Electronic supplementary material: Probabilistic participation in public goods games

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A Payoff Difference

Based on the previous results of voluntary PGGs (Hauert *et al.*, 2002), we calculate the advantage of a potential defector over a potential cooperator.

The variables $x_p := p_x\tilde{x}$, $y_p := p_y(1-\tilde{x})$ and z correspond to the relative frequencies of the three pure strategies of voluntary PGGs. According to Hauert *et al.* (2002), the average payoffs P_c and P_d of a cooperator and a defector are respectively given by

$$P_c = \sigma z^{N-1} + (r-1)(1-z^{N-1}) - r\frac{y_p}{1-z}\left(1-\frac{1-z^N}{N(1-z)}\right), \tag{A.1}$$

$$P_d = \sigma z^{N-1} + r\frac{x_p}{1-z}\left(1-\frac{1-z^N}{N(1-z)}\right), \tag{A.2}$$

where σ is a fixed payoff of loner-like behaviour.

Using P_c and P_d, the expected payoffs $P_{\tilde{c}}$ and $P_{\tilde{d}}$ are given by

$$P_{\tilde{c}} = p_x P_c + (1-p_x)\sigma, \tag{A.3}$$

$$P_{\tilde{d}} = p_y P_d + (1-p_y)\sigma. \tag{A.4}$$

Hence, estimating the advantage of a potential defector over a potential cooperator yields

$$\begin{aligned}
&(1-z)(P_{\tilde{d}} - P_{\tilde{c}})\\
&= \sigma(p_x-p_y)(1-z) + (x_p+y_p)(p_yP_d - p_xP_c)\\
&= \sigma(p_x-p_y)(1-z) - (p_x-p_y)(x_pP_c + y_pP_d) + p_xp_y(P_d-P_c)\\
&= (p_x-p_y)(\sigma - (x_pP_c + y_pP_d + z\sigma)) + p_xp_y(P_d-P_c)\\
&= (p_x-p_y)(\sigma-\bar{P}) + p_xp_y(P_d-P_c),
\end{aligned} \tag{A.5}$$

where we use $x_p + y_p + z = 1$ and $\bar{P} = x_p P_c + y_p P_d + z\sigma$. When z is fixed, this eqn (A.5) enables us to compare the dynamics of the current model with that of voluntary PGGs. In particular, for $p_x = p_y$, z is constant $(1 - p_x = z = 1 - p_y)$ and eqn (A.5) is then reduced to

$$P_{\tilde{d}} - P_{\tilde{c}} = (1 - z)(P_d - P_c). \tag{A.6}$$

Further, substituting eqns (A.1) and (A.2) for $P_{\tilde{d}} - P_{\tilde{c}}$, leads to the advantage function:

$$\begin{aligned}
&P_{\tilde{d}} - P_{\tilde{c}} \\
&= \sigma(p_x - p_y) + (p_y P_d - p_x P_c) \\
&= \sigma(p_x - p_y) + \sigma(p_x - p_y) z^{N-1} - (r-1)p_x(1 - z^{N-1}) \\
&\qquad\qquad\qquad + \frac{r}{1-z}(p_y x_p + p_x y_p)\left(1 - \frac{1 - z^N}{N(1-z)}\right) \\
&= (\sigma(p_x - p_y) - (r-1)p_x)(1 - z^{N-1}) + r\frac{p_x p_y}{1-z}\left(1 - \frac{1 - z^N}{N(1-z)}\right) \\
&=: \widetilde{F}(z(\tilde{x})),
\end{aligned}$$

where we use $p_y x_p + p_x y_p = p_y p_x \tilde{x} + p_x p_y (1 - \tilde{x}) = p_x p_y$.

We introduce some notations, $a := (\sigma - r + 1)p_x - \sigma p_y$ and $b := r p_x p_y$. Then the derivative of $\widetilde{F}$ with respect to z is given by

$$\begin{aligned}
\frac{d\widetilde{F}}{dz} &= \frac{d}{dz}\left(-a z^{N-1} + \frac{b}{N}\sum_{k=0}^{N-2}(N - 1 - k)z^k\right) \\
&= -a(N-1)z^{N-2} + \frac{b}{N}\sum_{k=1}^{N-2}(N-1-k)k z^{k-1} \quad (N \geq 3),
\end{aligned}$$

and $\widetilde{F}'(z) = -a$ $(N = 2)$. In the parameter space (p_x, p_y), it is always the case that $a \leq 0$ and $b \geq 0$. In particular, $a = 0 \wedge b = 0 \Leftrightarrow (p_x, p_y) = (0, 0)$. Therefore, we have $d\widetilde{F}/dz \geq 0$ $(d\widetilde{F}/dz = 0 \Leftrightarrow (p_x, p_y) = (0, 0))$ and

$$\frac{d\widetilde{F}}{d\tilde{x}} = \frac{d\widetilde{F}}{dz}\frac{dz}{d\tilde{x}} \begin{cases} > & 0 \quad (p_x < p_y) \\ = & 0 \quad (p_x = p_y) \\ < & 0 \quad (p_x > p_y), \end{cases}$$

where $dz/d\tilde{x} = p_y - p_x$.

