Nonsymmetrical Bifurcations in Arterial Branching

M. ZAMIR

Departments of Applied Mathematics and Biophysics, University of Western Ontario, London, Ontario, Canada N6A 5B9

ABSTRACT The results of optimality studies of the branching angles of arterial bifurcations are extended to nonsymmetrical bifurcations. Predicted nonsymmetrical bifurcations are found to be not unlike those observed in the cardiovascular system.

INTRODUCTION

It has long been suspected that the branching angles of blood arteries in the cardiovascular system are based on well-defined physiological principles. The cardiovascular system being basically a fluid-conducting system, these physiological principles are generally sought in the context of fluid dynamics. Several principles have so far been examined (Murray, 1926 *b*; Kamiya and Togawa, 1972; Zamir, 1976 *a*), and the results were recently put together for comparison (Zamir, 1976 *b*). Two of these principles propose that the branching angles are such that the lumen surface or lumen volume of the vessels involved in an arterial junction is minimum, and by simple analysis it then follows that the optimum branching angles are determined by the radii of the vessels involved. Another two principles propose that the branching angles are such that the pumping power required to drive the flow or the drag force on the endothelial surface is minimum, and then find, as a consequence, that the optimum angles depend on the flow in as well as the radii of the vessels involved.

In the case of a symmetrical bifurcation the flow in each of the branches is known to be half that in the parent vessel, and therefore, comparison of the four optimality principles is possible. In the case of nonsymmetrical bifurcations however, such comparison is not possible unless one knows the relation between the flow in and the radius of a blood artery in general. This question has been studied in some detail in the past (Murray, 1926 *a*; Rodbard, 1975; Zamir, 1977), and a possible conclusion seems to be that the radius of a blood artery is proportional to the cube root of the flow which that artery is designed to convey, but the constant of proportionality may be different in different parts of the system. The purpose of the present study is to use this result in combination with those on branching angles in order to extend and apply these optimality studies to nonsymmetrical bifurcations. This is an important step because the overwhelming majority of arterial bifurcations in the cardiovascular system are nonsymmetrical.

The application of optimality principles to nonsymmetrical bifurcations exposes these principles to a much larger ground on which they can be tested in order to establish the degree of validity of these principles, and of the relation between flow and radius, in the cardiovascular system. This knowledge is important to our understanding of the physiological basis of the cardiovascular system and its anomalies. These physiological aspects and some clinical implications have already been elaborated upon in the references cited above.

BASIC ANALYSIS AND NOTATION

Consider a nonsymmetrical bifurcation which by definition consists of a parent artery and two branches of which one has a larger radius than the other. Examples are shown in Fig. 5. Throughout this study a subscript '0' shall be used to refer to the parent artery, subscript '1' to the larger of the two branches, and subscript '2' to the smaller branch. Thus, if the radius of an artery is denoted by r, then the statement,

$$r_1 \geq r_2, \tag{1}$$

shall be true of all bifurcations we consider, including the special case of a symmetrical bifurcation where $r_1 = r_2$.

The sum of the cross sectional areas of the two branches divided by that of the parent artery is generally referred to as the "area ratio" β, thus

$$\beta = \frac{r_1^2 + r_2^2}{r_0^2}. \tag{2}$$

In analogy we define the "asymmetry ratio" α as the cross-sectional area of the smaller branch divided by that of the larger, i.e.,

$$\alpha = r_2^2/r_1^2. \tag{3}$$

By definition the value of this parameter shall be between zero and one, being equal to one in the special case of a symmetrical bifurcation, thus

$$0 < \alpha \leq 1. \tag{4}$$

If f denotes the flow in an artery, then the flow balance in a bifurcation is given by

$$f_0 = f_1 + f_2, \tag{5}$$

and introducing the result that the radius of an artery is proportional to the cube root of the flow which the artery is designed to convey (Murray, 1926 *a*; Rodbard, 1975; Zamir, 1977), i.e.,

$$f \propto r^3, \tag{6}$$

Eq. 5 becomes

$$r_0^3 = r_1^3 + r_2^3. \tag{7}$$

The angles θ_1 and θ_2 which the branches make with the direction of the parent artery are predicted to be as follows by the four different optimality principles (Zamir, 1976 *b*):

for minimum lumen surface

$$\cos\theta_1 = \frac{r_0^2 + r_1^2 - r_2^2}{2r_0r_1},\ \cos\theta_2 = \frac{r_0^2 + r_2^2 - r_1^2}{2r_0r_2}; \tag{8}$$

