STUDIES IN THE MEANING AND RELATIONSHIPS OF BIRTH AND DEATH RATES.

II.

Density of Population and Death Rate (Farr's Law).

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This subject was first considered statistically by the late Dr Farr. It is one of the brilliant attempts to extract the real meaning of figures so frequent in his work, but though this theory has not shared in the complete neglect that has been the lot of his attempt to put a quantitative measure to the course of epidemics, it has suffered as much from the kind of patronage with which it is usually discussed. On at least one of the great medical officers of health of his time, however,—the late Dr J. B. Russell—the theory exercised a strong fascination. My own copy of Farr's *Vital Statistics* came from Dr Russell's library, and the whole passage referring to the law is lined with his characteristic nervous pencil marks, while in much of his work on vital statistics the influence can easily be traced.

The neglect of the subject is of two-fold origin. In the first place the law appeared quite artificial. In the second place the statistics of the decade on which it was founded happened to be specially suitable for its discovery, while subsequent figures did not appear to afford the same support.

The law itself, if the death rate be denoted by R and the density of population (say the number of persons per square mile) by D, is that

$$R = cD^m,$$

when c and m are constants.

But how is the death rate to be measured? By Farr the crude death rate was used and found to give a good measure of the facts.

Later when corrected death rates were substituted, and that seemed to be the proper course, the law obviously did not hold, and even with crude death rates, its success as a descriptive formula was not nearly so marked. Thus in the absence of any *a priori* justification the law was relegated to a somewhat obscure position. Before proceeding to its justification, however, it is necessary to have a clear idea of what kind of evidence can be produced. The law must be a law of average, for on account of the arbitrary nature of the boundaries of the registration districts, the number of persons living on an acre is merely a rough approximation. The groups of localities which supply the figures must further be large, as some with better conditions will have lower death rates, and others with worse, a higher. Nor can even a large city be divided into small districts and these considered. A city population must be a whole population; the slum is not wholly recruited from the slum by any means. A district consisting chiefly of persons engaged in trades and minor occupations may have a very high density and yet a low death rate. All, or at least the great majority of the inhabitants are respectable, those who are not, are driven elsewhere, yet the latter must be considered as part of the same population: from this class, also, though some ascend in the social scale, they do not constitute a separate population. It is obvious therefore that to obtain a suitable average a few groups only must be chosen. Dr Farr made seven, Dr Tatham sixteen; the former may be too few, the latter seems too many. The effect of density is not merely as density. The country preserves life even in the presence of excess or dissipation: the town does not. Further, in the period of growth, children in the city do not get anything like the same chance as their fellows in the country, even though housing may be better and food more abundant. In addition filth in the country is at its worst in most cases but a local nuisance, spreading enteric fever and diarrhoea at times, but not having the power of rendering a whole district foetid. All these influences act concurrently and cumulatively to depress health the more closely people are crowded together, and as life is a physico-chemical process this effect must be measurable and should be capable of expression in some formula which goes back to chemistry and physics. Such a formula is that of Dr Farr. Nothing comparable to it was known in his day, so that as a mathematical formula can easily be found to describe almost any statistics, his formula seemed just such an one and no better than many others. It is, however, no longer alone.

This subject I investigated many years ago without making any advance. The difficulty of defining a death rate was too great. In my last paper, however, I have given a method for obtaining the death rate on a stationary population, and the application of this method justifies the law.

In order to illustrate the subject as fully as possible, two tables have been constructed, one showing the figures used by Dr Farr which refer to the decade 1861–1870, the second the comparative table for the decade 1891–1900 as given by Dr Tatham. Dr Farr used the crude death rate. Fortunately, he has also published the death rates at each age period for the groups of populations on which he based his law. This allows death rates to be calculated on the same standard population which has been used in framing the figures of the second table, and from these life table death rates, which are strictly comparable with those in the second table, have been calculated. The constants of the curves of the form $R = cD^m$ have been evaluated by the method of least squares for both periods, for the crude death rates, the corrected death rates, and the life table death rates.

