Supporting Text

1 Full Derivation of the Systematic Perturbation Expansion

1.1 Notation

To shorten the notation, we write $da(x,t) = r(x,t)dt + d\eta(x,t)$, where

$$r(x,t) \;=\; f(\mathcal{D} \star a)(x,t) - \mu a(x,t) - \alpha(\mathcal{C} \star a)(x,t)a(x,t),$$

so that we may write $ds(x,t) = da(x,t) - dq(t) = r(x,t)dt - dq(t) + d\eta(x,t)$. We shorten the notation further by writing $a_i = a(x_i,t)$, $s_i = s(x_i,t)$, $ds_i = ds(x_i,t)$, $d\eta_i = d\eta(x_i,t)$ and $r_i = r(x_i,r)$. Furthermore, as we need not consider time lags while deriving the perturbation expansion, we will drop the time t from the arguments of all functions.

As described above, we denote by G_n the nth-order central moment,

$$G_n(x_1, \ldots x_n) \;=\; E[s_1 \cdots s_n],$$

and by $G_n^*(x_1, \ldots x_n)$ the nth-order residual moment, which can be considered as the novel part of the nth-order moment. We denote by Z_n the nth-order raw moment

$$Z_n(x_1, \ldots x_n) \;=\; E[a_1 \cdots a_n].$$

As none of the moments depend on the absolute position, we may set the first spatial coordinate, e.g., to zero without loss of generality. To do so, we define the moments g_n, g_n^*, and z_n by

$$\begin{aligned}
G_n(x_1, \ldots x_n) &= g_n(x_2 - x_1, \ldots x_n - x_1), \\
G_n^*(x_1, \ldots x_n) &= g_n^*(x_2 - x_1, \ldots x_n - x_1), \\
Z_n(x_1, \ldots x_n) &= z_n(x_2 - x_1, \ldots x_n - x_1).
\end{aligned}$$

We note that as the arguments of the moments with capital letters can be freely permuted, so can the ones with small letters. Also, the sign of any argument can be changed, if the argument is at the same time subtracted from the other arguments, so that, e.g.,

$$g_n(x_1, \ldots x_{n-1}) \quad = \quad g_n(-x_1, x_2 - x_1, \ldots x_{n-1} - x_1).$$

For functions with several arguments, we denote the convolution with respect to the first argument simply by $\star$, so that, e.g.,

$$(\mathcal{D} \star G_2)(x_1, x_2) = \int C(x_1 - x)G(x, x_2)dx.$$

We note the identities

$$\begin{aligned}
(\mathcal{D} \star G_2)(x_1, x_2) &= (\mathcal{D} \star g_2)(x_1 - x_2), \\
(\mathcal{D} \star G_3)(x_1, x_2, x_3) &= (\mathcal{D} \star g_3)(x_1 - x_2, x_3 - x_2), \\
(\mathcal{D} \star G_4)(x_1, x_2, x_3, x_4) &= (\mathcal{D} \star g_4)(x_1 - x_2, x_3 - x_2, x_4 - x_2).
\end{aligned}$$

1.2 The Covariance of the Noise

To assess the covariance of the noise, we consider a discretization of the domain to sites of size dx. Assuming that a site has at time t k individuals (as dx is small, we either have $k = 0$ or $k = 1$), Table 1 lists the possible events that may take place before time $t+dt$. As the noise $d\eta$ is independent at different sites and times, we obtain $E[d\eta(x)d\eta(x', t')] = V(t)\delta(x-x')\delta(t-t')dt$, where

$$V(t) \quad = \quad (f + \mu)q(t) + \alpha q(t)^2 + R(t). \tag{1}$$

1.3 The Decompositions of the Central Moments

We start by deriving Eqs. **4-6** in the main text that decompose the central moments into a sum of the lower-order moments and the residual moment. The decompositions are constructed in such a way that the residual moments satisfy two conditions, which we call conditions (C1) and (C2). The condition (C1) requires that the residual moment is smooth, so that it does not contain any delta distributions. The condition (C2) requires that the residual moment vanishes if the maximum distance between any of the points diverges.

1.3.1 The second moment

The second raw moment can be written as

$$Z_2(x_1, x_2) \quad = \quad Z_2^*(x_1, x_2) + q\delta(x_1 - x_2), \tag{2}$$

where

$$Z_2^*(x_1, x_2) \quad = \quad q^2 + G_2^*(x_1, x_2)$$

refers to the case in which the two points do not coincide. To verify condition (C1), we consider a discretization of the domain to sites of size dx, where dx is so small that each site can be assumed to contain at most one individual. Thus, a site at x_i has an individual with probability qdx, in which case $a_i = 1/dx$, and is empty $(a_i = 0)$ with probability $1 - qdx$. If $x_1 = x_2$, we have $E(a_1 a_2) = qdx(1/dx)^2$. As $(1/dx)$ translates at the limit $dx \to 0$ to a delta distribution, we obtain $Z_2(x_1, x_2) = q\delta(x_1 - x_2)$. To see that the condition (C2) holds, we note that if $|x_1 - x_2| \to \infty$, then a_1 and a_2 become independent, and we should have $Z_2(x_1, x_2) \to q^2$, which is true with Eq. **2**. As $Z_2(x_1, x_2) = q^2 + G_2(x_1, x_2)$, we obtain Eq. **4** in the main text.

1.3.2 The third moment

The third raw moment can be written as

$$\begin{aligned} Z_3(x_1, x_2, x_3) \quad &= \quad Z_3^*(x_1, x_2, x_3) + \Sigma_{\{i,j,k\} \in P} Z_2^*(x_i, x_k)\delta(x_i - x_j) \\ &+ \quad q\delta(x_1 - x_2)\delta(x_1 - x_3), \end{aligned} \tag{3}$$

where

$$\begin{aligned} Z_3^*(x_1, x_2, x_3) \quad &= \quad q^3 + q\Sigma_{\{i,j,k\} \in P} G_2^*(x_i, x_j) \\ &+ \quad G_3^*(x_1, x_2, x_3) \end{aligned} \tag{4}$$

refers to the case in which none of the three points coincide. To see that condition (C1) holds, we need to consider two cases, as either all of the points or two of the points may coincide. First, if all the three points coincide, the reasoning goes exactly as above, and we obtain $Z_3(x_1, x_2, x_3) = q\delta(x_1 - x_2)\delta(x_1 - x_3)$. There are three permutations (P) by which two of the points can coincide. For example, if $x_1 = x_2 \neq x_3$, we have $Z_3(x_1, x_2, x_3) = (q/dx)E[a_3|a_1 = 1/dx]$. To assess the conditional expectation, we note that

$$
\begin{aligned}
Z_2^*(x_1, x_3) &= E[a_1 a_3] \\
&= E[a_3|a_1 = 0]0P(a_1 = 0) + E[a_3|a_1 = 1/dx](1/dx)P(a_1 = 1/dx) \\
&= qE[a_3|a_1 = 1/dx],
\end{aligned}
$$

and thus $E[a_3|a_1 = 1/dx] = Z_2^*(x_1, x_3)/q$, which gives $Z_3(x_1, x_2, x_3) = (1/dx)Z_2^*(x_1, x_3) \rightarrow Z_2^*(x_1, x_3)\delta(x_1 - x_2)$.

