

We Age Because We Grow

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Appendix (by Arthur J. Robson)

1 Detailed Assumptions for the Model

The gross energy production rate, G , of an individual is an increasing function of the quantity, K , and the quality, Q , of somatic capital. The energy net of fertility is then an increasing function, F , say, of this gross energy, G , and a decreasing function of fertility, s .

Assumption 1. $G : [0, \infty)^2 \rightarrow [0, \infty)$, $G \in C^2([0, \infty)^2)$. $G(0, Q) = G(K, 0) = 0$, $G_K(K, Q) > 0$, $G_Q(K, Q) > 0$, for all $K > 0$ and $Q > 0$.

$F \in C^2((0, \infty) \times [0, \infty))$. In addition, $\lim_{(G,s) \rightarrow (0,0)} F(G, s) = 0 \equiv F(0, 0)$. Also $F_G(G, s) \in (0, \ell]$, for some $\ell < \infty$, $F_s(G, s) < 0$, $F_{Gs}(G, s) > 0$, $F_{ss}(G, s) < 0$, for all $G > 0$ and $s \geq 0$. For analytic simplicity, finally, $F_s(G, s) \rightarrow -\infty$, as $s \rightarrow \infty$, for all $G \geq 0$, and $\lim_{(G',s') \rightarrow (0,s)} F(G', s') = -\infty$, for all $s > 0$.

Investment in somatic capital is irreversible and capital does not depreciate. Further, there is an initial period of maximal growth, until age t^* , say, followed by a growth plateau—

Assumption 2. *The capital stock, $K(t)$, at age t , evolves as*

$$\frac{dK(t)}{dt} = v(t) \text{ where } K(0) = K_0 > 0,$$

where investment $v(t) \in [0, \bar{v}]$, for some $\bar{v} \in (0, \infty)$. Indeed, there exists $t^* \geq 0$ such that

$$v = \bar{v}, \text{ for all } t \in [0, t^*] \text{ but } v = 0, \text{ for all } t > t^*.$$

The energy cost of this investment in quantity is αv , for some $\alpha > 0$.

In the absence of investment in quality, quality Q depreciates at a constant rate, ρ . Investment to offset or reverse such depreciation, w , has an energetic cost, where this cost is an increasing function of the quantity of somatic capital, K . Formally:

Assumption 3. *The quality of somatic capital at age t , $Q(t)$, evolves as:*

$$\begin{aligned} \frac{dQ(t)}{dt} &= w(t) - \rho; \text{ where } \rho > 0, \text{ for all } Q(t) > 0, \text{ but} \\ \frac{dQ(t)}{dt} &= 0, \text{ whenever } Q(t) = 0 \text{ and } Q(0) = Q_0. \end{aligned}$$

The energetic cost of investment in quality is

$$\beta d(w, K), \text{ where } \beta > 0, d : [0, \infty)^2 \rightarrow [0, \infty), d \in C^2([0, \infty)^2),$$

$$\text{and } d(0, K) = d(w, 0) = d_w(0, K) = 0 \text{ for all } w, K \geq 0$$

$$\text{Furthermore } d_{ww}(w, K) > 0 \text{ for all } w \geq 0 \text{ and } K > 0$$

$$\text{whereas } d_K(w, K) > 0, \text{ for all } w > 0 \text{ and } K \geq 0.$$

Lower mortality comes at a greater cost and, indeed, at a greater marginal cost, as follows:

Assumption 4. If $\mu(t)$ is the rate of mortality at age t , and $p(t) \in [0, 1]$ is the probability of survival to age t , then

$$\frac{1}{p(t)} \frac{dp(t)}{dt} = -\mu(t) \text{ where } p(0) = 1.$$

The energetic cost of μ is

$$e : (\underline{\mu}, \infty) \rightarrow [0, \infty), e \in C^2(\underline{\mu}, \infty),$$

$$e'(\mu) < 0, e''(\mu) > 0, \text{ for all } \mu \in (\underline{\mu}, \infty).$$

$$e(\mu) \rightarrow \infty, \text{ as } \mu \rightarrow \underline{\mu}; \text{ whereas } e(\mu) \rightarrow 0, \text{ as } \mu \rightarrow \infty.$$