B Arrangement of Dynamical Regimes

We separate the parameter space (p_x, p_y) depending on whether the signs of $\widetilde{F}(z(\tilde{x}))$ at each $\tilde{x} = 0, 1$ are the same. Consequently we obtain the following four regions:

$$
\begin{array}{ll}
\text{(i)} & \{\widetilde{F}(z(0)) \geq 0\} \cap \{\widetilde{F}(z(1)) \geq 0\} \\
\text{(ii)} & \{\widetilde{F}(z(0)) < 0\} \cap \{\widetilde{F}(z(1)) > 0\} \\
\text{(iii)} & \{\widetilde{F}(z(0)) \leq 0\} \cap \{\widetilde{F}(z(1)) \leq 0\} \\
\text{(iv)} & \{\widetilde{F}(z(0)) > 0\} \cap \{\widetilde{F}(z(1)) < 0\},
\end{array}
$$

where $z(0) = 1 - p_y$ and $z(1) = 1 - p_x$. Monotonically increasing $\widetilde{F}(z(\tilde{x})) \Leftrightarrow p_x < p_y$ yields that $p_x < p_y$ holds in the region (ii). Likewise, monotonically decreasing $\widetilde{F}(z(\tilde{x})) \Leftrightarrow p_x > p_y$ yields that $p_x > p_y$ holds in the region (iv).

On the diagonal $p_x = p_y$, $\widetilde{F}(z)$ is equal to $(1-z)F(z)$ (eqns (A.6)). According to Hauert *et al.* (2002), $F(0) > 0$ and $F(1) = 0$ hold. Then, if $F(z)$ has a unique interior root $\hat{z}$ in $(0, 1)$ $(r > 2)$, $F(z) > 0$ for $0 < z < \hat{z}$ and $F(z) < 0$ for $\hat{z} < z < 1$. If $F(z)$ has no root $\hat{z}$ there $(r \leq 2)$, $F(z) > 0$ for all $0 < z < 1$. Let **Q** be the point $(1-\hat{z}, 1-\hat{z})$ in (p_x, p_y). The properties of $F(z)$ lead to that for $r > 2$, at the point **Q** the diagonal $p_x = p_y$ is divided into the two segment covered by (i) and (iii), and for $r \leq 2$, the diagonal is included in (i). We specifically denote the two points set $\{(0, 0), \mathbf{Q}\}$ as (v) and exclude this from (i) and (iii).

By using $F(z)$, we obtain

$$
\begin{aligned}
\widetilde{F}(z(0)) &= \sigma(p_x - p_y)(1 - (1 - p_y)^{N-1}) + p_x F(1 - p_y), \\
\widetilde{F}(z(1)) &= (\sigma - r + 1)(p_x - p_y)(1 - (1 - p_x)^{N-1}) + p_y F(1 - p_x).
\end{aligned}
$$

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be the boundary curves of the four regions (i–iv), defined by $g_1(p_x, p_y) := \widetilde{F}(z(1)) = 0$ and $g_2(p_x, p_y) := \widetilde{F}(z(0)) = 0$, respectively. The monotonicity of $\widetilde{F}(z(\tilde{x}))$ yields that if $\widetilde{F}(z(0)) = \widetilde{F}(z(1))$, $z(0) = z(1)$ $(p_x = p_y)$ holds. Hence, the intersection of $\mathbf{C}_1$ and $\mathbf{C}_2$ exists on the diagonal. Since $g_1(1-z, 1-z) = g_2(1-z, 1-z) = F(z)$, The intersection consists of the single point $(0, 0)$ for $r \leq 2$ or the two points: $(0, 0)$ and **Q** for $r > 2$.

In order to investigate the arrangement of the regions in the vicinity of **Q**, we compute the Jacobian of (g_1, g_2) at **Q** as follows:

$$
\begin{aligned}
\left.\frac{\partial(g_1, g_2)}{\partial(p_x, p_y)}\right|_{\mathbf{Q}} &= \left|\begin{array}{cc} \frac{\partial g_1}{\partial p_x} & \frac{\partial g_1}{\partial p_y} \\ \frac{\partial g_2}{\partial p_x} & \frac{\partial g_2}{\partial p_y} \end{array}\right|_{\mathbf{Q}} \\
&= \left|\begin{array}{cc} (\sigma - r + 1)(1 - \hat{z}^N) - (1 - \hat{z})F'(\hat{z}) & -(\sigma - r + 1)(1 - \hat{z}^N) \\ \sigma(1 - \hat{z}^N) & -\sigma(1 - \hat{z}^N) - (1 - \hat{z})F'(\hat{z}) \end{array}\right| \\
&= (1 - \hat{z})F'(\hat{z})\{(1 - \hat{z})F'(\hat{z}) + (r - 1)(1 - \hat{z}^N)\} \\
&= (1 - \hat{z})F'(\hat{z})\{(r - 1)(N - 1)(1 - \hat{z})\hat{z}^{N-2} + (r - 2)(1 - \hat{z}^{N-1})\}.
\end{aligned}
$$

Because $0 < \hat{z} < 1$, $F'(\hat{z}) < 0$ (Hauert *et al.*, 2002), $r > 2$ and $N \geq 3$, the Jacobian is nonzero. Therefore, $\mathbf{C}_1$ and $\mathbf{C}_2$ are transversely crossing at **Q**, and the four regions (i–iv) exist in any small neighborhood of **Q**.