To verify condition (C2), we may now assume that the three points are disjoint. If e.g. x_1 diverges from the other points ($|x_1 - x_i| \rightarrow \infty$ for $i = 2, 3$), we should have $Z_3^*(x_1, x_2, x_3) \rightarrow qZ_2^*(x_2, x_3) = q^3 + qG_2^*(x_2, x_3)$, as is the case with Eq. **3** if $G_3^*(x_1, x_2, x_3) \rightarrow 0$. Note that x_i diverging includes also the possibility that all of the points diverge, in which case $Z_3(x_1, x_2, x_3) \rightarrow q^3$. As

$$
Z_3(x_1, x_2, x_3) = q^3 + q\Sigma_{\{i,j,k\}\in P} G_2(x_i, x_j) + G_3(x_1, x_2, x_3),
$$

we obtain Eq. **5** in the main text.

1.3.3 The fourth moment

The fourth raw moment can be written as

$$
\begin{aligned}
Z_4(x_1, x_2, x_3, x_4) &= Z_4^*(x_1, x_2, x_3, x_4) \\
&+ q\delta(x_1 - x_2)\delta(x_1 - x_3)\delta(x_1 - x_4)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \Sigma_{\{i,j,k,l\}\in P_4} Z_2^*(x_i, x_j)\delta(x_j - x_k)\delta(x_j - x_l) \\
&+ \quad \Sigma_{\{i,j,k,l\}\in P_6} Z_3^*(x_i, x_k, x_l)\delta(x_i - x_j) \\
&+ \quad \Sigma_{\{i,j,k,l\}\in P_3} Z_2^*(x_i, x_k)\delta(x_i - x_j)\delta(x_k - x_l), \qquad (5)
\end{aligned}
$$

where

$$
\begin{aligned}
Z_4^*(x_1, x_2, x_3, x_4) &= q^4 + q\Sigma_{\{i,j,k,l\}\in P_4} G_3^*(x_j, x_k, x_l) \\
&+ \Sigma_{\{i,j,k,l\}\in P_3} G_2^*(x_i, x_j) G_2^*(x_k, x_l) \\
&+ q^2 \Sigma_{\{i,j,k,l\}\in P_6} G_2^*(x_i, x_j) \\
&+ G_4^*(x_1, x_2, x_3, x_4) \qquad (6)
\end{aligned}
$$

again refers to the case in which none of the points coincide. To see that condition (C1) holds, we need to consider four possibilities. First, if all the four points coincide, we obtain $Z_4(x_1, x_2, x_3, x_4) = q\delta(x_1 - x_2)\delta(x_1 - x_3)\delta(x_1 - x_4)$. Second, three of the points may coincide, for which option there are P_4 permutations. For example, if $x_1 \neq x_2 = x_3 = x_4$, we have $Z_4(x_1, x_2, x_3, x_4) = (q/dx^2)E[a_1|a_2 = 1/dx] = (1/dx^2)Z_2^*(x_1, x_2) \to Z_2^*(x_1, x_2)\delta(x_2 - x_3)\delta(x_2 - x_4)$. Third, two of the points may coincide, while the other two are separate. For this option the relevant permutations are given by P_6. For example, if $x_1 = x_2$ while $x_3 \neq x_4$, $x_3 \neq x_1$ and $x_4 \neq x_1$, we have $Z_4(x_1, x_2, x_3, x_4) = (q/dx)E[a_3a_4|a_1 = 1/dx]$. To assess the conditional expectation, we note that

$$
\begin{aligned}
Z_3^*(x_1, x_3, x_4) &= E[a_1a_3a_4] \\
&= E[a_3a_4|a_1 = 0]0P(a_1 = 0) + E[a_3a_4|a_1 = 1/dx](1/dx)P(a_1 = 1/dx) \\
&= qE[a_3a_4|a_1 = 1/dx].
\end{aligned}
$$

Thus, $Z_4(x_1, x_2, x_3, x_4) = (1/dx)Z_3^*(x_1, x_3, x_4) \to \delta(x_1 - x_2)Z_3^*(x_1, x_3, x_4)$. The fourth alternative is that there are two pairs of coinciding points, for which option the relevant permutations are given by P_3. For example, if $x_1 = x_2 \neq x_3 = x_4$, we have $Z_4(x_1, x_2, x_3, x_4) =$

$(q/dx)E[a_3^2|a_1 = 1/dx]$. To asses the conditional expectation, we note that

$$E[a_3|a_1 = 1/dx] \quad = \quad P[a_3 = 0|a_1 = 1/dx]0 + P[a_3 = 1/dx|a_1 = 1/dx](1/dx),$$

and thus

$$P[a_3 = 1/dx|a_1 = 1/dx] \quad = \quad dxE[a_3|a_1 = 1/dx] = (dx/q)Z_2^*(a_1, a_3),$$

which gives

$$\begin{aligned} E[a_3^2|a_1 = 1/dx] &= P[a_3 = 1/dx|a_1 = 1/dx](1/dx)^2 \\ &= (1/q)(1/dx)Z_2^*(a_1, a_3). \end{aligned}$$

Thus, $Z_4(x_1, x_2, x_3, x_4) = (1/dx)^2 Z_2^*(a_1, a_3) \rightarrow Z_2^*(a_1, a_3)\delta(x_1 - x_2)\delta(x_3 - x_4)$.

