Incipient criticality in ecological communities – Supplementary Material

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Derivation of Main Result

Note: references to equations in the main text are numbered normally, whereas equations in the supplementary material are numbered with the prefix "SM".

Here we present the derivations of our central results. In the main text it was stated that $n_i \geq 1, i = 1, 2, ...S(A)$. Here we relax this assumption and we assume that $n_i \geq n_0 > 0$, where n_0 is interpreted as an A-independent lower cutoff for the n_is.

We approximate the sum over discrete distributions such as $P(n|A)$ with integrals from n_0 to ∞. This is an excellent approximation because the integrals are only used for deducing the scaling properties of the various functions. We do not choose 0 as the lower integration limit becuase this results in divergent integrals for standard RSA distributions such as the Fisher log series.

On substituting the scaling hypothesis, Eqn. 4, into the normalization condition, Eqn. 1, one obtains:

$$g(A) \int_{n_0}^{\infty} dn\, F(n/f(A)) = 1. \qquad \textbf{[SM.1]}$$

With a change of variable $x = n/f(A)$, the above equation can be rewritten as:

$$g(A)f(A) \int_{n_0/f(A)}^{\infty} dx\, F(x) = 1. \qquad \textbf{[SM.2]}$$

Likewise, one may write an expression for the average abundance per species:

$$\langle n \rangle_A = \int_{n_0}^{\infty} dn\, n\, P(n|A) = g(A)(f(A))^2 \int_{n_0/f(A)}^{\infty} dx\, x\, F(x). \qquad \textbf{[SM.3]}$$

In physics, scaling holds in the thermodynamic limit or infinite system size. Here we expect scaling to hold, if at all, for large values of A. We therefore turn to a consideration of the behavior of the integrals in Eqns. SM.2 and SM.3 in this limit. $f(A)$ is an increasing function of A. It is the characteristic scale of the abundance in an area A and diverges as $A \to \infty$. Consider first the integral in Eqn. SM.2 in the scaling regime and divide it into two parts:

$$\int_{n_0/f(A)}^{\infty} dx\, F(x) = \int_{n_0/f(A)}^{b} dx\, F(x) + \int_{b}^{\infty} dx\, F(x) \qquad \textbf{[SM.4]}$$

$$\sim \int_{n_0/f(A)}^{b} dx\, x^{-\Delta} + C_1. \qquad \textbf{[SM.5]}$$

Our focus is on elucidating the A dependence of the integral. We select a small cut-off value b, independent of A, so that for any $x < b$, we are in the limit in which Eq. 5 ($x^{\Delta}F(x) \sim constant$) holds. The first term on the right hand side of Eqn. SM.5 controls the scaling behavior and thus the behavior of $F(x)$ at intermediate values of x does not play a role. The second term on the right hand side of the Eqn. SM.5 does not depend on A and is a constant that we have denoted by the quantity C_1. The same procedure is applied to Eqn.SM.3 yielding:

$$\int_{n_0/f(A)}^{\infty} dx\, x\, F(x) = \int_{n_0/f(A)}^{b} dx\, x\, F(x) + \int_{b}^{\infty} dx\, x\, F(x) \qquad \textbf{[SM.6]}$$

$$\sim \int_{n_0/f(A)}^{b} dx\, x^{1-\Delta} + C_2. \qquad \textbf{[SM.7]}$$

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Abbreviations: SAD, Species Abundance Distribution; RSA, Relative Species Abundance; BCI, Barro Colorado Island

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In the large A limit, we now probe the convergence properties of the integrals on the right hand side of Eqns. SM.5 and SM.7. Let us consider the integral in Eqn. SM.5 first. From elementary calculus ($\Delta \neq 1$):

$$\int_{n_0/f(A)}^{b} dx\, x^{-\Delta} = \frac{b^{1-\Delta} - (f(A)/n_0)^{\Delta-1}}{1-\Delta} \quad \textbf{[SM.8]}$$

As $f(A) \to \infty$ this integral will diverge if $\Delta > 1$, or will converge to a constant value when $\Delta < 1$. Likewise, for the integral on the right hand side of Eqn. SM.7 we have:

$$\int_{n_0/f(A)}^{b} dx\, x^{1-\Delta} = \frac{b^{2-\Delta} - (f(A)/n_0)^{\Delta-2}}{2-\Delta} \quad \textbf{[SM.9]}$$

which diverges if $\Delta > 2$, and converges to a constant value when $\Delta < 2$. Thus we have 3 cases, depending on the value of Δ.

