Depolymerization-driven Flow in Nematode Spermatozoa Relates Crawling Speed to Size and Shape

Supplemental Material

The relation of the model to mass conservation

In this Supplemental Material section, we derive the two-phase model that we use by starting from a three-phase model that takes into account the cytosol and the MSP in dimer and polymer form. We work in terms of mass conservation and show that the equations written in terms of volume fraction, as shown in the body of the paper, are equivalent to the equations that track the mass of each phase. We assume that the cell membrane is impermeable to MSP, but that fluid is driven into or out of the cell by pressure gradients. The fluid velocity in our two-phase model is shown to account for both the movement of the fluid and the diffusion of the soluble MSP dimer.

In any given volume element inside the cell, a fraction of the space is filled by the fluid cytosol. We denote this volume fraction as ϕ_f. The remaining space is filled by either MSP polymer or soluble MSP dimer. The volume fractions for these two phases are ϕ_s and ϕ_d, respectively. (The s denotes that the polymer component is treated as a solid.) If these three phases are the only phases present in the cytoplasm, then

$$\phi_f + \phi_s + \phi_d = 1 \tag{S1}$$

The density of the cytosol is ρ_f and the density of the MSP is ρ_s. We also define the velocity of the fluid and the velocity of the polymer as $\mathbf{V}_f$ and $\mathbf{V}_s$, respectively. Conservation of the mass in each phase leads to three continuity equations,

$$\frac{\partial}{\partial t}\left(\rho_f \phi_f\right) = -\nabla \cdot \left(\rho_f \phi_f \mathbf{V}_f\right) \tag{S2}$$

$$\frac{\partial}{\partial t}\left(\rho_s \phi_s\right) = -\nabla \cdot \left(\rho_s \phi_s \mathbf{V}_s\right) - k_s \rho_s \phi_s \tag{S3}$$

$$\frac{\partial}{\partial t}\left(\rho_s \phi_d\right) = -\nabla \cdot \left(\rho_s \phi_d \mathbf{V}_f - D_d \rho_s \nabla \phi_d\right) + k_s \rho_s \phi_s \tag{S4}$$

where k_s is the disassembly rate for the polymer. Based on experimental evidence, we have assumed here that polymerization of MSP only occurs at

the membrane, and, therefore, polymerization comes in as a boundary condition. Disassembly is assumed to occur everywhere. We have also used that the MSP dimer is adevected with the fluid velocity and that it also diffuses with diffusion coefficient, D_d. If the cytosol and MSP are incompressible, the densities in these phases are constant, and we can divide them out, leading to three equations for the volume fractions,

$$\frac{\partial \phi_f}{\partial t} = -\nabla \cdot (\phi_f \mathbf{V}_f) \tag{S5}$$

$$\frac{\partial \phi_s}{\partial t} = -\nabla \cdot (\phi_s \mathbf{V}_s) - k_s \phi_s \tag{S6}$$

$$\frac{\partial \phi_d}{\partial t} = -\nabla \cdot (\phi_d \mathbf{V}_f - D_d \nabla \phi_d) + k_s \phi_s \tag{S7}$$

Eq. S6 is the continuity equation that we will use to solve for the dynamics of the MSP polymer (Eq. 5 in the main text). By adding Eqs. S5- S7, we get a conservation equation for the total volume in a given volume element,

$$\nabla \cdot (\phi_f \mathbf{V}_f + \phi_s \mathbf{V}_s + \phi_d \mathbf{V}_d - D_d \nabla \phi_d) = 0 \tag{S8}$$

We are interested in deriving a model that treats the cytosolic fluid and the MSP dimer as a single phase. To do this, we use that $\phi_f + \phi_d = 1 - \phi_s$ and define the effective fluid velocity as $\mathbf{V}'_f = \mathbf{V}_f - D_d \nabla \phi_d / (1 - \phi_s)$. Eq. S8 becomes

$$\nabla \cdot \big((1 - \phi_s) \mathbf{V}'_f + \phi_s \mathbf{V}_s \big) = 0 \tag{S9}$$

which is the effective two-phase conservation equation (Eq. 4 in the main text).

At the cell membrane, we assume that the membrane is impermeable to MSP. Therefore, the normal component of the mass flux of MSP (for both the polymer and the dimer) at the boundary must move with the cell membrane. If the membrane is permeable to fluid, then there is a discrepency between the fluid mass flux and the motion of the cell membrane that is proportional to the pressure difference across the membrane. Assuming steady crawling with velocity $\mathbf{V}_0$, we get the boundary conditions for all three phases,

$$\hat{\mathbf{n}} \cdot (\rho_s \phi_s \mathbf{V}_s) + J(\mathbf{x}_b) = \rho_s \phi_s \mathbf{V}_0 \cdot \hat{\mathbf{n}} \tag{S10}$$

$$\hat{\mathbf{n}} \cdot (\rho_s \phi_d \mathbf{V}_f - D_d \rho_s \nabla \phi_d) - J(\mathbf{x}_b) = \rho_s \phi_d \mathbf{V}_0 \cdot \hat{\mathbf{n}} \tag{S11}$$

$$\hat{\mathbf{n}} \cdot (\rho_f \phi_f \mathbf{V}_f) = \rho_f \phi_f \mathbf{V}_0 \cdot \hat{\mathbf{n}} + k_f \rho_f \left(p - p_0 \right) \tag{S12}$$

Here $J(\mathbf{x}_b)$ is a spatially dependent polymerization rate that converts MSP dimer into polymer. $\mathbf{x}_b$ denotes the coordinates of the boundary. k_f is the filtration coefficient, p is the pressure inside the cell, and p_0 is the external pressure. Dividing out the densities and adding these three equations, leads to the boundary condition on the total volume flux (Eq. 11 in the main text),

$$\hat{\mathbf{n}} \cdot \left(\phi_s \mathbf{V}_s + (1-\phi_s)\,\mathbf{V}'_f\right) = \mathbf{V}_0 \cdot \hat{\mathbf{n}} + k_f\,(p - p_0) \tag{S13}$$

Therefore, the equations in the main body of the paper for the dynamics of the volume fraction, ϕ, (Eq. 5), the conservation equation (Eq. 4), and the boundary condition on the volume flux (Eq. 11) are equivalent to equations derived from mass conservation (Eqs. S6, S9, & S13). Note that here ϕ_s and $\mathbf{V}'_f$ correspond to ϕ and $\mathbf{V}_f$, respectively, in the main text. Force balance in the bulk and on the boundary leads to equations for the velocities as well as a boundary condition on ϕ_s, which close the system of equations (See main text).