It will be noticed that the values of m roughly correspond for each separate case in two periods, but in the case of the life table death rates, they correspond within three places of decimals, the furthest that could be statistically expected. We thus have a quite definite law acting independently of the changes which have taken place due to sanitary progress. Improve all round and the exponent does not vary, but only the multiplying constant. The former constant m therefore represents the law, and the latter c may be called the co-efficient of intensity of unhealthiness in the country. This co-efficient c will vary as sanitary conditions improve or the reverse, though the law will remain the same.

When the columns showing the results obtained by fitting similar curves to the crude and corrected death rates are compared, it is seen that the crude death rate fits less well than the life table death rate and that the corrected death rates are very badly represented by the formula. This is what would be expected from the fact shown in the previous paper that life table death rate can be obtained by multiplying the corrected death rate by one constant and adding a second. It will be noticed that the crude death rate curve of Dr Farr has an exponent of ·1193 which is much nearer the probable true exponent ·100 than that of the crude death rate for the decade 1891–1900 which is ·1276. This is explained by the fact that in the earlier period the crude death rate was 22·42 as against a life table death rate of 24·06, while in the latter period the corresponding figures are 18·19 and 21·77. Dr Farr had thus a better opportunity of formulating a law than his successors. With the crude death rate diverging more and more from the life table death rate it became more and more difficult to accept the relationship demanded by the formula. A law of which the main feature, namely, the exponent m, varied could hardly lay claim to be a law at all, any more than the law of gravitation could be justified if the relationship were not constantly the inverse square. Using a death rate which is more comparable between populations, namely, that which would hold if the population were stationary, the exponent takes the same value. It will be noticed that the co-efficient c decreases from 12·42 in the first period to 10·83 in the latter. In other words, density has only ·875 times the effect in producing mortality it had in 1860–1870, so much have sanitary conditions improved.

But the law remains apparently. Sanitation may diminish c, but the ill effects of concentration do not seem capable of being changed merely by sanitation. What the figures just given clearly mean is, that on the whole, conditions of life in modern England seem to be so uniformly the result of the action of the modern developments of industrialism, etc., as to be comprehended in a formula. The prospect that the town may become as healthy as the country, given proper precautions of living, does not seem possible if any law like that of Farr is found to hold permanently. In any case, decrease of density is essential.

But there is one exception to this law in both periods, namely, that of London. In 1861–70, the life table death rate of London was 26 per mille as against 32 expected by the formula. Unless this can be explained the formula falls. But I think it can be explained. Modern England was in 1860, and still is, a recent phenomenon compared with London. Liverpool, Manchester, etc., are but mushroom growths of yesterday. London began to pay its 'prentice fee' as a city in the middle of the seventeenth century. More than a century ago it had a million inhabitants. Sanitation was unknown. Countless thousands tried to live in it and failed. It was in the contemporary documents the 'wen' or the 'vampyre' that sucked England's blood. It was fifty years later than the rest of England in having a birth rate in excess of its death rate, and now it has its reward, the result of two centuries of natural selection in its crudest form. The death rate of London to-day is in no sense a measure of its sanitation. This will be referred to again in a subsequent paper.

I said earlier in this paper that Farr's law did not stand alone. In later papers certain examples of similar relationships will be referred to, but one specially is mentioned here. It is given in a remarkable communication by Mr A. E. Kennealy[1] entitled "An Approximate Law of Fatigue in the Speeds of Racing Animals." This came into my hands a number of years ago and it immediately suggested "Farr's Law," but the difficulty which was still unsolved was as already mentioned the measure of the death rates. Mr Kennealy's paper contains the results of an investigation into the speeds of animals. It is shown that each racing record whether for horse or man, trotting, pacing, walking, running or swimming, obeys a formula of the same form. The figures compared in each instance are the record times achieved for each different distance. As is well known the rate of running for a hundred yards is greater than that for a mile, but that the record time for 20 yards, 40 yards, 100 yards, one mile, ten miles, etc., for each separate sport for practically all the racing records of the world should be comprehended in the same formula

$$T = cL^{\frac{9}{8}},$$

when T is the time taken to cover a distance L, and c is a constant, was hardly to be expected. The constancy of the value $m = 9/8$ is surprising.

This formula when V is the average velocity can be put in the form

$$V = cL^{-\frac{1}{8}} \text{ since } V = \frac{L}{T}.$$

In this form it may perhaps represent the same kind of relationship as Farr's formula.