Case 1: $\Delta < 1$**.** In this case the integrals on the right hand side of Eqns. SM.5 and SM.7 are constant and independent of A, so that Eqns. SM.2 and SM.3 become:

$$g(A)f(A) \sim 1 \quad \textbf{[SM.10]}$$
$$\langle n \rangle_A \sim g(A)(f(A))^2, \quad \textbf{[SM.11]}$$

from which we deduce:

$$f(A) \sim < n >_A \quad \textbf{[SM.12]}$$
$$g(A) \sim 1/f(A). \quad \textbf{[SM.13]}$$

Case 2: $1 < \Delta < 2$**.** Now the integral on the right hand side of Eqn. SM.7 remains independent of A, but the integral on the right hand side of Eqn. SM.5 diverges as $(f(A))^{\Delta-1}$. This results in Eqns. SM.2 and SM.3 becoming:

$$g(A)(f(A))^{\Delta} \sim 1 \quad \textbf{[SM.14]}$$
$$\langle n \rangle_A \sim g(A)(f(A))^2, \quad \textbf{[SM.15]}$$

from which we deduce:

$$f(A) \sim < n >_A^{1/(2-\Delta)} \quad \textbf{[SM.16]}$$
$$g(A) \sim (f(A))^{-\Delta} \quad \textbf{[SM.17]}$$

Case 3: $\Delta > 2$**.** Now the integral in Eqn. SM.7 diverges as $(f(A))^{\Delta-2}$ while the integral in Eqn. SM.5 diverges as $(f(A))^{\Delta-1}$. Thus Eqns. SM.2 and SM.3 become:

$$g(A)(f(A))^{\Delta} \sim 1 \quad \textbf{[SM.18]}$$
$$\langle n \rangle_A \sim constant, \quad \textbf{[SM.19]}$$

from which we deduce:

$$S(A) \sim A \quad \textbf{[SM.20]}$$
$$g(A) \sim (f(A))^{-\Delta} \quad \textbf{[SM.21]}$$

In this last case, information on the scaling of $f(A)$ may be obtained by examining the second moment $\langle n^2 \rangle_A$. It is useful to define a β-diversity function $\beta(\mathbf{r})$ as the probability of finding two conspecific individuals separated by a vector $\mathbf{r}$. It has been demonstrated (1) that $\langle n^2 \rangle_A / \langle n \rangle_A = \sum_{\mathbf{r}} \beta(\mathbf{r})$ which scales as

$$\frac{\langle n^2 \rangle_A}{\langle n \rangle_A} \sim \text{const} \qquad \Delta > 3 \quad \textbf{[SM.22]}$$
$$\frac{\langle n^2 \rangle_A}{\langle n \rangle_A} \sim f(A)^{3-\Delta} \qquad 2 < \Delta < 3 \quad \textbf{[SM.23]}$$
$$\frac{\langle n^2 \rangle_A}{\langle n \rangle_A} \sim f(A) \qquad \Delta < 2 \quad \textbf{[SM.24]}$$

These relationships allow one to treat $\Delta > 2$ using β-diversity data.

We now show that, when the scaling collapse holds, the moment ratios are power laws of $\langle n \rangle$ and for $1 < \Delta < 2$ that we obtain Eq. 3 in the main text. If the scaling collapse holds, using the same techniques as before:

$$\langle n^k \rangle_A = \int_{n_0}^{\infty} n^k \, P(n|A) dn \quad \textbf{[SM.25]}$$

$$= g(A) f(A)^{k+1} \int_{n_0/f(A)}^{\infty} dx \, x^k \, F(x) \quad \textbf{[SM.26]}$$

$$= g(A) f(A)^{k+1} \left(\int_{n_0/f(A)}^{b} dx \, x^k \, F(x) + \int_{b}^{\infty} dx \, x^k \, F(x) \right) \quad \textbf{[SM.27]}$$

$$= g(A) f(A)^{k+1} \left(\int_{n_0/f(A)}^{b} dx \, x^{k-\Delta} + \int_{b}^{\infty} dx \, x^k \, F(x) \right) \quad \textbf{[SM.28]}$$

$$= g(A) f(A)^{k+1} \left(\frac{b^{k-\Delta+1} - \left(\frac{n_0}{f(A)}\right)^{k-\Delta+1}}{k - \Delta + 1} + C_k \right) \quad \textbf{[SM.29]}$$

When A is large enough, then $f(A)$ is the dominant term in the expression in parenthesis; the value of $\langle n^k \rangle_A$ depends on the sign of $k - \Delta + 1$. We have:

$$\langle n^k \rangle_A \sim f(A)^{\Delta} \qquad k < \Delta - 1, \quad \textbf{[SM.30]}$$

$$\langle n^k \rangle_A \sim f(A)^{k} \qquad k > \Delta - 1. \quad \textbf{[SM.31]}$$

The moment ratios are thus calculated as:

$$n_k(A) = \frac{\langle n^k \rangle_A}{\langle n^{k-1} \rangle_A} \quad \textbf{[SM.32]}$$

$$= f(A) \frac{\left(\frac{a^{k-\Delta+1} - \left(\frac{n_0}{f(A)}\right)^{k-\Delta+1}}{k-\Delta+1} + C_k \right)}{\left(\frac{a^{k-\Delta} + \left(\frac{n_0}{f(A)}\right)^{k-\Delta}}{k-\Delta} + C_{k-1} \right)} \quad \textbf{[SM.33]}$$