for minimum lumen volume

$$\cos\theta_1 = \frac{r_0^4 + r_1^4 - r_2^4}{2r_0^2r_1^2},\ \cos\theta_2 = \frac{r_0^4 + r_2^4 - r_1^4}{2r_0^2r_2^2}; \tag{9}$$

for minimum pumping power

$$\cos\theta_1 = \frac{(f_0^4/r_0^8) + (f_1^4/r_1^8) - (f_2^4/r_2^8)}{2f_0^2f_1^2/r_0^4r_1^4},$$

$$\cos\theta_2 = \frac{(f_0^4/r_0^8) + (f_2^4/r_2^8) - (f_1^4/r_1^8)}{2f_0^2f_2^2/r_0^4r_2^4}; \tag{10}$$

for minimum drag

$$\cos\theta_1 = \frac{(f_0^2/r_0^4) + (f_1^2/r_1^4) - (f_2^2/r_2^4)}{2f_0f_1/r_0^2r_1^2},$$

$$\cos\theta_2 = \frac{(f_0^2/r_0^4) + (f_2^2/r_2^4) - (f_1^2/r_1^4)}{2f_0f_2/r_0^2r_2^2}. \tag{11}$$

Using Eq. 6 for the relation between flow and radius we note that the optimum angles for minimum power become the same as those for minimum volume, and the optimum angles for minimum drag become the same as those for minimum lumen surface.

RESULTS

Combining Eqs. 2, 3, and 7 we obtain

$$\beta = \frac{1 + \alpha}{(1 + \alpha^{3/2})^{2/3}}. \tag{12}$$

This is the predicted value of the area ratio β in an arterial bifurcation in terms of its asymmetry ratio α, assuming the relation between flow and radius in Eq. 6. The result is illustrated graphically in Fig. 1. The value of β ranges from 1 in the case of a highly nonsymmetrical bifurcation ($\alpha \approx 0$) to 1.26 in the case of a symmetrical bifurcation ($\alpha = 1$).

Eqs. 8–11 combined with Eqs. 3 and 6 give the results for branching angles as follows:

for minimum pumping power and lumen volume

$$\cos\theta_1 = \frac{(1 + \alpha^{3/2})^{4/3} + 1 - \alpha^2}{2(1 + \alpha^{3/2})^{2/3}},$$

$$\cos\theta_2 = \frac{(1 + \alpha^{3/2})^{4/3} + \alpha^2 - 1}{2\alpha(1 + \alpha^{3/2})^{2/3}}; \tag{13}$$

for minimum drag and lumen surface

$$\cos\theta_1 = \frac{(1+\alpha^{3/2})^{2/3} + 1 - \alpha}{2(1+\alpha^{3/2})^{1/3}},$$

$$\cos\theta_2 = \frac{(1+\alpha^{3/2})^{2/3} + \alpha - 1}{2\alpha^{1/2}(1+\alpha^{3/2})^{1/3}}. \tag{14}$$

These are the predicted optimum angles which the branches make with the direction of the parent artery, based again on the relation between flow and radius in Eq. 6. These results are illustrated graphically in Figs. 2, 3, and 4.

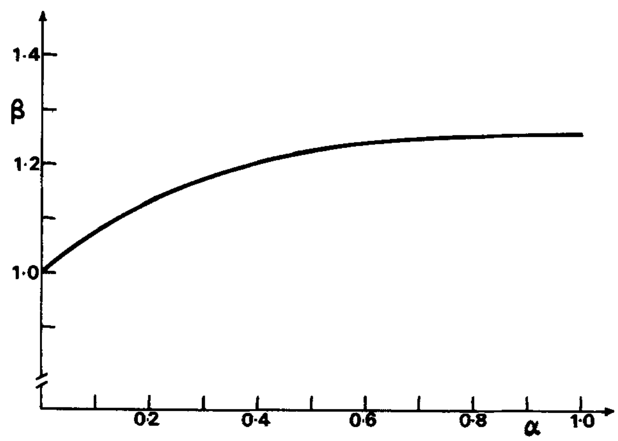

FIGURE 1. Relation between the area ratio β and the asymmetry ratio α based on the relation between flow and radius in Eq. 6.

