Example to show the procedures of calculating dynamic stability

Local dynamic stability was determined[3] based on the maximum finite-time Lyapunov exponent, λ_{max}. The procedures to calculate the λ_{max} are shown below using Lorenz attractor as an example. The Lorenz equations are,

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= (-xz + \rho x - y)\\ \dot{z} &= (xy - \beta z)\end{aligned}$$

where $\sigma = 16.0$, $\rho = 45.92$ and $\beta = 4.0$ in this example. Figure 1 is the traditional Lorenz attractor with these parameters, initial conditions [x,y,z] = [20,20,20] and time range from 0 to 10 seconds (timestep = 1 msec).

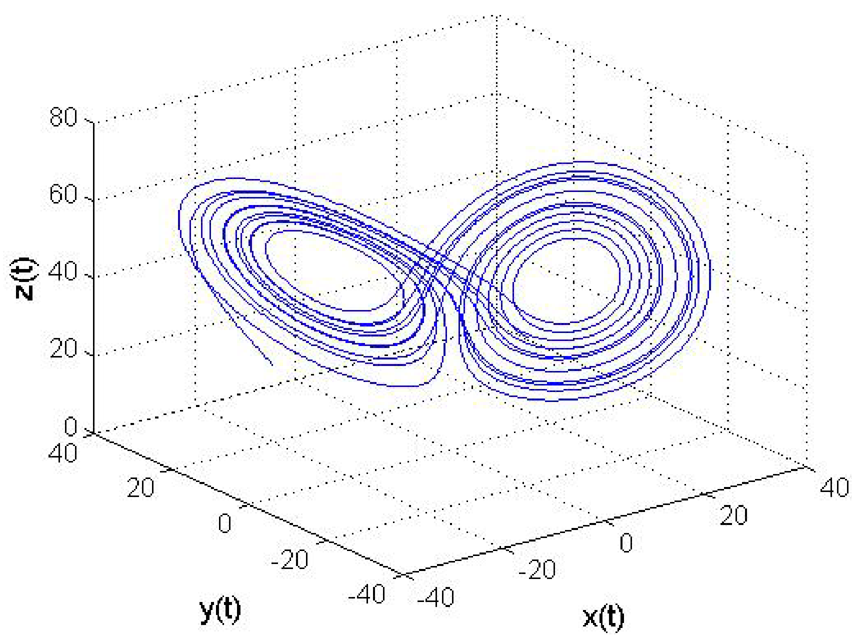

Figure 1. Lorenz attractor

Norm of vectors:

It is unreasonable to assume that one can measure all of the dynamic states of the biomechanical system. Therefore, in this example we will demonstrate how a measurable subset of the dynamic states can be used to estimate the nonlinear behavior[1] The stability analyses were performed on the Euclidean norm of the three variables determining the dynamics at each time interval,

$$X(t) = \sqrt{x^2(t) + y^2(t) + z^2(t)}$$

The norm of the example Lorenz attractor is illustrated in Figure 2 plotted *X(t)* as a function of time. This can be used to represent the measured experimental data.

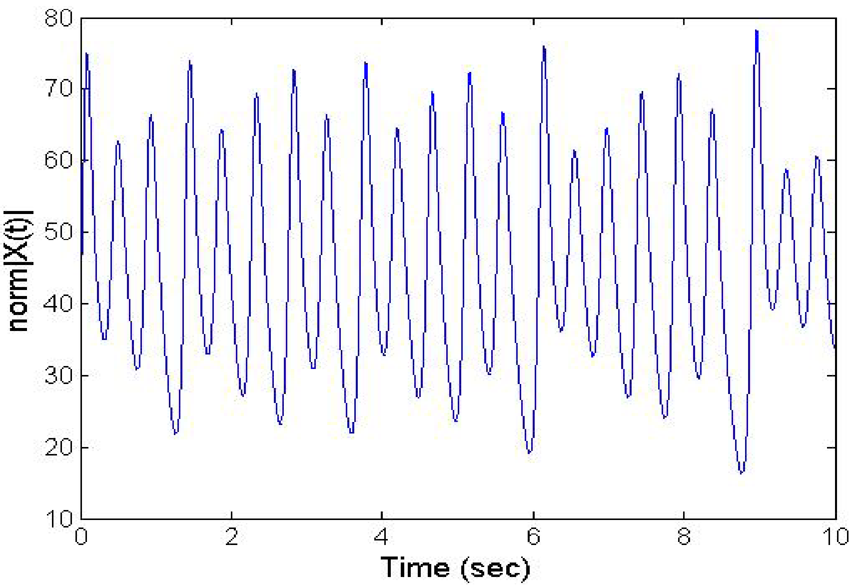

Figure 2. Norm of the Lorenz attractor. This represents a sample measurable state.