To see that condition (C2) holds, we may now assume that none of the points coincide. There are two alternatives by which the maximum distance between the points may diverge, as either one point may diverge from all the other points, or then a pair of points may diverge from another pair of points. First, assume that the distance between a single point (say x_1, there are P_4 permutations) and the other points diverges (not restricting whether the three remaining points stay close to each other or diverge from each other). We should have $Z_4(x_1, x_2, x_3, x_4)^* \rightarrow qZ_3^*(x_2, x_3, x_4)$, which is consistent with Eq. **5**. Second, it may be that two pairs of points diverge from each other (not restricting whether the points within the pairs stay close to each other or not). For example, if x_1 and x_2 diverge from x_3 and x_4 (there are P_3 permutations), we should have $Z_4(x_1, x_2, x_3, x_4)^* \rightarrow Z_2^*(x_1, x_2)Z_2^*(x_3, x_4)$, again consistent with Eq. **5**. As

$$\begin{aligned} Z_4(x_1, x_2, x_3, x_4) &= q^4 + q^2\Sigma_{\{i,j,k,l\}\in P_6} G_2(x_i, x_j) \\ &+ q\Sigma_{\{i,j,k,l\}\in P_4} G_3(x_j, x_k, x_l) + G_4(x_1, x_2, x_3, x_4), \end{aligned}$$

as

$$\Sigma_{\{i,j,k,l\}\in P_4} G_3(x_j, x_k, x_l) \quad = \quad q\Sigma_{\{i,j,k,l\}\in P_4} \delta(x_j - x_k)\delta(x_j - x_l)$$

$$
\begin{aligned}
&+ \quad \Sigma_{\{i,j,k,l\}\in P_4} G_3^*(x_j, x_k, x_l) \\
&+ \quad \Sigma_{\{i,j,k,l\}\in P_6} [G_2^*(x_i, x_k) + G_2^*(x_i, x_l)]\delta(x_i - x_j),
\end{aligned}
$$

and as

$$
\begin{aligned}
& \Sigma_{\{i,j,k,l\}\in P_6} Z_3^*(x_i, x_k, x_l)\delta(x_i - x_j) \\
= \quad & q^3 \Sigma_{\{i,j,k,l\}\in P_6} \delta(x_i - x_j) \\
+ \quad & \Sigma_{\{i,j,k,l\}\in P_6} G_3^*(x_i, x_k, x_l)\delta(x_i - x_j) \\
+ \quad & q\Sigma_{\{i,j,k,l\}\in P_6} [G_2^*(x_i, x_k) + G_2^*(x_i, x_l) + G_2^*(x_k, x_l)]\delta(x_i - x_j),
\end{aligned}
$$

we obtain Eq. **6** in the main text.

1.4 The Exact Equations for Central Moments

1.4.1 Mean

Using the notation of Eq. **8** in the main text,

$$
da = [f(\mathcal{D} \star a) - \mu a - \alpha(\mathcal{C} \star a)a]dt + d\eta,
$$

from which we get

$$
\begin{aligned}
dq &= E[da(x)] \\
&= E[f(\mathcal{D} \star a) - \mu a - \alpha(\mathcal{C} \star a)a]dt,
\end{aligned}
$$

and thus the exact equation for first moment (Eq. **1** in the main text) is

$$
\frac{dq}{dt} = (f - \mu)q - \alpha q^2 - R,
$$

where

$$
\begin{aligned}
R &= \alpha E[(\mathcal{C} \star s)(x)s(x)] \\
&= \alpha(\mathcal{C} \star G_2)(x, x) \\
&= \alpha(\mathcal{C} \star g_2)(0).
\end{aligned}
$$

1.4.2 Second moment

We have

$$\begin{aligned} dG_2(x_1,x_2) &= G_2(x_1,x_2,t+dt) - G_2(x_1,x_2) \\ &= E[(s_1+ds_1)(s_2+ds_2)] - G_2(x_1,x_2) \\ &= E[s_1 ds_2] + E[s_2 ds_1] + E[ds_1 ds_2]. \end{aligned}$$

First,

$$\begin{aligned} E[s_1 ds_2] &= E[s_1 r_2 dt] \\ &= E[s_1(f(\mathcal{D}\star a)(x_2) - \mu a(x_2) - \alpha(\mathcal{C}\star a)(x_2)a(x_2))]dt. \end{aligned}$$

Thus,

$$\begin{aligned} E[s_1 ds_2]/dt &= f(\mathcal{D}\star G_2)(x_2,x_1) - \mu G_2(x_1,x_2) \\ &- \alpha q[G_2(x_1,x_2) + (\mathcal{C}\star G_2)(x_2,x_1)] - R_2(x_1,x_2), \end{aligned}$$

where

$$\begin{aligned} R_2(x_1,x_2) &= \alpha E[s(x_1)s(x_2)(\mathcal{C}\star s)(x_2)] \\ &= \alpha(\mathcal{C}\star G_3)(x_2,x_2,x_1) \\ &= \alpha(\mathcal{C}\star g_3)(0,x_1-x_2). \end{aligned}$$

Second,

$$E[ds_1 ds_2] = E[d\eta_1 d\eta_2] = V\delta(x_1-x_2)dt.$$

Thus,

$$\begin{aligned} \frac{dG_2(x_1,x_2)}{dt} &= f[(\mathcal{D}\star G_2)(x_1,x_2) + (\mathcal{D}\star G_2)(x_2,x_1)] - 2\mu G_2(x_1,x_2) \\ &- 2\alpha q G_2(x_1,x_2) - \alpha q[(\mathcal{C}\star G_2)(x_1,x_2) + (\mathcal{C}\star G_2)(x_2,x_1)] \\ &+ V\delta(x_1-x_2) - R_2(x_1,x_2) - R_2(x_2,x_1). \end{aligned}$$

Accounting for the symmetry, we obtain

$$\begin{aligned}\frac{dg_2(x)}{dt} &= 2f(\mathcal{D}\star g_2)(x) - 2\mu g_2(x) - 2\alpha q[g_2(x) + (\mathcal{C}\star g_2)(x)] \\ &+ V\delta(x) - 2r_2(x),\end{aligned}$$

where

$$\begin{aligned}r_2(x) &= R_2(x', x + x') \\ &= \alpha(\mathcal{C}\star g_3)(0, x).\end{aligned}$$

1.4.3 Third moment

We have

$$\begin{aligned}dG_3(x_1, x_2, x_3) &= G_3(x_1, x_2, x_3, t + dt) - G_3(x_1, x_2, x_3) \\ &= E[(s_1 + ds_1)(s_2 + ds_2)(s_3 + ds_3)] - G(x_1, x_2, x_3) \\ &= \Sigma_{\{i,j,k\}\in P} E[s_i s_j ds_k] + \Sigma_{\{i,j,k\}\in P} E[ds_i ds_j s_k] \\ &+ E[ds_1 ds_2 ds_3].\end{aligned}$$

First,

$$\begin{aligned}E[s_i s_j ds_k] &= E[s_i s_j r_k]dt - dqE[s_i s_j] \\ &= E[s_i s_j r_k]dt - dqG_2(x_i, x_j).\end{aligned}$$

