1 Electronic Appendices to "Space and contact networks: capturing the locality of disease transmission"

These Appendices provide the more mathematical details of the paper that would otherwise break up the continuity of the arguments in the main text. Electronic Appendix A considers in more detail how the connectivity (or transitivity) of the approximating network model architecture may be directly calculated from the spatial transmission kernel $U(r)$ used to construct the network. Electronic Appendix B highlights how the basic reproduction number R_0 may be calculated for the spatial and network SIR models with two competing hazards, whilst Appendices C and D demonstrate how analytical expressions for the spatial clustering parameter k may be estimated for a given $U(r)$ and we consider the cases where the assumption of separability may or may not be adopted. Appendix E considers how the functional form of the pair correlation model on the network may be demonstrated to give an excellent approximation to the spatial moment closure model in the regime where deviation from homogeneous mixing is limited.

2 Electronic Appendix A: Defining ϕ from $U(r)$

Consider a random distribution of hosts of uniform density ρ in a spatially homogeneous and isotropic arena. Following the arguments in Section 2, consider two individuals linked with probability $nU(\mathbf{r}-\mathbf{r}')$, where n is the expected number of neighbours per individual and $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of the two individuals. We set

$$n = \rho A \tag{1}$$

where A is the effective uniform neighbourhood area. Consider three points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ where $\mathbf{A}-\mathbf{B}=\mathbf{r}$, $\mathbf{C}-\mathbf{B}=\mathbf{r}'$ and $\mathbf{A}-\mathbf{C}=\mathbf{r}-\mathbf{r}'$. Recall that we define ϕ as the proportion of triples that form triangles so that

$$\phi = \frac{\int\int nU(\mathbf{r})nU(\mathbf{r}')nU(\mathbf{r}-\mathbf{r}')\,\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}'}{\int\int nU(\mathbf{r})nU(\mathbf{r}')\,\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}'} \tag{2}$$

gives the probability that if X-Y and Y-Z are linked then X-Z are also linked. For a spatially homogeneous landscape, we only need to consider $R=|\mathbf{r}-\mathbf{r}'|$

so that since n is a constant, (2) reduces to

$$\begin{aligned}\phi &= \frac{n\int\int U(\mathbf{r})U(\mathbf{r}')U(\mathbf{r}-\mathbf{r}')\,\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}'}{\int\int U(\mathbf{r})U(\mathbf{r}')\,\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}'} \\ &= \frac{n\int\int\int\int U(r)U(r')U(R)\,r\mathrm{d}r\mathrm{d}\theta r'\mathrm{d}r'\mathrm{d}\theta'}{\int U(\mathbf{r})\,\mathrm{d}\mathbf{r}\int U(\mathbf{r}')\,\mathrm{d}\mathbf{r}'} \\ &= 2\pi n\int\int\int U(R)U(r')U(r)\,r'\mathrm{d}r' r\mathrm{d}r\mathrm{d}\theta \qquad (3)\end{aligned}$$

since the kernel is normalised according to Equation (1) in the main text and the factor of 2π comes from the θ' integral in the numerator. Since we assume a uniform density arena, we set $\rho = 1$ so that $n = A$ from (1) and (3) becomes

$$\phi = 2\pi A\int\int\int U(R)U(r')U(r)\,r'\mathrm{d}r' r\mathrm{d}r\mathrm{d}\theta \qquad (4)$$

as stated in the main text.

3 Electronic Appendix B: R_0 for the spatial contact and network models

For the spatial model defined in Section 2, the hazard posed by an infectious individual to a susceptible individual at the origin is $\beta U(r)$, so that the probability that the individual does not become infected for this model with two competing hazards (with recovery at rate γ) is simply $\gamma/(\gamma + \beta U(r))$. The probability that the individual does become infected is therefore simply $1 - \gamma/(\gamma + \beta U(r)) = 1 - 1/(1 + (\beta/\gamma)U(r))$. Given the assumption of unit density, integrating this expression over all space simply gives the total number of infections generated per infected individual (R_0) as

$$R_0 = \int_\Omega \left(1 - \frac{1}{1 + (\beta/\gamma)U(r)}\right)\mathrm{d}\mathbf{r},$$

which is (5) in the main text.