C Average Payoff at Equilibrium

We here calculate the average population payoff in each moment at each type of equilibria of this model. Let $\bar{P}_z$ be the average population payoff at a point z.

Firstly, we consider the case of interior equilibrium, that is, the region (ii) and (iv), and denote the unique interior fixed point as $\tilde{z}$. We then suppose the case of $r > 2$ (**Q** exists). Considering a sufficient small neighborhood of **Q**, allows us to assume that the range of z contains the unique root $\hat{z}$ of $F(z)$.

Using eqn (A.5):

$$(1 - z)\widetilde{F}(z) = (p_x - p_y)(\sigma - \bar{P}_z) + p_x p_y F(z) \tag{C.1}$$

leads to

$$\widetilde{F}(\hat{z}) = 0 \Leftrightarrow \tilde{z} = \hat{z} \Leftrightarrow \bar{P}_{\tilde{z}} = \sigma.$$

Since the signs of $F(z)$ and $\widetilde{F}(z)$ change once from $(+)$ to $(-)$ and from $(-)$ to $(+)$ in the open interval $(0, 1)$ respectively, we obtain

$$\widetilde{F}(\hat{z}) > 0 \Leftrightarrow \tilde{z} < \hat{z} \Leftrightarrow F(\tilde{z}) > 0.$$

For the region (ii) ($p_x < p_y$), using $\widetilde{F}(\tilde{z}) = 0$ in eqn (C.1), we obtain $F(\tilde{z}) > 0$ $\Leftrightarrow \bar{P}_{\tilde{z}} < \sigma$, that is,

$$\widetilde{F}(\hat{z}) < 0 \Leftrightarrow \bar{P}_{\tilde{z}} > \sigma.$$

Likewise, for the region (iv) ($p_x > p_y$), $\widetilde{F}(\hat{z}) > 0 \Leftrightarrow \bar{P}_{\tilde{z}} > \sigma$ holds. Let $\mathbf{C}_3$ be the curve $\widetilde{F}(\hat{z}) = 0$. The above results can be summarized by stating that $\mathbf{C}_3$ divides each of the region (ii) and (iv) into two subregions which are characterized by $\bar{P}_{\tilde{z}} > \sigma$ and $\bar{P}_{\tilde{z}} < \sigma$, in the vicinity of **Q**. $\bar{P}_{\tilde{z}}$ is then equal to σ on $\mathbf{C}_3$.

Conversely, if $r \leq 2$ (**Q** does not exists), $F(z) > 0$ always holds in $(0, 1)$. Using eqn (C.1) yields that $p_x < p_y$ in (ii) $\Rightarrow \bar{P}_{\tilde{z}} < \sigma$, and $p_x > p_y$ in (iv)$\Rightarrow \bar{P}_{\tilde{z}} > \sigma$. Other two subregions: $\bar{P}^* > \sigma$ in (ii) and $\bar{P}^* < \sigma$ in (iv), do not appear. We then notice that $N = 2 \Rightarrow r \leq 2$ because of the precondition of the public goods game: $1 < r < N$.

Secondly, we remark the case of trivial equilibrium, that is, both $\widetilde{\mathrm{C}}$- and $\widetilde{\mathrm{D}}$-homogeneous states. In the former, substituting $y_p = 0$ ($\tilde{x} = 1$) for eqn (A.1) yields $P_c = \sigma z^{N-1} + (r-1)(1 - z^{N-1})$. Thus, since $p_x = 1 - z$ and $\sigma < r - 1$ (the precondition of PGG with loners), eqn (A.3) yields

$$\begin{aligned} P_{\tilde{c}} &= (\sigma - r + 1)\{z + (1-z)z^{N-1}\} + (r-1) \\ &\geq (\sigma - r + 1) \cdot 1 + (r-1) \\ &= \sigma, \end{aligned}$$

where $P_{\tilde{c}} = \sigma \Leftrightarrow p_x = 0$ which means that all of the population are actually pure loners. In the latter, likewise, substituting $x_p = 0$ ($\tilde{x} = 0$) for eqn (A.2) yields $P_d = \sigma z^{N-1}$. Thus, since $p_y = 1 - z$, eqn (A.4) yields

$$P_{\tilde{d}} = \sigma\{z + (1-z)z^{N-1}\} \leq \sigma,$$

where $P_{\tilde{d}} = \sigma \Leftrightarrow p_y = 0$.

References

[1] Hauert, C., De Monte, S., Hofbauer, J. and Sigmund, K. 2002 Replicator dynamics for optional public goods games. *J. Theor. Biol.* **218**, 187–194. (DOI: 10.1006/jtbi.2002.3067)