We have

$$\begin{aligned}E[s_i s_j r_k] &= (fq - \mu q - \alpha q^2)G(x_i, x_j) \\ &+ fE[s_i s_j(\mathcal{D}\star s)(x_k)] - (\mu + \alpha q)G_3(x_i, x_j, x_k) \\ &- \alpha qE[s_i s_j(\mathcal{C}\star s)(x_k)] - R_3(x_i, x_j, x_k) \\ &= (fq - \mu q - \alpha q^2)G_2(x_i, x_j)\end{aligned}$$

$$
\begin{aligned}
&+ \quad f(\mathcal{D}\star G_3)(x_k,x_i,x_j)-(\mu+\alpha q)G_3(x_i,x_j,x_k) \\
&- \quad \alpha q(\mathcal{C}\star G_3)(x_k,x_i,x_j)-R_3(x_i,x_j,x_k).
\end{aligned}
$$

where

$$
\begin{aligned}
R_3(x_i,x_j,x_k) &= \alpha E[s_i s_j s_k(\mathcal{C}\star s)(x_k)] \\
&= \alpha(\mathcal{C}\star G_4)(x_k,x_i,x_j,x_k).
\end{aligned}
$$

We write this as

$$
\begin{aligned}
E[s_i s_j r_k] &= (fq-\mu q-\alpha q^2)g_2(x_j-x_i) \\
&+ f(\mathcal{D}\star g_3)(x_k-x_i,x_j-x_i)-(\mu+\alpha q)g_3(x_j-x_i,x_k-x_i) \\
&- \alpha q(\mathcal{C}\star g_3)(x_k-x_i,x_j-x_i)-r_3(x_j-x_i,x_k-x_i),
\end{aligned}
$$

where

$$
r_3(x_A,x_B) = \alpha(\mathcal{C}\star g_4)(x_B,x_A,x_B).
$$

Second,

$$
E[ds_i ds_j s_k] = E[d\eta_i d\eta_j s_k]=E[V(x_j)s_k]\delta(x_i-x_j)dt,
$$

where

$$
V(x) = f(\mathcal{D}\star a)(x)+\mu a(x)+\alpha(\mathcal{C}\star a)(x)a(x),
$$

and we have used the fact that $E[d\eta_i d\eta_j s_k]=0$ if $x_j\neq x_i$. We have

$$
\begin{aligned}
E[V(x_j)s_k] &= fE[s_k(\mathcal{D}\star s)(x_j)]+(\mu+\alpha q)G_2(x_k,x_j) \\
&+ \alpha qE[s_k(\mathcal{C}\star s)(x_j)]+R_2(x_k,x_j) \\
&= f(\mathcal{D}\star G_2)(x_j,x_k)+(\mu+\alpha q)G_2(x_k,x_j)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \alpha q(\mathcal{C} \star G_2)(x_j, x_k) + R_2(x_k, x_j) \\
&= \quad f(\mathcal{D} \star g_2)(x_j - x_k) + (\mu + \alpha q) g_2(x_j - x_k) \\
&+ \quad \alpha q(\mathcal{C} \star g_2)(x_j - x_k) + r_2(x_j - x_k).
\end{aligned}
$$

Finally,

$$
E[ds_1 ds_2 ds_3] \quad = \quad dq\delta(x_1 - x_2)\delta(x_1 - x_3).
$$

Combining the above components and accounting for symmetry gives

$$
\begin{aligned}
\frac{dg_3(x_1, x_2)}{dt} \quad &= \quad \Sigma_{\{x_A, x_B\} \in P'}[(fq - \mu q - \alpha q^2) g_2(x_B) \\
&+ \quad f(\mathcal{D} \star g_3)(x_A, x_B) - (\mu + \alpha q) g_3(x_A, x_B) \\
&- \quad \alpha q(\mathcal{C} \star g_3)(x_A, x_B) - r_3(x_B, x_A) - \frac{dq}{dt} g_2(x_B)] \\
&+ \quad \Sigma_{\{x_A, x_B\} \in P'}[f(\mathcal{D} \star g_2)(x_A) + (\mu + \alpha q) g_2(x_A) \\
&+ \quad \alpha q(\mathcal{C} \star g_2)(x_A) + r_2(x_A)]\delta(x_B) \\
&+ \quad \frac{dq}{dt}\delta(x_1)\delta(x_2) \\
&= \quad \Sigma_{\{x_A, x_B\} \in P'}[f(\mathcal{D} \star g_3)(x_A, x_B) - (\mu + \alpha q) g_3(x_A, x_B) \\
&- \quad \alpha q(\mathcal{C} \star g_3)(x_A, x_B) - r_3(x_B, x_A) + R g_2(x_B)] \\
&+ \quad \Sigma_{\{x_A, x_B\} \in P'}[f(\mathcal{D} \star g_2)(x_A) + (\mu + \alpha q) g_2(x_A) \\
&+ \quad \alpha q(\mathcal{C} \star g_2)(x_A) + r_2(x_A)]\delta(x_B) \\
&+ \quad \frac{dq}{dt}\delta(x_1)\delta(x_2),
\end{aligned}
$$

where $P' = \{(x_2, x_1), (x_1, x_2), (x_1, x_1 - x_2)\}$.

1.5 The Exact Equations for the Residual Moments

To transform the differential equations for the central moments g_2 and g_3 to corresponding equations for the residual moments g_2^* and g_3^*, we use the equations

$$
\begin{aligned}
g_2(x) &= q\delta(x) + g_2^*(x), \\
g_3(x_1, x_2) &= q\delta(x_1)\delta(x_2) + g_3^*(x_1, x_2) + \Sigma_{\{x_A, x_B\} \in P'} g_2^*(x_A)\delta(x_B), \\
g_4(x_1, x_2, x_3) &= q\delta(x_1)\delta(x_2)\delta(x_3) + g_4^*(x_1, x_2, x_3) \\
&+ \Sigma_{\{x_A, x_B, x_C\} \in P_4'} g_2^*(x_A)\delta(x_B)\delta(x_C) \\
&+ \Sigma_{\{x_A, x_B, x_C\} \in P_3'} [q^2 + g_2^*(x_A)]\delta(x_B)\delta(x_C) \\
&+ \Sigma_{\{x_A, x_B, x_C\} \in P_6'} [q g_2^*(x_A) + g_3^*(x_A, x_B)]\delta(x_C) \\
&+ \Sigma_{\{x_A, x_B, x_C\} \in P_3'} g_2^*(x_B) g_2^*(x_C),
\end{aligned}
$$

where

$$
\begin{aligned}
P_3' &= \{\{x_2, x_1, x_3 - x_2\}, \{x_1, x_2, x_3 - x_1\}, \{x_1, x_3, x_2 - x_1\}\}, \\
P_4' &= \{\{x_1, x_2, x_3\}, \{x_2, x_1, x_3\}, \{x_3, x_1, x_2\}, \{x_1, x_2 - x_1, x_3 - x_1\}\}, \\
P_6' &= \{\{x_3 - x_2, -x_2, x_1\}, \{x_3 - x_1, -x_1, x_2\}, \{x_2 - x_1, -x_1, x_3\}, \{x_3, x_1, x_2 - x_1\}, \\
&\quad \{x_2, x_1, x_3 - x_1\}, \{x_1, x_2, x_3 - x_2\}\}.
\end{aligned}
$$