We can follow an analogous procedure for the network model. The probability that a susceptible neighbour of an infected individual does not become infected is $\gamma/(\gamma + \tau)$, so that the probability of becoming infected is simply $1 - 1/\left(1 + \left(\frac{\tau}{\gamma}\right)\right)$. For a mean neighbourhood size n, R_0 is therefore given by

$$R_0 = n\left(1 - \frac{1}{1 + \left(\frac{\tau}{\gamma}\right)}\right)$$

as stated in Section 2.

4 Electronic Appendix C: Estimating k for a given kernel

The convolution approximation can be written as

$$\overline{(U * c_{XY})} = k\overline{c}_{XY} \tag{5}$$

or in full

$$\int U(\mathbf{r})\,\mathrm{d}\mathbf{r} \int U(\mathbf{r}-\mathbf{r}')c_{XY}(\mathbf{r}',t)\,\mathrm{d}\mathbf{r}' = k(t)\int U(\mathbf{r})c_{XY}(\mathbf{r},t)\,\mathrm{d}\mathbf{r},$$

so that

$$k = \frac{\int\int U(\mathbf{r}-\mathbf{r}')U(\mathbf{r})c_{XY}(\mathbf{r}')\,\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}}{\int U(\mathbf{r})c_{XY}(\mathbf{r})\,\mathrm{d}\mathbf{r}}. \tag{6}$$

By substituting the separability assumption

$$c_{XY}(\mathbf{r},t) = A_{XY}(\mathbf{r})B_{XY}(t)$$

into the moment equations (8) in the paper, we can show that

$$c_{XY}(\mathbf{r},t) \propto A_{XY}(\mathbf{r}) \propto U(\mathbf{r}),$$

whereby substituting into (6) gives

$$k = \frac{\int\int U(\mathbf{r}-\mathbf{r}')U(\mathbf{r})U(\mathbf{r}')\,\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}}{\int U(\mathbf{r})^2\,\mathrm{d}\mathbf{r}}.$$

Letting $r = |\mathbf{r}|$, $r' = |\mathbf{r}'|$ and $R = |\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$ for a spatially homogeneous and isotropic landscape, using the definition of A gives

$$k = 2\pi A \int\int\int U(R)U(r')U(r)\,r'\mathrm{d}r'r\mathrm{d}r\mathrm{d}\theta \tag{7}$$

which is (12) in the main text.

5 Electronic Appendix D: Other expressions for k

A number of other expressions for k may also be derived. The first is obtained by first writing the convolution approximation out in full as

$$\int U(\mathbf{r}-\mathbf{r}')c_{XY}(\mathbf{r}',t)\,\mathrm{d}\mathbf{r}' = k(t)c_{XY}(\mathbf{r},t).$$

Invoking the separability assumption as per Appendix C in a homogeneous landscape then gives

$$\int U(R)U(r')\, r'\mathrm{d}r'\mathrm{d}\theta = kU(r),$$

whereupon simply integrating both sides over r gives

$$k = \frac{\int\int\int U(R)U(r')r'\mathrm{d}r'\mathrm{d}\theta\mathrm{d}r}{\int U(r)\,\mathrm{d}r}. \tag{8}$$

Note that both (7) and (8) assume a separable covariance to estimate k and whilst this holds extremely well for most of the epidemic, the early coupling between spatial and temporal components means that this assumption is less accurate at small r. To account for this, we consider replacing the previous assumption that

$$\frac{c_{XY}(r,t)}{U(r)} \propto \frac{A_{XY}(r)}{U(r)} = \text{constant} \tag{9}$$

by

$$\frac{c_{XY}(r,t)}{U(r)} = \omega_1 U(r) + \omega_2, \tag{10}$$

where ω_1 and ω_2 are constants and (9) is then a special case when $\omega_1 = 0$.