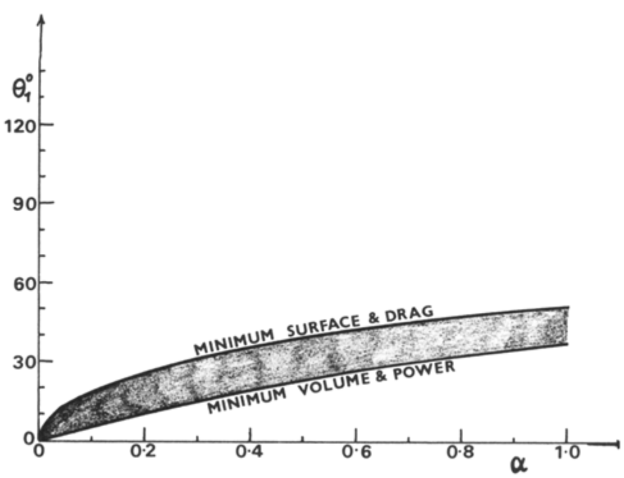

FIGURE 2. The angle θ_1 which the larger branch in a nonsymmetrical bifurcation makes with the direction of the parent artery as predicted by the indicated optimality principles, where α is the asymmetry ratio.

DISCUSSION AND CONCLUSIONS

The results of this study should not be treated as accurate predictions of what the branching angles of blood arteries in the cardiovascular system should be. The results can only be used qualitatively in an attempt to uncover the physiological principle or principles which underlie arterial branching in the cardiovascular system, and this is the context to which the following discussion and conclusions shall be confined.

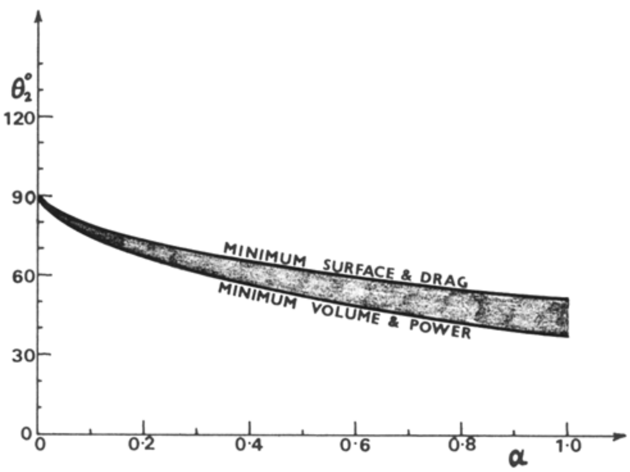

FIGURE 3. The angle θ_2 which the smaller branch in a nonsymmetrical bifurcation makes with the direction of the parent artery as predicted by the indicated optimality principles, where α is the asymmetry ratio.

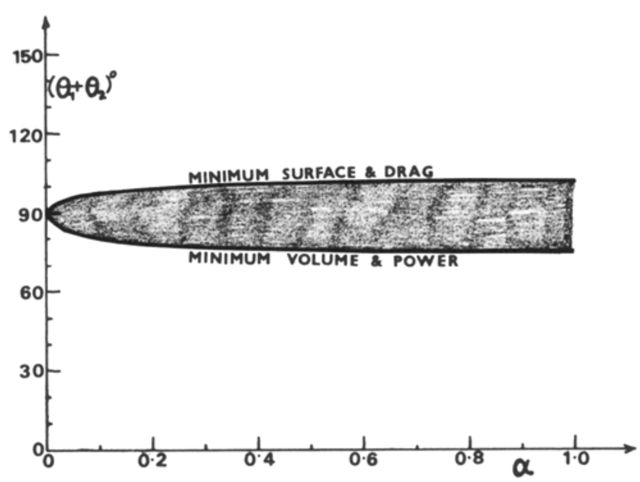

FIGURE 4. The total angle ($\theta_1 + \theta_2$) in a nonsymmetrical bifurcation as predicted by the indicated optimality principles, where α is the asymmetry ratio.

The most striking feature of the results in Figs. 2, 3, and 4 is the closeness of the angles predicted by all four optimality principles. This suggests that a bifurcation which falls within the shaded regions of these figures is not far from satisfying all four conditions of minimum lumen surface, lumen volume, pumping power, and viscous drag. Considering the difficulties involved in