For ease of reference, the Euler-Lotka equation is included in this Appendix—

$$\int_0^\infty e^{-rt} p(t) s(t) dt = 1. \quad (1)$$

Similarly included is the steady-state “budget balance condition”—

$$\int_0^\infty e^{-rt} p(t) (F(G(K(t), Q(t)), s(t)) - \alpha v(t) - \beta w(t) d(K(t)) - e(\mu(t))) dt \geq 0. \quad (2)$$

2 Interpretation of the Parameter β

This subsection continues the discussion of the size of the parameter β that is started in the second paragraph after the statement of Theorem 1. Consider then a germline which has quantity $k > 0$, and quality q . For simplicity, take the quantity and quality of this germline to be constant over time. The quality of the germline is the initial quality of an individual, so that $q = Q_0$. The cost of maintaining the quality of the germline, with same technology as for the somatic line, is then $\beta d(\rho, k)$, with marginal cost $\beta d_w(\rho, k)$. Suppose that individuals are created from the germline at a constant continuous rate $\delta > 0$.¹ It follows that the marginal benefit of a higher quality germline is then $\delta\psi(0, \beta)$, where $\psi(0, \beta)$ is marginal lifetime benefit from higher initial quality for an individual of age $t = 0$. This marginal valuation of quality is defined in the next subsection and is used in the proof of Theorem 1.² Altogether, the condition $\beta d_w(\rho, k) = \delta\psi(0, \beta)$ describes the optimal choice of the quality of the germline q , given this is also the initial quality of an individual, Q_0 .

How then is this story consistent with the stylized facts that the germline is very small, and that the initial quality of an individual, while high, is certainly finite? It can be shown that $\psi(0, \beta)$ has a well-defined limit as $\beta \rightarrow \infty$. It follows readily that if β satisfies $\beta d_w(\rho, k) = \delta\psi(0, \beta)$, and $\beta > \underline{\beta} > 0$, then $\beta \rightarrow \infty$ as $k \rightarrow 0$ but q is constant. Hence the assumption of Theorem 1 that β is large reflects these two stylized facts.

¹Suppose each individual uses a vanishingly small amount of the germline itself.

²The dependence of ψ on β is made explicit for the present purpose.

3 Proof of Theorem 1

Theorem 1 states necessary properties satisfied by any solution to the basic evolutionary problem. The plan of this proof is as follows. First, it is shown that any solution to the basic evolutionary problem defined in Theorem 1 is also a solution to a “transformed problem” of maximizing total expected discounted energy surplus, for a given value of r , together with the requirement that this maximum surplus be zero, subject to the constraints in the basic problem other than (2). It is then shown that any such solution path for the transformed problem has the properties described in Theorem 1. That is:

Lemma 1. *Any solution to the basic evolutionary problem of maximizing r subject to Assumptions 1-4, the Euler-Lotka equation (1) and the budget balance condition (2) is also a solution to the transformed problem—*

$$\max_{t^* \geq 0} W(t^*, r) \equiv U(r) \quad (3)$$

where

$$W(t^*, r) \equiv$$

$$\max_{s \geq 0, w \geq 0, \mu \geq 0} \left\{ \int_0^{t^*} e^{-rt} p(t) (F(G(K_0 + \bar{v}t), Q(t)), s(t)) - \alpha \bar{v} - \beta d(w(t), K_0 + \bar{v}t) - e(\mu(t)) dt \right. \\ \left. + \int_{t^*}^{\infty} e^{-rt} p(t) (F(G(K(t^*), Q(t)), s(t)) - \beta d(w(t), K(t^*)) - e(\mu(t))) dt \right\} \quad (4)$$

subject to

$$1 = \int_0^{\infty} e^{-rt} p(t) s(t) dt \quad (5)$$

$$\frac{dp(t)}{dt} = -p(t)\mu(t), \text{ for all } t \in [0, \infty), p(0) = 1 \quad (6)$$

$$\frac{dQ(t)}{dt} = w(t) - \rho, \text{ for all } Q(t) > 0, \text{ but} \quad (7)$$