Since the measured time-series data, *X(t)*, is a one dimensional column vectors it is necessary to reconstruct an n-dimensional state-space out of the data in order to accurately represent the nonlinear dynamics. One typical method of creating an n-dimensional state-space from scalar data is by method of delays (equation 2)[4]. Two critical parameters are necessary including the constant time delay T_d and the number of reconstructed embedding dimensions, n.

Time Delay:

The time delay T_d was estimated from the Average Mutual Information Function[2]. T_d was taken as the first minimum of the Average Mutual Informatio(AMI) function. Figure 3 shows that in this case (norm of the Lorenz attractor) the minimum AMI occurred at 90 samples i.e. 0.09sec.

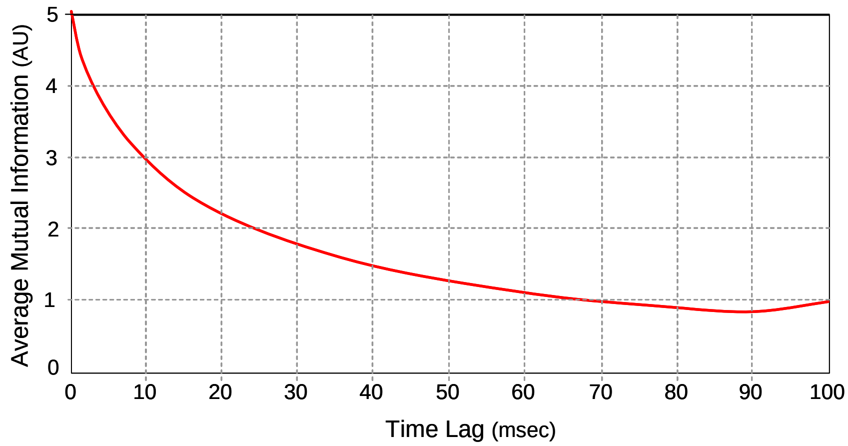

Figure 3. Time delay from AMI

Embedding Dimension:

Embedding dimension was based on a global false nearest neighbor analysis. Figure 4 shows the percentage of false neighbors is minimum at embedding dimension n = 3. Therefore for this example the embedding dimension n = 3 was used to reconstruct the state-space.

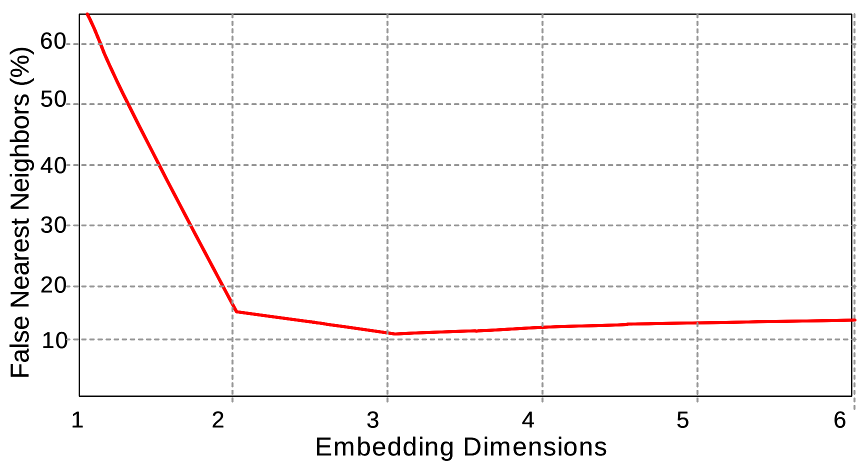

Figure 4. Embedding dimension from false nearest neighbors analysis

Reconstructed State-Space:

Figure 5.A below is the reconstructed state space of X(t) with an embedding dimension of n = 3. Figure 5.B shows the Euclidean distance between the nearest neighbors. The Euclidean

distance between nearest neighbors, $d_i(t)$, was computed for each data point, *i*, in the reconstructed state-space. The nearest neighbor of a data point was found by selecting a point on a separate trajectory such that the distance between the two points was minimum compared to the distance between the reference point and any other point on a different trajectory in the state-space.

