Mathematical Appendix

Let 1 = R denote the renewal equation (Eq. **3** in Appendix) and 0 = B denote the balance equation (Eq. **4** in Appendix). Let $R'$and B' be partial derivatives with respect to γ, holding r constant; let $r_R{}'$and $r_B{}'$be partial derivatives with respect to γ, holding π constant for $r_B{}'$.

$$r_R' = \frac{R'}{A_m} = \frac{1}{A_m} \int_0^{\omega} e^{-rx} \left\{ \frac{\partial l(x)}{\partial \gamma} m(x) + \frac{\partial m(x)}{\partial \gamma} l(x) \right\} dx \qquad \textbf{[A1]}$$

$$\begin{aligned} r_B' = \frac{B'}{C(A_y - A_c)} = \frac{1}{C(A_y - A_c)} \int_0^{\omega} e^{-rx} \frac{\partial l(x,\gamma)}{\partial \gamma} \left[\pi \left(E/N \right) y(x,\gamma) - c(x,\gamma) \right] dx \\ + \int_0^{\omega} e^{-rx} l(x,\gamma) \left[\pi \left(E/N \right) \frac{\partial y(x,\gamma)}{\partial \gamma} - \frac{\partial c(x,\gamma)}{\partial \gamma} \right] dx \end{aligned} \qquad \textbf{[A2]}$$

A_y and A_m are calculated similarly to A_c:

$$A_c = \frac{\int_0^{\omega} x e^{-rx} l(x,\gamma) c(x,\gamma) dx}{\int_0^{\omega} e^{-rx} l(x,\gamma) c(x,\gamma) dx} \qquad \textbf{[A3]}$$

The denominator in **A3** is C, the discounted lifetime value of consumption, which must equal the similarly calculated πY.

Fertility. Differentiate R = 1 and B = 0, and simplify to find

$$\frac{dr}{d\varepsilon(a)} = e^{-ra} l(a) / A_m + r_R' \frac{d\gamma}{d\varepsilon(a)} \qquad \textbf{[A4]}$$

$$\frac{dr}{d\varepsilon(a)} = r_B' \frac{d\gamma}{d\varepsilon(a)} \qquad \textbf{[A5]}$$

Substitute out $d\gamma/d\varepsilon(a)$ and solve for $dr/d\varepsilon(a)$ to find Eq. **5** in Appendix.

Mortality. Differentiate R = 1 and B = 0, and simplify to find

$$\frac{dr}{d\delta(a)} = \frac{d\gamma}{d\delta(a)} r_R' + F(a)/A_m \qquad \textbf{[A6]}$$

$$\frac{dr}{d\delta(a)} = T(a)/C(A_y - A_c) + \frac{d\gamma}{d\delta(a)} r'_B \qquad \textbf{[A7]}$$

Substitute out $d\gamma/d\delta(a)$ and solve for $dr/d\delta(a)$ to find Eq. **6** in Appendix.