1.5.1 Mean

We have

$$
\frac{dq}{dt} = (f - \mu)q - \alpha q^2 - R,
$$

where

$$
\begin{aligned}
R &= \alpha(\mathcal{C} \star g_2)(0) \\
&= \alpha[q\mathcal{C}(0) + (\mathcal{C} \star g_2^*)(0)].
\end{aligned}
$$

1.5.2 Second moment

We obtain

$$\begin{aligned}\frac{dg_2^*(x)}{dt} &= 2f(\mathcal{D}\star g_2^*)(x) - 2\mu g_2^*(x) - 2\alpha q[g_2^*(x) + (\mathcal{C}\star g_2^*)(x)] \\ &+ 2fq\mathcal{D}(x) - 2\alpha q^2\mathcal{C}(x) + 2R\delta(x) - 2r_2(x).\end{aligned}$$

As

$$r_2(x) = \alpha(\mathcal{C}\star g_3)(0,x) = R\delta(x) + r_2^*(x),$$

where

$$r_2^*(x) = \alpha[(\mathcal{C}\star g_3^*)(0,x) + \mathcal{C}(x)g_2^*(x) + \mathcal{C}(0)g_2^*(x)],$$

we obtain

$$\begin{aligned}\frac{dg_2^*(x)}{dt} &= 2f(\mathcal{D}\star g_2^*)(x) - 2\mu g_2^*(x) - 2\alpha q[g_2^*(x) + (\mathcal{C}\star g_2^*)(x)] \\ &+ 2fq\mathcal{D}(x) - 2\alpha q^2\mathcal{C}(x) - 2r_2^*(x).\end{aligned}$$

1.5.3 Third moment

We have

$$\begin{aligned}\frac{dg_3^*(x_1,x_2)}{dt} &= \Sigma_{\{x_A,x_B\}\in P'}\{[f(\mathcal{D}\star g_3)(x_A,x_B) - (\mu+\alpha q)g_3(x_A,x_B) \\ &- \alpha q(\mathcal{C}\star g_3)(x_A,x_B) - r_3(x_B,x_A) + Rg_2(x_B)] \\ &+ [f(\mathcal{D}\star g_2)(x_A) + (\mu+\alpha q)g_2(x_A) \\ &+ \alpha q(\mathcal{C}\star g_2)(x_A) + r_2(x_A) - \frac{dg_2^*(x_A)}{dt}]\delta(x_B)\}.\end{aligned} \tag{7}$$

We note that

$$r_3(x_B,x_A)$$

$$
\begin{aligned}
&= \alpha(\mathcal{C}\star g_4)(x_A,x_B,x_A)\\
&= \alpha q\mathcal{C}(0)\delta(x_B)\delta(x_A)+\alpha(\mathcal{C}\star g_4^*)(x_A,x_B,x_A)\\
&+ \alpha(\mathcal{C}\star g_2^*)(0)\delta(x_B)\delta(x_A)+\alpha\mathcal{C}(0)g_2^*(x_B)\delta(x_A)\\
&+ \alpha\mathcal{C}(x_A)g_2^*(x_A)\delta(x_B)+\alpha\mathcal{C}(0)g_2^*(x_A)\delta(x_B-x_A)\\
&+ \alpha q^2[\mathcal{C}(x_A)\delta(x_A-x_B)+\mathcal{C}(0)\delta(x_B)+\mathcal{C}(x_B)\delta(x_A)]\\
&+ \alpha[\mathcal{C}(x_B)g_2^*(x_B)\delta(x_B-x_A)+\mathcal{C}(0)g_2^*(x_A)\delta(x_B)+\mathcal{C}(x_B)g_2^*(x_B)\delta(x_A)]\\
&+ \alpha q[\mathcal{C}(x_A)g_2^*(x_A-x_B)+\mathcal{C}(x_B-x_A)g_2^*(x_A)+\mathcal{C}(0)g_2^*(x_B)]\\
&+ \alpha q[(\mathcal{C}\star g_2^*)(x_B)\delta(x_A)+(\mathcal{C}\star g_2^*)(0)\delta(x_B)+(\mathcal{C}\star g_2^*)(x_A)\delta(x_A-x_B)]\\
&+ \alpha[\mathcal{C}(x_A)g_3^*(x_A,x_B)+(\mathcal{C}\star g_3^*)(0,x_A)\delta(x_B)+(\mathcal{C}\star g_3^*)(0,x_B)\delta(x_A)]\\
&+ \alpha[\mathcal{C}(x_A-x_B)g_3^*(x_A,x_B)+\mathcal{C}(0)g_3^*(x_A,x_B)+(\mathcal{C}\star g_3^*)(0,x_A)\delta(x_A-x_B)]\\
&+ \alpha[(\mathcal{C}\star g_2^*)(x_A)g_2^*(x_A-x_B)+(\mathcal{C}\star g_2^*)(0)g_2^*(x_B)+(\mathcal{C}\star g_2^*)(x_B-x_A)g_2^*(x_A)],
\end{aligned}
$$

where we have used the fact that

$$
(\mathcal{C}\star g_3^*)(x,x) \;=\; (\mathcal{C}\star g_3^*)(0,x).
$$

We also note that

$$
(\mathcal{F}\star g_2)(x_A) \;=\; q\mathcal{F}(x_A)+(\mathcal{F}\star g_2^*)(x_A),
$$

that

$$
\begin{aligned}
(\mathcal{F}\star g_3)(x_A,x_B) &= q\mathcal{F}(x_A)\delta(x_B)+(\mathcal{F}\star g_3^*)(x_A,x_B)\\
&+ \mathcal{F}(x_A)g_2^*(x_B)+\mathcal{F}(x_A-x_B)g_2^*(x_B)+(\mathcal{F}\star g_2^*)(x_A)\delta(x_B),
\end{aligned}
$$

and that

$$
\begin{aligned}
g_3(x_A,x_B) &= q\delta(x_A)\delta(x_B)+g_3^*(x_A,x_B)\\
&+ g_2^*(x_B)\delta(x_A)+g_2^*(x_A)\delta(x_B)+g_2^*(x_A)\delta(x_A-x_B).
\end{aligned}
$$