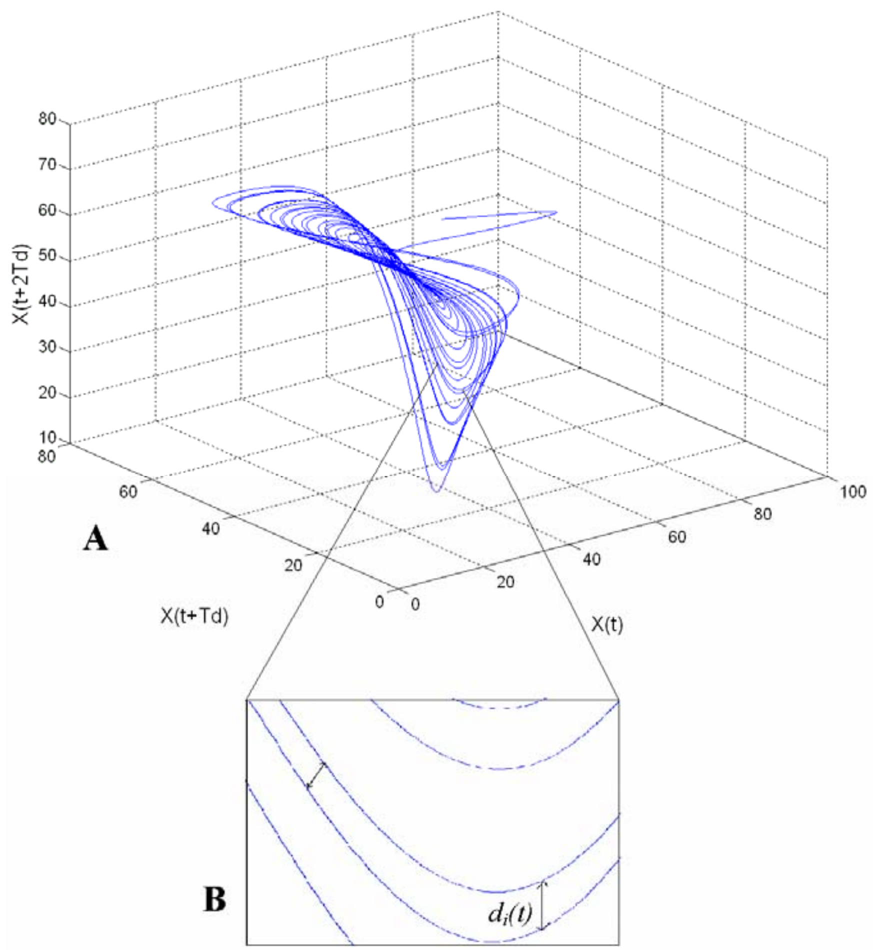

Figure 5. (A) Reconstructed state-space with 3 embedded dimensions. (B) Euclidean distance between nearest neighbors

Calculating Maximum Finite Time Lyapunov Exponent λ_{max} :

The average logarithmic divergence of all pairs of nearest neighbors, *i* are calculated from the reconstructed state space. The maximum finite-time Lyapunov exponent, λ_{max} was calculated as the slope of the logarithm of average divergence across the span of o to 1 cycle as shown in the Figure 6.

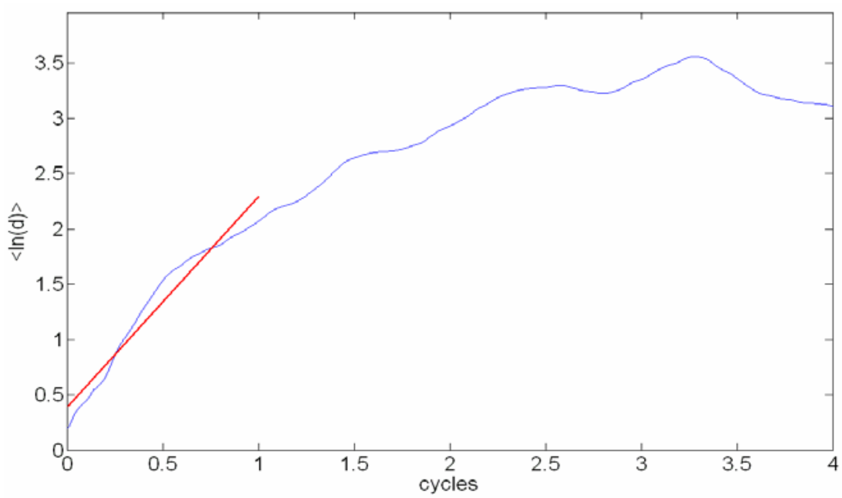

Figure 6. Average logarithmic divergence vs. time

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2. Fraser AM and Swinney HL. Independent coordinates for strange attractors from mutual information. Phys.Rev.A 1986;33:1134-40.
3. Rosenstein MT, Collins JJ, and Deluca CJ. A Practical Method for Calculating Largest Lyapunov Exponents from Small Data Sets. Physica D 1993;65:117-34.
4. Rosenstein MT, Collins JJ, and Deluca CJ. Reconstruction Expansion As A Geometry-Based Framework for Choosing Proper Delay Times. Physica D 1994;73:82-98.