These remarks suggest a fact which is fully discussed in a succeeding paper that the death rates of different age periods of life in different districts are really organically connected and cannot be compared without the exercise of great care, though on superficial observation they seem directly significant. It is interesting to note that this fact was perceived by the genius of Dr Farr fifty years before modern statistical methods had been introduced.

[1] *Proc. Amer. Acad. Arts and Sciences*, 1906, p. 275.

TABLE I.

Showing the figures relating to Density and death rate. 1861–70*.

No. of districts	Density (persons per square mile)	Corrected death rate (1)	Do. Fitted by least squares	Crude death rate (2)	Do. Fitted by Farr	Life table death rate (3)	Do. Fitted by least squares
53	166	15·30	16·70	16·75	18·90	19·90	20·73
345	186	17·02	17·00	19·16	19·16	21·07	20·96
137	379	20·52	18·99	21·88	20·87	23·47	22·51
47	1718	24·35	24·03	24·90	25·02	26·09	26·19
9	4499	27·94	27·92	28·08	28·08	28·54	28·84
1	12,357	33·98	32·67	32·49	32·70†	32·67	31·92
1	65,823	40·55	42·39	38·62	38·74	37·17	37·74
			$E\,\%=3{\cdot}79$ $\Delta=1{\cdot}17$		$E\,\%=2{\cdot}70$ $\Delta=\cdot90$		$E\,\%=2{\cdot}01$ $\Delta=\cdot61$

(1) $R=7{\cdot}534\,D^{\cdot15571}$. (2) $R=10{\cdot}234\,D^{\cdot11998}$. (3) $R=12{\cdot}419\,D^{\cdot10018}$.

$E\,\%$ = mean percentage error. Δ = square root of the mean of the squares of the errors.

* Farr, *Vital Statistics*, p. 175.

† A misprint in the original of 37·7 has been corrected.

TABLE II.

Showing the figures relating to density and death rate. 1891–1900*.

1 No. of districts	2 No. of inhabitants divided by 1,000	3 Density (persons per square mile)	4 Corrected death rate	5 Do. Fitted by least squares (1)	6 Crude death rate	7 Do. Fitted by least squares (2)	8 Life table death rate	9 Do. Fitted by least squares (3)
27	305	136	11·63	13·06	14·20	14·16	17·38	17·18
112	1676	161	12·54	13·43	15·05	14·51	18·01	18·12
121	2496	181	13·44	13·70	15·44	14·68	18·62	18·33
92	3849	261	14·52	14·56	15·46	15·38	19·36	19·02
53	2272	407	15·53	15·68	16·08	16·28	20·05	19·90
56	2577	457	16·53	15·99	16·67	16·52	20·24	20·13
31	1839	737	17·58	17·32	17·64	17·56	21·45	21·12
40	3690	1303	18·53	19·05	18·04	18·88	22·10	22·31
31	3159	1705	19·42	19·93	18·61	19·54	22·71	22·99
21	2240	2339	20·37	21·00	19·50	20·35	23·36	23·72
18	2777	4424	21·56	23·37	20·21	22·08	24·18	25·31
13	2119	4884	22·36	23·76	20·69	22·35	24·72	25·56
6	801	4194	23·48	23·16	22·05	21·93	25·49	25·10
5	762	2925	24·33	21·80	23·29	20·94	26·07	24·21
5	791	7480	26·54	25·51	24·74	23·60	27·58	26·68
4	288	55,563	34·82	35·66	32·67	30·49	33·25	32·58
				$E\,\%=4{\cdot}3$ $\Delta=1{\cdot}05$		$E\,\%=3{\cdot}8$ $\Delta=1{\cdot}14$		$E\,\%=2{\cdot}03$ $\Delta=\cdot63$

(1) $R=12{\cdot}40\,D^{\cdot16715}$. (2) $R=13{\cdot}57\,D^{\cdot12755}$. (3) $R=10{\cdot}83\,D^{\cdot10078}$.

* Dr Tatham: *Decennial Supplement, Registrar-General of England and Wales,* Part II. 1908, p. lxxi.