When A is large enough, then using Eqs SM.30 and SM.31 we see that this last expression takes 3 different values depending on the relative values of k and Δ. Specifically:

$$n_k(A) \sim 1 \qquad k < \Delta - 1, \quad \textbf{[SM.34]}$$

$$n_k(A) \sim f(A)^{k-\Delta+1} \qquad \Delta - 1 < k < \Delta, \quad \textbf{[SM.35]}$$

$$n_k(A) \sim f(A) \qquad k > \Delta. \quad \textbf{[SM.36]}$$

On inserting Eq. 9, we obtain Eq. 3 in the text for the case $1 < \Delta < 2$.

We turn now to the special case of $\Delta = 1$, which separates two distinct scaling regimes (Eqns. 8 and 9) and, just as in physics, is associated with logarithmic corrections. We begin by expanding x^{Δ} in series around $\Delta = 1$:

$$x^{\Delta} = x \, x^{\Delta-1} = x \left(1 + (\Delta - 1) \ln x + \frac{1}{2} (\Delta - 1)^2 \ln^2 x + \cdots \right) \quad \textbf{[SM.37]}$$

and if we take only the first order in $\Delta - 1 \equiv a$ into account, then we have

$$F(x) \underset{x \sim 0}{\sim} \frac{1 - a \ln x}{x}, \quad \textbf{[SM.38]}$$

proceeding as before, from the normalization condition, Eq 1 in the main text, we obtain

$$1 = g(A) \int_{1}^{\infty} \mathrm{d}n \, F\left(\frac{n}{f}\right) \quad \textbf{[SM.39]}$$

$$\sim g(A) \, f(A) \left(\int_{1/f(A)}^{1} \mathrm{d}x \, \frac{1 - a \ln x}{x} + \text{const} \right) \quad \textbf{[SM.40]}$$

$$\sim g(A) \, f(A) \ln f(A) \left(K + \frac{a}{2} \ln f(A) \right) \quad \textbf{[SM.41]}$$

$$\sim f(A) \, g(A) \ln f(A) \left(1 + \frac{a'}{2} \ln f(A) \right) \quad \textbf{[SM.42]}$$

with $x = n/f(A)$, $a' \equiv a/K$ and K is a constant. With the same procedure we obtain also, when $p > 0$:

$$\langle n^p \rangle = \int_1^\infty \mathrm{d}n\, n^p\, P(n|A) \qquad \textbf{[SM.43]}$$

$$\sim g(A)\, f(A)^{p+1} \left(\int_{1/f}^1 \mathrm{d}x x^p \frac{1 - a \ln x}{x} + \text{const} \right) \qquad \textbf{[SM.44]}$$

$$\sim g(A)\, f(A)^{p+1} \qquad \textbf{[SM.45]}$$

where the last equation follows because the integral between parentheses do not diverge. Now, using Eq.SM.42 and the fact that $\langle n \rangle \sim g(A)\, f(A)^2$ we obtain:

$$\ln\langle n \rangle = z - \ln z - \ln\left(1 + \frac{a'}{2} z\right) \qquad \textbf{[SM.46]}$$

where $z \equiv \ln f$. Since $z \gg \ln z$ we can expand in $\ln z/z$ and get

$$z = \ln\langle n \rangle + \ln\ln\langle n \rangle + \ln\left(1 + \frac{a'}{2}\ln\langle n \rangle\right) + \mathcal{O}\left(\frac{\ln\ln\langle n \rangle}{\ln n}\right), \qquad \textbf{[SM.47]}$$

which gives

$$f \sim \langle n \rangle \ln\langle n \rangle \left(1 + \frac{a'}{2}\ln\langle n \rangle\right). \qquad \textbf{[SM.48]}$$

which is Eq. 12 in the main text. The resulting scaling in the case $\Delta = 1$ is thus

$$\ln\langle n \rangle_A \left(1 + \frac{a'}{2}\ln\langle n \rangle\right) C(n|A) = F_1(n/f(A)). \qquad \textbf{[SM.49]}$$

From Eq.SM.45 we also obtain the moment ratios:

$$\frac{\langle n^{p+1} \rangle}{\langle n^p \rangle} = f(A) = \langle n \rangle \ln\langle n \rangle \left(1 + \frac{a'}{2}\ln\langle n \rangle\right) \qquad \text{for all } p \geq 0 \qquad \textbf{[SM.50]}$$

References

1. Zillio, T, Volkov, I, Banavar, JR, Hubbell, SP, Maritan, A (2005) Spatial scaling in model plant communities. *Phys. Rev. Lett.* 95:098101.