$$\frac{dQ(t)}{dt} = 0, \text{ whenever } Q(t) = 0, Q(0) = Q_0$$

and where

$$\max_{t^* \geq 0} W(t^*, r) = U(r) = 0. \quad (8)$$

Proof. Suppose that t^* , $s'(\cdot)$, $w'(\cdot)$, and $\mu'(\cdot)$, together with the implied $v'(\cdot)$, $K'(\cdot)$, $Q'(\cdot)$, and $p'(\cdot)$, are a solution to the basic evolutionary problem, of maximizing r subject to Assumptions 2-4, (1), and (2), yielding a maximized growth rate of r' . However, suppose this is not a solution to the transformed problem described by (3), (4), (5), (6), and (7), and (8). It follows that there exist t^* , $s(\cdot)$, $w(\cdot)$, and $\mu(\cdot)$, together with the implied $K(\cdot)$, $Q(\cdot)$, and $p(\cdot)$, satisfying (5), (6), and (7), such that

$$\left\{ \int_0^{t^*} e^{-r't} p(t) (F(G(K_0 + \bar{v}t), Q(t)), s(t)) - \alpha \bar{v} - \beta d(w(t), K_0 + \bar{v}t) - e(\mu(t)) dt + \int_{t^*}^{\infty} e^{-r't} p(t) (F(G(K(t^*), Q(t)), s(t)) - \beta d(w(t), K(t^*)) - e(\mu(t))) dt \right\} > 0.$$

(It need not be that t^* , $s(\cdot)$, $w(\cdot)$, and $\mu(\cdot)$ differ from t^* , $s'(\cdot)$, $w'(\cdot)$, and $\mu'(\cdot)$.) There is then a $r = r' + \varepsilon$, for $\varepsilon > 0$, that is feasible for the basic problem. To show this, choose $\tilde{s}(t) = s(t) + \delta(\varepsilon)$, for $\delta(\varepsilon) > 0$ such that $\int_0^{\infty} e^{-r't} p(t) \tilde{s}(t) dt = 1$, and such that the strict inequality above still holds, while leaving t^* , $w(\cdot)$, and $\mu(\cdot)$ and the implied $K(\cdot)$, $Q(\cdot)$, and $p(\cdot)$ unaltered, for example. This is the required contradiction. ■

The proof now constructs a time path that can be shown to be a partial solution to the transformed problem. Consider, indeed, any fixed $t^* \geq 0$, with the associated investment

and capital profiles $v(\tau)$ and $K(\tau)$. That is, $v(\tau) = \bar{v}$, and $K(\tau) = K_0 + \bar{v}\tau$, for all $\tau \in [t, t^*]$; $v(\tau) = 0$ and $K(\tau) = K(t^*) = K^*$, say, for all $\tau > t^*$. Take any $t \geq 0$, and parameter $\eta > 0$. Define $\tilde{F}(K, Q, \eta) = \max_{s \geq 0} (F(G(K, Q), s) + \eta s)$. Thus $-F_s(G, s) = \eta$, if $s > 0$; $-F_s(G, s) \geq \eta$, if $s = 0$. Also note that, by the envelope theorem, $\tilde{F}_K = F_G G_K$ and $\tilde{F}_Q = F_G G_Q$.

Lemma 2. *For any $Q > 0$, given β is large enough, there exists a time path $(Q(\tau), L(\tau), \psi(\tau)) \geq 0$ for any initial $t \geq 0$, satisfying, for some $T > 0$,*

$$Q(t) = Q, \quad \frac{dQ(\tau)}{d\tau} = w(\tau) - \rho, \quad \text{where } \beta d_w(w, K) = \psi, \quad \text{for all } \tau \in [t, T], \quad \text{and } Q(\tau) = 0, \quad \text{for all } \tau \geq T$$

$$\frac{dL(\tau)}{d\tau} = (r + \mu)L - \tilde{F} + \alpha v + \beta d + e,$$

where $L(\tau) > 0$ and $-e'(\mu) = L$, for all $t \in [0, T)$, and $L(\tau) = 0$ for all $\tau \geq T$

$$\frac{d\psi}{d\tau} = (r + \mu)\psi - \tilde{F}_Q \quad \text{for all } \tau \in [t, T], \quad \text{and } \psi(\tau) = 0 \quad \text{for all } \tau \geq T.$$