We write Eq. **7** as

$$\frac{dg_3^*(x_1,x_2)}{dt} = J\delta(x_1)\delta(x_2) + I_1(x_2)\delta(x_1) + I_2(x_1)\delta(x_2) + I_{12}(x_1)\delta(x_1-x_2) + \text{smooth part},$$

where the coefficients of the delta distributions should vanish. To see this is the case, we note that

$$\begin{aligned} J &= -3q(\mu+\alpha q) - 3\alpha q\mathcal{C}(0) - 3\alpha(\mathcal{C}\star g_2^*)(0) + 3q(\mu+\alpha q) + 3R \\ &= 0, \end{aligned}$$

and that $I_1(x) = I_2(x) = I_{12}(x) = I(x)$, where

$$\begin{aligned} I(x) &= f[q\mathcal{D}(x) + (\mathcal{D}\star g_2^*)(x)] - 3(\mu+\alpha q)g_2^*(x) \\ &- \alpha q^{\mathcal{C}}(x) - \alpha q(\mathcal{C}\star g_2^*)(x) \\ &- 2\alpha\mathcal{C}(0)g_2^*(x) - \alpha\mathcal{C}(x)g_2^*(x) - \alpha q^2\mathcal{C}(0) - 2\alpha q^2\mathcal{C}(x) \\ &- 2\alpha\mathcal{C}(x)g_2^*(x) - \alpha\mathcal{C}(0)g_2^*(x) - \alpha q(\mathcal{C}\star g_2^*)(0) - 2\alpha q(\mathcal{C}\star g_2^*)(x) \\ &- 3\alpha(\mathcal{C}\star g_3^*)(0,x) + \alpha q[q\mathcal{C}(0) + (\mathcal{C}\star g_2^*)(0)] \\ &+ fq\mathcal{D}(x) + f(\mathcal{D}\star g_2^*)(x) + (\mu+\alpha q)g_2^*(x) \\ &+ \alpha q^2\mathcal{C}(x) + \alpha q(\mathcal{C}\star g_2^*)(x) \\ &+ \alpha(\mathcal{C}\star g_3^*)(0,x) + \alpha\mathcal{C}(x)g_2^*(x) + \alpha\mathcal{C}(0)g_2^*(x) \\ &- 2f(\mathcal{D}\star g_2^*)(x) + 2\mu g_2^*(x) + 2\alpha q[g_2^*(x) + (\mathcal{C}\star g_2^*)(x)] \\ &- 2fq\mathcal{D}(x) + 2\alpha q^2\mathcal{C}(x) \\ &+ 2\alpha[(\mathcal{C}\star g_3^*)(0,x) + \mathcal{C}(x)g_2^*(x) + \mathcal{C}(0)g_2^*(x)] \\ &= 0. \end{aligned}$$

Combining the smooth terms we finally get

$$\frac{dg_3^*(x_1,x_2)}{dt}$$

$$
\begin{aligned}
&= \Sigma_{\{x_A,x_B\}\in P'}\{f(\mathcal{D}\star g_3^*)(x_A,x_B) + f\mathcal{D}(x_A)g_2^*(x_B) + f\mathcal{D}(x_A-x_B)g_2^*(x_B)\\
&- (\mu+\alpha q)g_3^*(x_A,x_B) - \alpha q(\mathcal{C}\star g_3^*)(x_A,x_B) - \alpha q\mathcal{C}(x_A)g_2^*(x_B) - \alpha q\mathcal{C}(x_A-x_B)g_2^*(x_B)\\
&- \alpha(\mathcal{C}\star g_4^*)(x_A,x_B,x_A) - \alpha q[\mathcal{C}(x_A)g_2^*(x_A-x_B) + \mathcal{C}(x_B-x_A)g_2^*(x_A) + \mathcal{C}(0)g_2^*(x_B)]\\
&- \alpha[\mathcal{C}(x_A)g_3^*(x_A,x_B) + \mathcal{C}(x_A-x_B)g_3^*(x_A,x_B) + \mathcal{C}(0)g_3^*(x_A,x_B)]\\
&- \alpha[(\mathcal{C}\star g_2^*)(x_A)g_2^*(x_A-x_B) + (\mathcal{C}\star g_2^*)(0)g_2^*(x_B) + (\mathcal{C}\star g_2^*)(x_B-x_A)g_2^*(x_A)]\\
&+ Rg_2^*(x_B)\\
&= \Sigma_{\{x_A,x_B\}\in P'}\{f(\mathcal{D}\star g_3^*)(x_A,x_B) - (\mu+\alpha q)g_3^*(x_A,x_B) - \alpha q(\mathcal{C}\star g_3^*)(x_A,x_B)\\
&+ f[\mathcal{D}(x_A)+\mathcal{D}(x_A-x_B)]g_2^*(x_B) - 2\alpha q[\mathcal{C}(x_A)+\mathcal{C}(x_A-x_B)]g_2^*(x_B)\\
&- \alpha[(\mathcal{C}\star g_2^*)(x_A) + (\mathcal{C}\star g_2^*)(x_A-x_B)]g_2^*(x_B) - r_3^*(x_A,x_B)\},
\end{aligned}
$$

where

$$
\begin{aligned}
r_3^*(x_A,x_B) &= \alpha(\mathcal{C}\star g_4^*)(x_A,x_B,x_A)\\
&+ \alpha[\mathcal{C}(x_A)g_3^*(x_A,x_B) + \mathcal{C}(x_A-x_B)g_3^*(x_A,x_B) + \mathcal{C}(0)g_3^*(x_A,x_B)].
\end{aligned}
$$

1.6 The Fourier Transformed Exact Equations

To work out the systematic perturbation equation from the exact equations, it is convenient to transform the equations to the Fourier domain. To do so, we denote by $\tilde{\cdot}$ the Fourier transform over all spatial coordinates, so that

$$
\begin{aligned}
\tilde{g}_2^*(\omega) &= \int g_2^*(x)e^{-2\pi i\omega x}dx,\\
\tilde{g}_3^*(\omega_1,\omega_2) &= \int\int g_3^*(x_1,x_2)e^{-2\pi i(\omega_1x_1+\omega_2x_2)}dx_2dx_1,\\
\tilde{g}_4^*(\omega_1,\omega_2,\omega_3) &= \int\int\int g_4^*(x_1,x_2,x_3)e^{-2\pi i(\omega_1x_1+\omega_2x_2+\omega_3x_3)}dx_3dx_2dx_1.
\end{aligned}
$$

We note the symmetries $\tilde{g}_3^*(\omega_1,\omega_2) = \tilde{g}_3^*(\omega_2,\omega_1) = \tilde{g}_3^*(\omega_1+\omega_2,-\omega_2)$.