To determine ω_1 and ω_2, consider (10) when $r = 0$ namely

$$\frac{c_{XY}(0,t)}{U(0)} = \omega_1 U(0) + \omega_2. \tag{11}$$

But $c_{XY}(0,t) = 1$ when X=Y since individuals are always at their own location, so that (11) becomes

$$\frac{1}{U(0)} = \omega_1 U(0) + \omega_2. \tag{12}$$

If we now take $\partial/\partial r$ of (10), evaluating at $r = 0$ and making the further approximation that $(\partial c_{XY}(r,t)/\partial r)_{r=0} \approx (\mathrm{d}U(r)/\mathrm{d}r)_{r=0}$ gives

$$1 = 2\omega_1 U(0) + \omega_2. \tag{13}$$

Simultaneously solving (12) and (13) gives

$$\omega_1 = \frac{1}{U(0)}\left(1 - \frac{1}{U(0)}\right) \tag{14}$$

and

$$\omega_2 = \frac{2}{U(0)} - 1, \tag{15}$$

so that putting (12) and (13) into (10), substituting into the convolution approximation and integrating over r gives

$$k = \frac{\int\int\int U(R)U(r') \left[U(r') \left\{U(0) - 1\right\} - U(0) \left\{U(0) - 2\right\}\right] r' \mathrm{d}r' \mathrm{d}\theta \mathrm{d}r}{\int U(r) \left[U(r) \left\{U(0) - 1\right\} - U(0) \left\{U(0) - 2\right\}\right] \mathrm{d}r} \tag{16}$$

as an alternative estimate of k.

6 Electronic Appendix E: Mapping the spatial and network equations under moment closure

To demonstrate that the pair equations map identically onto (13) in the paper to first-order in the covariances, consider for simplicity and illustrative purposes the SI model (14) (main text) (setting $\gamma = 0$) and the change in the number of S-S pairs

$$\frac{\mathrm{d}[SS](t)}{\mathrm{d}t} = -2\tau[SSI]. \tag{17}$$

Substituting in the pair approximation (16) in the paper in the limit $n \to N \to \infty$ gives

$$\frac{\mathrm{d}[SS](t)}{\mathrm{d}t} = -\frac{2\tau[SS][SI]}{[S]} \left((1-\phi) + \frac{\phi[SI]}{[S][I]} \right),$$

whereby substituting the mappings (18)-(20) in the paper leads to

$$\frac{\mathrm{d}\overline{c}_{SS}(t)}{\mathrm{d}t} = -2\beta \left(\overline{c}_{SS} \left\{ \overline{I} + \frac{\overline{c}_{SI}}{\overline{S}} \right\} + \phi\overline{c}_{SI} \left\{ \overline{S} + \frac{\overline{c}_{SI}}{\overline{I}} + \frac{\overline{c}_{SS}}{\overline{S}} + \frac{\overline{c}_{SS}\overline{c}_{SI}}{\overline{S}^2\overline{I}} \right\} \right), \tag{18}$$

or to first-order in the covariances

$$\frac{\mathrm{d}\overline{c}_{SS}(t)}{\mathrm{d}t} = -2\beta \left(\overline{I}\overline{c}_{SS} + \phi\overline{S}\overline{c}_{SI} \right). \tag{19}$$

This is identical to the corresponding equation in (13) (main text) if we map

$$\phi = k, \tag{20}$$

which is (22) in the main text.

The neglect of covariance terms of quadratic-order and higher amounts to assuming that

$$\overline{X}\,\overline{Y} \gg \overline{c}_{XY}, \tag{21}$$

or $\overline{XY} \approx \overline{X}\,\overline{Y}$ and this is true for mixing close to homogeneous. Thus, when spatial correlations are fairly weak, the first term in each of the curly brackets in (18) strongly dominates and reduces to (19) to a very good approximation.

Repeating this procedure for $\overline{c}_{II}(t)$ and $\overline{c}_{SI}(t)$ is a simple extension and we reach identical conclusions. The extension to the SIR case is similarly trivial.