Mean-Curvature Motion

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The mathematical background of MCM can be derived from the partial differential equation of normal linear diffusion

$$\frac{\partial I}{\partial t} = \mathrm{div}(\nabla I) \tag{1}$$

which describes the temporal evolution of an image $I = I(\mathbf{x}, t)$ starting from the image $I_0 = I(\mathbf{x}, 0)$ at the time t=0. ∇I denotes the gradient of the image with respect to the coordinates $\mathbf{x}$ in a space of arbitrary dimensions. As well known, the process equilibrates the gray level differences such that the image becomes smoother with increasing time (1). Using $\mathrm{Hess}(I)$ to denote the matrix containing all spatial second order derivatives, the equation can be rewritten as follows

$$\frac{\partial I}{\partial t} = \mathrm{div}(\nabla I) = |\nabla I|\,\mathrm{div}\left(\frac{\nabla I}{|\nabla I|}\right) + \left(\frac{\nabla I}{|\nabla I|}, \mathrm{Hess}(I)\frac{\nabla I}{|\nabla I|}\right) \tag{2}$$

where the second term describes smoothing in the direction of the gradient. Suppressing this term, one obtains the equation

$$\frac{\partial I}{\partial t} = |\nabla I|\,\mathrm{div}\left(\frac{\nabla I}{|\nabla I|}\right) \tag{3}$$

for a nonlinear anisotropic diffusion process (2), which was derived to improve edge preservation by smoothing the image only along level lines and not across edges.

Regarding a boundary passing through the point $\mathbf{x}$ of the image $I(\mathbf{x}, t)$, either a level line in two dimensions or a isosurface in three dimensions, equation 3 can also be interpreted as a propagation of this boundary in the direction of its inward normal with a speed proportional to its local curvature (1). As can be realized after some calculation, the expression $\mathrm{div}(\nabla I/|\nabla I|)$ describes the curvature of a line in two dimensions

$$\mathrm{div}\left(\frac{\nabla I}{|\nabla I|}\right) = \frac{I_{xx} I_y^2 - 2 I_{xy} I_x I_y + I_{yy} I_x^2}{\left(I_x^2 + I_y^2\right)^{3/2}} \tag{4}$$

as well as the mean curvature of a surface in three dimensions

$$\mathrm{div}\left(\frac{\nabla I}{|\nabla I|}\right) = \frac{I_{xx}\left(I_y^2 + I_z^2\right) + I_{yy}\left(I_x^2 + I_y^2\right) + I_{zz}\left(I_x^2 + I_y^2\right) - 2 I_{xy} I_x I_y - 2 I_{xz} I_x I_z - 2 I_{yz} I_y I_z}{\left(I_x^2 + I_y^2 + I_z^2\right)^{3/2}} \tag{5}$$

For this reason, "mean-curvature motion" has been introduced to name the process corresponding to equation 3 (2).

For the present implementation we used a discretization scheme similar to that proposed by Osher and Sethian (3). Using the abbreviations

$$D^{+x} I = \frac{1}{h}\left(I(x+h, y, z, t) - I(x, y, z, t)\right)$$

$$D^{-x} I = \frac{1}{h}\left(I(x, y, z, t) - I(x-h, y, z, t)\right)$$

$$D^{0x} I = \frac{1}{2h}\left(I(x+h, y, z, t) - I(x-h, y, z, t)\right)$$

$$D^{0x0x} I = \frac{1}{h^2}\left(I(x+h, y, z, t) - 2I(x, y, z, t) + I(x-h, y, z, t)\right),$$

with corresponding definitions for y and z, the image evolution was calculated in the 2-D and 3-D case according to

$$D^{+t} I = \Delta t\Big[\left(D^{0x} I\right)^2 D^{0y0y} I + \left(D^{0y} I\right)^2 D^{0x0x} I - \max\left(D^{0x} I\, D^{0y} I, 0\right)\left(D^{-x+y} I + D^{+x-y} I\right) - \min\left(D^{0x} I\, D^{0y} I, 0\right)\left(D^{+x+y} I + D^{-x-y} I\right)\Big] \Big/ \left[\left(D^{0x} I\right)^2 + \left(D^{0y} I\right)^2\right]$$

and

$$D^{+t} I = \Delta t\Big[\left(D^{0x} I\right)^2\left(D^{0y0y} I + D^{0z0z} I\right) + \left(D^{0y} I\right)^2\left(D^{0x0x} I + D^{0z0z} I\right) + \left(D^{0z} I\right)^2\left(D^{0x0x} I + D^{0y0y} I\right)$$

$$-\max\left(D^{0x}I \;\; D^{0y}I,0\right)\left(D^{-x+y}I + D^{+x-y}I\right) - \min\left(D^{0x}I \;\; D^{0y}I,0\right)\left(D^{+x+y}I + D^{-x-y}I\right)$$

$$-\max\left(D^{0x}I \;\; D^{0z}I,0\right)\left(D^{-x+z}I + D^{+x-z}I\right) - \min\left(D^{0x}I \;\; D^{0z}I,0\right)\left(D^{+x+z}I + D^{-x-z}I\right)$$

$$-\max\left(D^{0y}I \;\; D^{0z}I,0\right)\left(D^{-y+z}I + D^{+y-z}I\right) - \min\left(D^{0y}I \;\; D^{0z}I,0\right)\left(D^{+y+z}I + D^{-y-z}I\right)\Big]\Big/$$

$$\Big/\left[\left(D^{0x}I\right)^2 + \left(D^{0y}I\right)^2 + \left(D^{0z}I\right)^2\right]$$

respectively.

References

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3. Osher, S. & Sethian, J. A. (1988) *Journal of Computational Physics* **79,** 12-49.