Proof. For simplicity, it is assumed here that $t^* < T$ and is indeed small enough that $\tilde{F} - \alpha v = 0$ for the first time at T . Since this holds for the optimal t^* , this assumption is without true loss of generality. A further minor complication that prevents direct application of a standard result for existence of a solution to a system of ordinary differential equations is that $\mu \rightarrow \infty$ as $\tau \rightarrow T$. Consider then an artificial cost of mortality

$$e_{\bar{\mu}} : (\underline{\mu}, \infty) \rightarrow [0, \infty), \quad \text{for any } \bar{\mu} > \underline{\mu}, \quad \text{where}$$

$$e_{\bar{\mu}}(\mu) = e(\mu) - e(\bar{\mu}), \quad \text{for all } \mu \in (\underline{\mu}, \bar{\mu}), \quad \text{but } e_{\bar{\mu}}(\mu) = 0, \quad \text{for all } \mu \geq \bar{\mu}$$

The artificial problem becomes—

$$Q(t, \bar{\mu}) = Q, \quad \frac{dQ(\tau, \bar{\mu})}{d\tau} = w(\tau) - \rho, \quad \text{where } \beta d_w(w, K) = \psi, \text{ for all } \tau \in [t, T], \text{ and } Q(T, \bar{\mu}) = 0$$

$$\frac{dL(\tau, \bar{\mu})}{d\tau} = (r + \mu)L - \tilde{F} + \alpha v + \beta d + e_{\bar{\mu}},$$

$$\text{where } -e'_{\bar{\mu}}(\mu) = -e'(\mu) = L, \text{ if such } \mu \leq \bar{\mu} \text{ but } \mu = \bar{\mu} \text{ otherwise, for all } \tau \in [t, T],$$

$$\text{and } L(T, \bar{\mu}) = 0$$

$$\frac{d\psi(\tau, \bar{\mu})}{d\tau} = (r + \mu)\psi - \tilde{F}_Q \text{ for all } \tau \in [t, T], \text{ and } \psi(T, \bar{\mu}) = 0.$$

It is straightforward then to show that this system of equations satisfies the appropriate Lipschitz condition and so has a unique solution for $(Q(\tau, \bar{\mu}), L(\tau, \bar{\mu}), \psi(\tau, \bar{\mu}))$ on $[t, T]$, for an arbitrary $T > t$, disregarding the condition that $Q(t) = Q$. This solution can be constructed backwards from T . In the limiting case where $\beta \rightarrow \infty$ there is obviously a unique strictly monotonically decreasing solution to the first equation, given by $w = 0$ and $Q(\tau) = Q - (\tau - t)\rho$ so that $T = \frac{Q}{\rho} + t$. If β is large enough, it must still be possible to choose $T(\bar{\mu}) > t$ such that $Q(t) = Q$, for any $Q > 0$.

It is also straightforward to show that, when β is large enough, there exists a time path $(Q(\tau), L(\tau), \psi(\tau))$ and terminal time T such that

$$(Q(\tau, \bar{\mu}), L(\tau, \bar{\mu}), \psi(\tau, \bar{\mu})) \rightarrow (Q(\tau), L(\tau), \psi(\tau)) \text{ and } T(\bar{\mu}) \rightarrow T, \text{ as } \bar{\mu} \rightarrow \infty, \text{ for all } \tau \in [t, T].$$

This pointwise limit $(Q(\tau), L(\tau), \psi(\tau))$, with the additional condition that $(Q(\tau), L(\tau), \psi(\tau)) = 0$, for $\tau \geq T$, must then be a solution to the original problem. ■

The proof now proceeds by showing that the time path found above yields a partial solution to the transformed problem.

Indeed, the time path found in Lemma 1 implies a value of $\frac{1}{p(t)} \int_t^T e^{-rt} p(\tau) s(\tau) d\tau = R(Q, \eta, t)$, say, where $R(\cdot)$ is continuous. $R(\cdot)$ is also strictly increasing in η , since $s > 0$ is increasing in η ; further, \tilde{F} is increasing in η , so $L(\tau)$ is increasing in η , μ is decreasing and $p(\tau)$ is increasing in η . Since, in addition, $R(Q, 0, t) = 0$ and $R(Q, \eta, t) \rightarrow \infty$, as $\eta \rightarrow \infty$, there exists a unique $\eta(Q, R, t) > 0$ associated with each $Q > 0$, $R > 0$, and $t \geq 0$. Thus, a unique L can be obtained not only for each (Q, η, t) but also for each (Q, R, t) , justifying not only the notation $L(Q, \eta, t)$ but also $L(Q, R, t)$.