1.6.1 First moment

For the first moment we obtained

$$\frac{dq}{dt} \quad = \quad (f-\mu)q - \alpha q^2 - R,$$

where we write R now as

$$R \quad = \quad \alpha[q\mathcal{C}(0) + \int \tilde{\mathcal{C}}(\omega)\tilde{g}_2^*(\omega)]d\omega.$$

1.6.2 Second moment

For the second moment we obtain

$$\frac{d\tilde{g}_2^*(\omega)}{dt} \quad = \quad 2\tilde{A}(\omega)\tilde{g}_2^*(\omega) + 2fq\tilde{\mathcal{D}}(\omega) - 2\alpha q^2\tilde{\mathcal{C}}(\omega) - 2\tilde{r}_2^*(\omega),$$

where

$$\tilde{A}(\omega,t) \quad = \quad f\tilde{\mathcal{D}}(\omega) - \mu - \alpha q(t)[1+\tilde{\mathcal{C}}(\omega)], \tag{8}$$

and

$$\tilde{r}_2^*(\omega) \quad = \quad \alpha\left[\int_{\omega'} \tilde{\mathcal{C}}(\omega')\tilde{g}_3^*(\omega',\omega)d\omega' + (\tilde{\mathcal{C}}\star\tilde{g}_2^*)(\omega) + \mathcal{C}(0)\tilde{g}_2^*(\omega)\right].$$

1.6.3 Third moment

Let us define $P'' = \{\omega_1, \omega_2, \omega_1+\omega_2\}$. For any kernel $\mathcal{F}$, we have the identities

$$\begin{aligned}
& \Sigma_{\{x_A,x_B\}\in P'} \int\int (\mathcal{F}\star g_3^*)(x_A,x_B)e^{-2\pi i(\omega_1 x_1+\omega_2 x_2)}dx_2dx_1 \\
= \quad & \Sigma_{\omega\in P''}\tilde{\mathcal{F}}(\omega)\tilde{g}_3^*(\omega_1,\omega_2), \\
& \Sigma_{\{x_A,x_B\}\in P'} \int\int \mathcal{F}(x_A)g_3^*(x_A,x_B)e^{-2\pi i(\omega_1 x_1+\omega_2 x_2)}dx_2dx_1 \\
= \quad & 2(\tilde{\mathcal{F}}\star\tilde{g}_3^*)(\omega_1,\omega_2) + (\tilde{\mathcal{F}}\star\tilde{g}_3^*)(\omega_2,\omega_1), \\
& \Sigma_{\{x_A,x_B\}\in P'} \int\int \mathcal{F}(x_A-x_B)g_3^*(x_A,x_B)e^{-2\pi i(\omega_1 x_1+\omega_2 x_2)}dx_2dx_1 \\
= \quad & (\tilde{\mathcal{F}}\star\tilde{g}_3^*)(\omega_2,\omega_1)(\tilde{\mathcal{F}}\star\tilde{g}_3^*)(-\omega_2,\omega_1+\omega_2) + (\tilde{\mathcal{F}}\star\tilde{g}_3^*)(-\omega_1,\omega_1+\omega_2).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& \Sigma_{\{x_A,x_B\}\in P'}\int\int[\mathcal{F}(x_A)+\mathcal{F}(x_A-x_B)]g_2^*(x_B)e^{-2\pi i(\omega_1x_1+\omega_2x_2)}dx_2dx_1 \\
= & \Sigma_{\{\omega,\omega'\}\in P'''}\tilde{\mathcal{F}}(\omega)\tilde{g}_2^*(\omega'),
\end{aligned}$$

where $P'''=\{(\omega,\omega')|\omega,\omega'\in P'',\omega'\neq\omega\}$. We thus obtain

$$\begin{aligned}
\frac{d\tilde{g}_3^*(\omega_1,\omega_2)}{dt} &= [\Sigma_{\omega\in P''}\tilde{A}(\omega)]\tilde{g}_3^*(\omega_1,\omega_2)+\Sigma_{\{\omega,\omega'\}\in P'''}\tilde{B}(\omega)\tilde{g}_2^*(\omega') \\
&- \tilde{r}_3^*(\omega_1,\omega_2),
\end{aligned}$$

where

$$\tilde{B}(\omega) = f\tilde{\mathcal{D}}(\omega)-2\alpha q\tilde{\mathcal{C}}(\omega)-\alpha\tilde{\mathcal{C}}(\omega)\tilde{g}_2^*(\omega),$$

and

$$\begin{aligned}
& \tilde{r}_3(\omega_1,\omega_2) \\
= & \alpha\int\tilde{\mathcal{C}}(\omega)[\tilde{g}_4^*(\omega,\omega_1,\omega_2-\omega)+\tilde{g}_4^*(\omega,\omega_2,\omega_1-\omega)+\tilde{g}_4^*(\omega,-\omega_2,\omega_1+\omega_2-\omega)]d\omega \\
+ & 2\alpha(\tilde{\mathcal{C}}\star\tilde{g}_3^*)(\omega_1,\omega_2)+2\alpha(\tilde{\mathcal{C}}\star\tilde{g}_3^*)(\omega_2,\omega_1) \\
+ & \alpha(\tilde{\mathcal{C}}\star\tilde{g}_3^*)(-\omega_2,\omega_1+\omega_2)+\alpha(\tilde{\mathcal{C}}\star\tilde{g}_3^*)(-\omega_1,\omega_1+\omega_2) \\
+ & 3\alpha\mathcal{C}(0)\tilde{g}_3^*(\omega_1,\omega_2).
\end{aligned}$$

1.7 The Perturbation Expansion

The final step in deriving the perturbation expansion is to expand all terms into power series of L^{-d} and to match the terms of the same orders. To do this, we write $\mathcal{D}(x)=\mathcal{D}_0(x/L)/L^d$, $\mathcal{C}(x)=\mathcal{C}_0(x/L)/L^d$, where $\mathcal{C}_0$ and $\mathcal{D}_0$ are independent of the length scale $L=L_D=L_C$. We note that $\tilde{\mathcal{D}}(\omega)=\tilde{\mathcal{D}}_0(L\omega)$ and $\tilde{\mathcal{C}}(\omega)=\tilde{\mathcal{C}}_0(L\omega)$. The expansions are given as