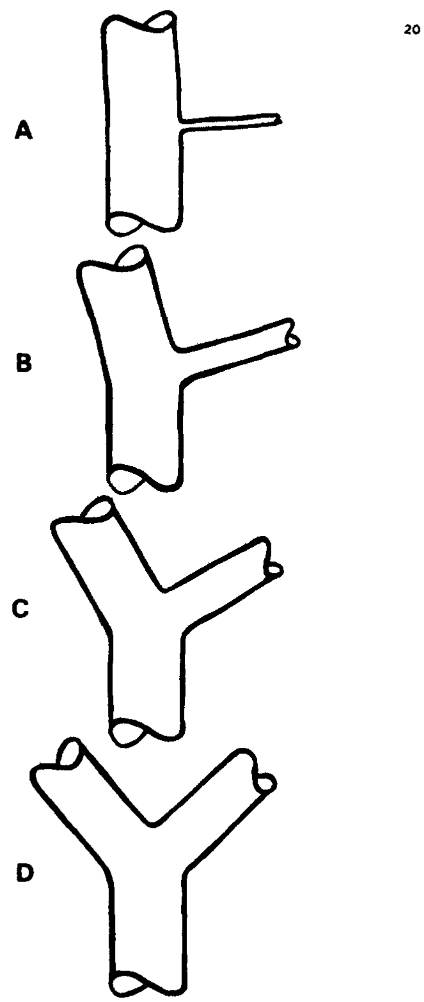

FIGURE 5. Examples of the predicted nonsymmetrical bifurcations in which the lumen surface, the lumen volume, the pumping power, and the viscous drag are all very nearly minimum. The asymmetry ratio is (A) 0.01, (B) 0.15, (C) 0.45, and (D) 0.80.

obtaining observational data relating to flow, radii, and branching angles, it may be long before such data can be made accurate enough and representative enough to discriminate between the predictions made by the two pairs of optimality principles. Therefore, at this stage it is perhaps unreasonable to do more than consider the whole of the shaded regions in Figs. 2, 3, and 4.

Qualitatively speaking, the nonsymmetrical bifurcations which lie within the shaded regions of Figs. 2, 3, and 4, and the corresponding area ratios in Fig. 1, are well within the physiological range. To illustrate this point, a few "typical" bifurcations were constructed from points which lie exactly midway between the two prediction curves in each of Figs. 2, 3, and 4. The results are shown in Fig. 5. These bifurcations are certainly not unlike those found in the cardiovascular system. The retinal circulation offers the best and perhaps the most reliable pictures of arterial bifurcations, because three-dimensional problems in the retina are minimal and the observed branching angles cannot be much different from the real ones. Two such examples which are large enough for measurement were found in Wise et al. (1971). These were measured and compared with the corresponding predicted bifurcations. The results are shown in Fig. 6, with the predicted bifurcations enlarged for clarity. Many other examples can be found in pictures of the retinal circulation though they are generally too small for accurate measurement.

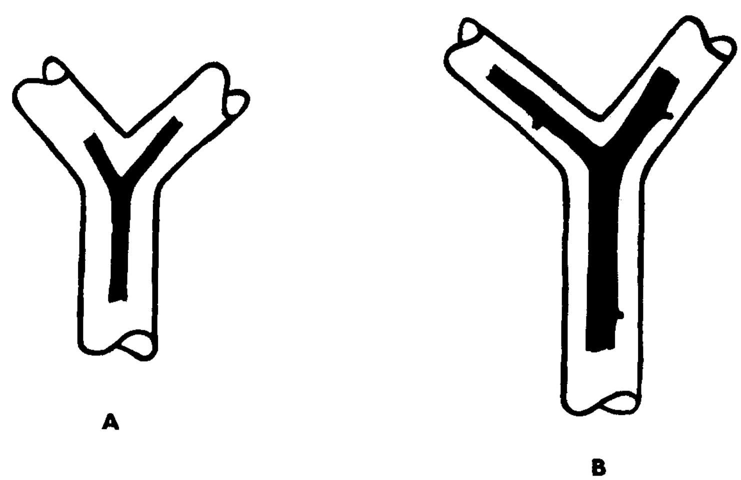

FIGURE 6. Two arterial bifurcations from the retinal circulation (solid) enclosed by the corresponding predicted bifurcations. The retinal bifurcations are based on pictures by Wise et al. (1971) of (A) a mature owl monkey and (B) a human. The predicted bifurcations are enlarged for clarity but they are based on the same asymmetry ratio as that of the retinal bifurcation in each case: (A) $\alpha = 0.90$ and (B) $\alpha = 0.71$. The predicted angles are 42° and 46° compared with the measured angles of 34° and 37° in A, and 37° and 51° compared with 33° and 52° in B.