Now define

$$V(Q, R, t) = L(Q, R, t) - \eta(Q, R, t)R.$$

It follows that

Lemma 3. *Further, there exists $\bar{\beta} > 0$, independent of α and \bar{v} , such that, if $\beta > \bar{\beta}$, then $V(Q, R, t)$ is the value function for the subproblem given by (4), (5), (6), and (7). The optimal $\mu = \arg \max_{\mu \in [\underline{\mu}, \bar{\mu}]} [-e(\mu) - \mu V + \mu V_R R]$; the optimal $s = \arg \max_{s \geq 0} \tilde{F}$; and the optimal w satisfies $\beta d_w(w, K) = \psi$.*

Proof. i) Given the assumptions on e , $-e'(\mu) = L$, if $L > 0$; $\mu = \infty$, if $L \leq 0$; and $L = V + \eta R$, the optimality of μ follows since

$$-e(\mu) - (\mu + r)V + \mu V_R R \geq -e(\mu') - (\mu' + r)V + \mu' V_R R, \text{ for all } \mu' \in [\underline{\mu}, \infty),$$

if it can be shown that $V_R(Q, R, t) = -\eta$. But this is a consequence of

$$L_\eta(Q, \eta, t) = R$$

since this implies

$$L_R(Q, R, t) = L_\eta(Q, \eta, t) \frac{\partial \eta}{\partial R} = R \frac{\partial \eta}{\partial R} \text{ so that } V_R = L_R - \frac{\partial \eta}{\partial R} R - \eta = -\eta.$$

But the result that $L_\eta(Q, \eta, t) = R$ follows from the envelope theorem.³ That is,

$$\begin{aligned} L_\eta(Q, \eta, t) &= R + \frac{1}{p(t)} \int_t^T e^{-r(\tau-t)} \frac{\partial p(\tau)}{\partial \eta} (\tilde{F} - \alpha v - \beta d - e) d\tau \\ &\quad + \frac{1}{p(t)} \int_t^T e^{-r(\tau-t)} p(\tau) \tilde{F}_Q \frac{\partial Q}{\partial \eta} d\tau \\ &\quad - \frac{1}{p(t)} \int_t^T e^{-r(\tau-t)} p(\tau) \beta d_w \frac{\partial w}{\partial \eta} d\tau \\ &\quad + \frac{1}{p(t)} \int_t^T e^{-r(\tau-t)} p(\tau) (-e'(\mu)) \frac{\partial \mu}{\partial \eta} d\tau \\ &\quad + \frac{e^{-r(T-t)} p(T) L(T)}{p(t)} \frac{\partial T}{\partial \eta} \\ &= R - e^{-r(\tau-t)} \frac{\partial p(\tau)}{\partial \eta} L \Big|_t^T \frac{1}{p(t)} - e^{-r(\tau-t)} p(\tau) \frac{\partial Q}{\partial \eta} \psi \Big|_t^T \frac{1}{p(t)} = R, \end{aligned}$$

since $L(T) = 0$; $-e'(\mu) = L$; $\frac{\partial p(t)}{\partial \eta} = 0$, given $p(t)$ is fixed; $\frac{\partial Q(t)}{\partial \eta} = 0$, given $Q(t) = Q$ is fixed;

$$\begin{aligned} \frac{d}{d\tau} \left(e^{-r(\tau-t)} \frac{\partial p}{\partial \eta} L \right) &= -e^{-r(\tau-t)} \left(\frac{\partial p}{\partial \eta} r + \frac{\partial p}{\partial \eta} \mu + p \frac{\partial \mu}{\partial \eta} \right) L + e^{-r(\tau-t)} \left((\mu + r)L - (\tilde{F} - \alpha v - \beta d - e) \right) \frac{\partial p}{\partial \eta} \\ &= -e^{-r(\tau-t)} p \frac{\partial \mu}{\partial \eta} L - e^{-r(\tau-t)} \left(\tilde{F} - \alpha v - \beta d - e \right) \frac{\partial p}{\partial \eta} \end{aligned}$$

and, similarly,

$$\frac{d}{d\tau} \left(e^{-r(\tau-t)} \psi \frac{\partial Q}{\partial \eta} p \right) = -e^{-r(\tau-t)} \tilde{F}_Q \frac{\partial Q}{\partial \eta} p + e^{-r(\tau-t)} p \psi \frac{\partial w}{\partial \eta}$$