$$q = \Sigma_{i=0}^{\infty}q^{(i)}L^{-di},$$

$$
\begin{aligned}
\tilde{g}_2^*(\omega) &= \Sigma_{i=0}^{\infty}\tilde{g}_2^{*(i)}(L\omega)L^{-di}, \\
\tilde{g}_3^*(\omega_1,\omega_2) &= \Sigma_{i=0}^{\infty}\tilde{g}_3^{*(i)}(L\omega_1,L\omega_2)L^{-di}, \\
\tilde{A}(\omega) &= \Sigma_{i=0}^{\infty}\tilde{A}^{(i)}(L\omega)L^{-di}, \\
\tilde{B}(\omega) &= \Sigma_{i=0}^{\infty}\tilde{B}^{(i)}(L\omega)L^{-di}, \\
R &= \Sigma_{i=1}^{\infty}R^{(i)}L^{-di}, \\
\tilde{r}_2^*(\omega) &= \Sigma_{i=1}^{\infty}\tilde{r}_2^{*(i)}(L\omega)L^{-di}, \\
\tilde{r}_3^*(\omega_1,\omega_2) &= \Sigma_{i=1}^{\infty}\tilde{r}_3^{*(i)}(L\omega_1,L\omega_2)L^{-di},
\end{aligned}
$$

where we have accounted for the fact that the residual terms R, $\tilde{r}_2^*$ and $\tilde{r}_3^*$ do not have a zeroth-order term.

1.7.1 Mean-field

Collecting the terms of the zeroth order gives the mean-field model,

$$
\frac{dq^{(0)}}{dt} = (f-\mu)q^{(0)} - \alpha {q^{(0)}}^2.
$$

1.7.2 First order

Collecting the terms of order L^{-d} gives

$$
\begin{aligned}
\frac{dq^{(1)}}{dt} &= (f-\mu)q^{(1)} - 2\alpha q^{(0)}q^{(1)} - R^{(1)}, \\
R^{(1)} &= \alpha q^{(0)}\mathcal{C}_0(0) + \alpha\int\tilde{\mathcal{C}}_0(\omega)\tilde{g}_2^{*(0)}(\omega)d\omega, \\
\frac{d\tilde{g}_2^{*(0)}(\omega)}{dt} &= 2\tilde{A}^{(0)}(\omega)\tilde{g}_2^{*(0)}(\omega) + 2fq^{(0)}\tilde{\mathcal{D}}_0(\omega) - 2\alpha {q^{(0)}}^2\tilde{\mathcal{C}}_0(\omega), \\
\tilde{A}^{(0)}(\omega) &= f\tilde{\mathcal{D}}_0(\omega) - \mu - \alpha q^{(0)}[1+\tilde{\mathcal{C}}_0(\omega)].
\end{aligned}
$$

1.7.3 Second order

Finally, collecting the terms of order L^{-2d} gives

$$
\frac{dq^{(2)}}{dt} = (f-\mu)q^{(2)} - \alpha {q^{(1)}}^2 - 2\alpha q^{(0)}q^{(2)} - R^{(2)},
$$

$$
\begin{aligned}
R^{(2)} &= \alpha q^{(1)} \mathcal{C}_0(0) + \alpha \int \tilde{\mathcal{C}}_0(\omega) \tilde{g}_2^{*(1)}(\omega) d\omega, \\
\frac{d\tilde{g}_2^{*(1)}(\omega)}{dt} &= 2\tilde{A}^{(0)}(\omega)\tilde{g}_2^{*(1)}(\omega) + 2\tilde{A}^{(1)}(\omega)\tilde{g}_2^{*(0)}(\omega) \\
&+ 2fq^{(1)}\tilde{\mathcal{D}}_0(\omega) - 4\alpha q^{(0)} q^{(1)} \tilde{\mathcal{C}}_0(\omega) - 2\tilde{r}_2^{*(1)}(\omega), \\
\tilde{A}^{(1)}(\omega) &= -\alpha q^{(1)}[1 + \tilde{\mathcal{C}}_0(\omega)], \\
\tilde{r}_2^{*(1)}(\omega) &= \alpha \left[\int_{\omega'} \tilde{\mathcal{C}}_0(\omega') \tilde{g}_3^{*(0)}(\omega', \omega) d\omega' + (\tilde{\mathcal{C}}_0 \star \tilde{g}_2^{*(0)})(\omega) + \mathcal{C}_0(0) \tilde{g}_2^{*(0)}(\omega) \right], \\
\frac{d\tilde{g}_3^{(0)}(\omega_1, \omega_2)}{dt} &= [\Sigma_{\omega \in P''} \tilde{A}^{(0)}(\omega)] \tilde{g}_3^{(0)}(\omega_1, \omega_2) + \Sigma_{\{\omega, \omega'\} \in P'''} \tilde{B}^{(0)}(\omega) \tilde{g}_2^{*(0)}(\omega'), \\
\tilde{B}^{(0)}(\omega) &= f\tilde{\mathcal{D}}_0(\omega) - 2\alpha q^{(0)} \tilde{\mathcal{C}}_0(\omega) - \alpha \tilde{\mathcal{C}}_0(\omega) \tilde{g}_2^{*(0)}(\omega).
\end{aligned}
$$

The equations are closed and can be solved numerically, which gives the expansion $q(t) = q^{(0)}(t) + q^{(1)}(t)/L^d + q^{(2)}(t)/L^{2d} + \ldots$.

2 Asymptotic Exactness of Symmetric Closures

Here we show that the symmetric power-1, power-2, and power-3 closures are asymptotically exact, but that the asymmetric closures fail to do so. The formulae for the closures are from ref. 1.

2.1 Power-1

The general power-1 closure is given by

$$
z_3^*(x_1, x_2) = \frac{1}{\beta}[\alpha q z_2^*(x_2 - x_1) + \beta q z_2^*(x_2) + \gamma q z_2^*(x_1) - (\alpha + \gamma) q^3].
$$

As discussed in Section 1.3, the approximation is asymptotically exact if and only if the closure is consistent with Eq. **3** in the sense that the remaining residual moment satisfies the conditions (C1) and (C2). It is easy to see that this is the case if and only if the closure is symmetric, i.e., $\alpha = \beta = \gamma$.

2.2 Power-2

Similarly, the general power-2 closure is given by

$$z_3^*(x_1, x_2) = \frac{\alpha z_2^*(x_1) z_2^*(x_2) + \beta z_2^*(x_1) z_2^*(x_2 - x_1) + \gamma z_2^*(x_2) z_2^*(x_2 - x_1) - \beta q^4}{q(\alpha + \gamma)},$$

which is consistent with Eq. **3** if and only if $\alpha = \beta = \gamma$.

2.3 Power-3

The symmetric power-3 closure is given by

$$z_3^*(x_1, x_2) = \frac{z_2^*(x_1) z_2^*(x_2) z_2^*(x_2 - x_1)}{q^3},$$

which is consistent with Eq. **3**.

1. Murrell, D.J., Dieckmann, U. & Law, R. (2004). *J. Theor. Biol.* **229,** 421-432.