It is of interest to note that the total bifurcation angle ($\theta_1 + \theta_2$) changes very little with change in the degree of asymmetry. Thus, in Fig. 4 considering the shaded region as a whole again, we observe that all bifurcations ranging from the symmetrical to the highly nonsymmetrical have a total bifurcation angle between 75 and 100°. This general characteristic is seen clearly in Fig. 5 where the total bifurcation angle is found to be ~90° in all four cases. Again, qualitatively speaking, this general feature can be observed in the cardiovascular system, particularly so in places such as the retina where the angles can be observed without much distortion, and in places where the angles are not governed by anatomical considerations.

Another general characteristic to be observed is the fact that, in a large number of nonsymmetrical bifurcations, the larger of the two branches "appears" to have the same diameter as that of the parent artery. Three of the four bifurcations in Fig. 5 portray this feature and, again, the same feature can be observed in the cardiovascular system. The reason for this "appearance" lies in the relation given in Eq. 7 between the radii of the two branches. This relation is illustrated in Fig. 7 for different values of the asymmetry ratio α. It is seen that for an asymmetry ratio of as much as 0.5 where the radius of the smaller branch is ~65% of that of the parent artery, the radius of the larger branch is still ~90% of the parent radius.

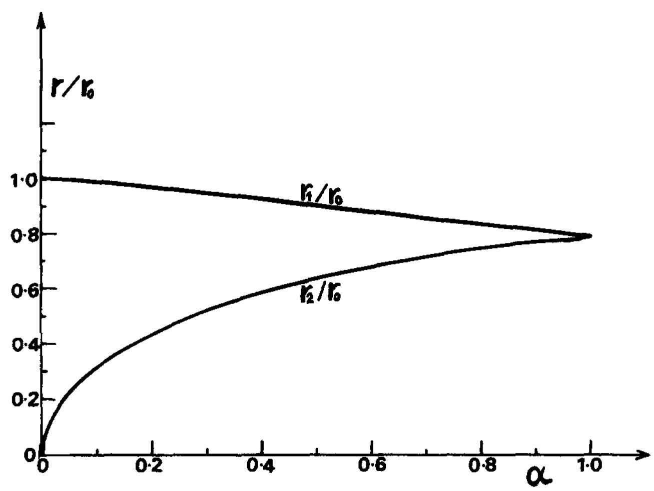

FIGURE 7. The relative radii of the branches in a nonsymmetrical bifurcation as predicted by a compromise between the principles of minimum lumen surface, lumen volume, pumping power, and viscous drag, where α is the asymmetry ratio.

In summary, nonsymmetrical bifurcations predicted by four optimality principles and an optimum relation between the radius and flow in a blood artery are qualitatively not unlike those found in the cardiovascular system. It seems reasonable to conclude that at least some and possibly all of these principles of minimum lumen surface, lumen volume, pumping power, and viscous drag, are among the physiological principles which underlie arterial branching in the cardiovascular system. Figs. 2, 3, and 4 can be used to map out measured arterial junctions to determine whether or not they fall within the shaded regions. Such simple tests may establish more quantitatively the degree of involvement of these optimality principles and of the assumed relation between flow and radius in the cardiovascular system. The required empirical data is at present highly scarce. An experimental program designed to yield such data is underway.

The author is supported in part by the Ontario Heart Foundation and in part by the National Research Council of Canada.

Received for publication 8 May 1978.

REFERENCES

KAMIYA, A., and T. TOGAWA. 1972. Optimal branching of the vascular tree. *Bull. Math. Biophys.* **34**:431-438.

MURRAY, C. D. 1926 *a*. The physiological principle of minimum work. I. The vascular system and the cost of blood volume. *Proc. Natl. Acad. Sci. U.S.A.* **12**:207-214.

MURRAY, C. D. 1926 *b*. The physiological principle of minimum work applied to the angle of branching of arteries. *J. Gen. Physiol.* **9**:835-841.

RODBARD, S. 1975. Vascular caliber. Cardiology. **60**:4-49.

WISE, G. N., C. T. DOLLERY, and P. HENKIND. 1971. The Retinal Circulation. Harper & Row, Publishers, Inc., New York.

ZAMIR, M. 1976 *a*. The role of shear forces in arterial branching. *J. Gen. Physiol.* **67**:213-222.

ZAMIR, M. 1976 *b*. Optimality principles in arterial branching. *J. Theor. Biol.* **62**:227-251.

ZAMIR, M. 1977. Shear forces and blood vessel radii in the cardiovascular system. *J. Gen. Physiol.* **69**:449-461.