³An heuristic proof of this version of the envelope theorem is given here. There are a number of similar appeals to the envelope theorem in this appendix, the heuristic proofs of which are omitted. Rigorous proofs can be obtained from results by Coddington and Levinson (1955, Chapter 1.7), concerning the dependence of the solution to a differential equation on various parameters.

$V(Q, R, t)$ is the value function because

ii) Given $F(G, \cdot)$ is concave and $-\eta = V_R(Q, R, t)$, the optimality of s follows since $F(G, s) - V_R s \geq F(G, s') - V_R s'$, for all $s' \geq 0$.

iii) By the envelope theorem,

$$V_Q(Q, R, t) = \frac{1}{p(t)} \int_t^T e^{-r(\tau-t)} \left[\tilde{F}_Q \frac{\partial Q(\tau)}{\partial Q} - \beta d_w \frac{\partial w(\tau)}{\partial Q} \right] p(\tau) d\tau.$$

The optimality of the choice of w then follows since

$$V_Q(Q, R, t) = \beta d_w \text{ where } \beta d_w = \psi.$$

Consider, indeed,

$$\frac{d}{d\tau} \left[e^{-r(\tau-t)} p \psi \frac{\partial Q(\tau)}{\partial Q} \right] = -e^{-r(\tau-t)} p \tilde{F}_Q \frac{\partial Q(\tau)}{\partial Q} + e^{-r(\tau-t)} p \psi \frac{\partial w(\tau)}{\partial Q}$$

Hence

$$p(t)\psi(t) = \int_t^T e^{-r(\tau-t)} \left[\tilde{F}_Q \frac{\partial Q(\tau)}{\partial Q} - \beta d_w \frac{\partial w(\tau)}{\partial Q} \right] p(\tau) d\tau,$$

as required. ■

Consider now the choice of *optimal* t^* as in (3). Note that $V(Q_0, 1, r) = W(t^*, r)$, as in (4).

Lemma 4. *There exist $\bar{\alpha} > 0$, independent of the value of $\beta > \bar{\beta}$ and of \bar{v} , such that, if $\alpha < \bar{\alpha}$, then there exists $t^* > 0$ maximizing $W(t^*, r)$.*

Proof. The envelope theorem implies that

$$W_{t^*}(t^*, r) = \bar{v} \int_{t^*}^T e^{-r\tau} p(\tau) \left[\tilde{F}_K - \beta d_K \right] d\tau - \alpha e^{-rt^*} p(t^*) \bar{v},$$

Hence

$$W_{t^*}(T, r) = -\alpha e^{-rT} p(T) \bar{v} < 0.$$

On the other hand,

$$W_{t^*}(0, r) = \bar{v} \int_0^T e^{-r\tau} p(\tau) [F_G G_K - \beta d_K] d\tau - \alpha \bar{v} > 0,$$

whenever $\alpha < \bar{\alpha}$, and $\beta > \bar{\beta}$. (Note that $\beta d_K(w, K) \rightarrow 0$ as $\beta \rightarrow \infty$ because $\frac{d_K(w, K)}{d_w(w, K)} \rightarrow \frac{d_w K(0, K)}{d_{ww}(0, K)} = 0$, as $w \rightarrow 0$, by l'Hôpital's rule.) Hence there exists an optimal $t^* \in (0, T)$, satisfying $\frac{1}{p(t^*)} \int_{t^*}^T e^{-r(\tau-t^*)} p(\tau) [F_G G_K - \beta d_K] d\tau = \alpha$. ■

We are now in a position to establish the key properties of the time paths claimed in Theorem 1.⁴

Lemma 5. *The properties of the time paths of fertility, $s(t)$, and mortality, $\mu(t)$, that are described in Theorem 1 now follow.*

