fact that $M$ can be expressed as a power law of both $N_n$ and $N_{nn}$, it is not hard to relate $M$ to the total number of cells $N$. Indeed, inverting (1) we get

$$N_n = \left(\frac{M}{k_n}\right)^{\frac{1}{\alpha_n}} , \tag{11a}$$

$$N_{nn} = \left(\frac{M}{k_{nn}}\right)^{\frac{1}{\alpha_{nn}}} . \tag{11b}$$

Recalling that $N = N_n + N_{nn}$, we obtain

$$N = \left(\frac{M}{k_n}\right)^{\frac{1}{\alpha_n}} + \left(\frac{M}{k_{nn}}\right)^{\frac{1}{\alpha_{nn}}} . \tag{12}$$

The formula above is not a power law. Since the inverse of a power law is also a power law (with inverse exponent), clearly $M(N)$ cannot be in general a power law either. However, since $\alpha_n > \alpha_{nn}$ and $k_n < k_{nn}$ it follows that as $M$ becomes small $N_n$ becomes much larger than $N_{nn}$, and likewise as $M$ becomes large $N_{nn} >> N_n$. More precisely,

$$\lim_{M\to 0} M = k_n N^{\alpha_n} , \tag{13a}$$

$$\lim_{M\to \infty} M = k_{nn} N^{\alpha_{nn}} . \tag{13b}$$

For most of the brains studied, and for each of the structures considered, one kind of cell clearly predominates. For the cerebellum, neurons dominate, comprising about 70% to 90% of the total number of cells, while for the cortex and remaining areas the non-glial cells are 15% to 60% and 12% to 50% respectively of the total number of cells. It then follows that when we plot $M$ as a function of $N$ it to a very good approximation conforms to a power law of the type $M \propto N^{\alpha}$, where the exponent $\alpha$ is very similar to $\alpha_n$ in the case of the cerebellum, and somewhat close to $\alpha_{nn}$ in the case of the cortex and the remaining areas. The deviation in the latter case is due to the fact that non-neuronal cells are not overwhelmingly more abundant in either structure. Nonetheless, the function $M(N)$ can still be well approximated by a power law since the difference between $\alpha_n$ and $\alpha_{nn}$ (which is the difference of the slopes of the asymptotes) is not too great.