Proof. Take the optimal t^* and define $y(t) = \tilde{F}(K, Q, \eta) - \alpha v(t) - \beta d(w(t), K(t))$, $v(t) = \bar{v}$. Now $\frac{dG}{dt} = G_K \bar{v} + G_Q(w - \rho) > 0$ and $\frac{dy}{dt} = F_G(G_K \bar{v} + G_Q(w - \rho)) - \beta d_w(w(t), K(t)) \frac{dw(t)}{dt} - \beta d_K(w(t), K(t)) \bar{v} = (\tilde{F}_K - \psi \frac{d_K}{d_w}) \bar{v} + G_Q(w - \rho) - \psi \frac{dw(t)}{dt} > 0$, for all $t \in [0, t^*)$, whenever β and \bar{v} are large enough. (Recall that $\frac{d_K(w, K)}{d_w(w, K)} \rightarrow \frac{d_w K(0, K)}{d_{ww}(0, K)} = 0$ as $w \rightarrow 0$, by l'Hôpital's rule. Note also that $\frac{dw(t)}{dt} = \frac{\frac{d\psi}{dt} d_w}{\psi d_{ww}}$ and $\frac{d_w(w, K)}{d_{ww}(w, K)} \rightarrow 0$ as $w \rightarrow 0$, so $\frac{dw(t)}{dt} \rightarrow 0$ as $\beta \rightarrow \infty$.) Although $G(K(t), Q(t))$ is continuous, $y(t^*+) - y(t^*-) = \bar{v}$, whereas $\frac{dG}{dt} = G_Q(w - \rho) < 0$

⁴Note that it is not necessary to invoke (8) to establish Theorem 1. This condition simply serves to tie down the maximum value of r . If this condition is applied, it can be shown that the basic evolutionary problem and the transformed problem have precisely the same solutions.

and $\frac{dy}{dt} = \tilde{F}_Q(w - \rho) - \beta d_w(w(t), K(t)) \frac{dw(t)}{dt} = \tilde{F}_Q(w - \rho) - \psi \frac{dw(t)}{dt} < 0$, for all $t > t^*$, whenever β is large enough.

The path of $s(t)$ can now be characterized. That is, there must now exist a $t_L \in [0, t^*)$ such that $s(t) = 0$ for all $t \leq t_L$ and, since $\frac{ds}{dt} = -\frac{F_{Gs}(G, s)}{F_{ss}(G, s)} \frac{dG}{dt}$, if $s > 0$, then $\frac{ds}{dt} > 0$, for all $t \in (t_L, t^*)$. Further, since $-F_s(G(K(t), Q(t)), 0) \rightarrow \infty$, as $t \rightarrow T$, there exists $t_H \in (t^*, T)$ such that $-F_s(G(K(t), Q(t)), 0) = \eta$. Hence $s > 0$ and $\frac{ds}{dt} < 0$, for all $t \in (t^*, t_H)$, but $s(t) = 0$ for all $t \in [t_H, T)$.

The path of $\mu(t)$ follows since $L(t) = \frac{1}{p(t)} \int_t^T e^{-r(\tau-t)} p(\tau) (y(\tau) - e(\mu)) d\tau$ is hump-shaped, where the maximum of $L(t)$ is not after the maximum of $y(t)$ at t^* . Indeed, as long as $t \neq t^*$, $\frac{dL(t)}{dt} = (\mu(t) + r)L(t) - y(t) + e(\mu)$ and $\frac{d^2L(t)}{dt^2} = \frac{d\mu}{dt}L(t) + (\mu + r)\frac{dL}{dt} - \frac{dy}{dt} + e'(\mu)\frac{d\mu}{dt} = (\mu + r)\frac{dL}{dt} - \frac{dy}{dt}$. Thus, if $\frac{dL}{dt} \geq 0$ for any $t > t^*$, this implies $L(T) > 0$, a contradiction. Thus $\frac{dL}{dt} < 0$ for all $t > t^*$. In addition, if $\frac{dL}{dt} = 0$ at any $t < t^*$, then $\frac{d^2L}{dt^2} < 0$, so that there can be at most one critical point in this range, which must be a maximum. Given that $y(t)$ jumps up at t^* , $\frac{dL}{dt}$ jumps down. Altogether, then, $L(t)$ must have a unique maximum at some $\hat{t} \leq t^*$. Hence mortality $\mu(t)$ is U-shaped with a unique minimum at \hat{t} . ■