Web-based Supplementary Materials for Nested Markov Compliance Class Model in the Presence of Time-Varying Noncompliance by Lin, Ten Have, and Elliott.

Web Appendix A: Priors and Conditional Draws of the Gibbs Sampler

Let $\mathbf{Y}_i$ and $\mathbf{C_i}$ denote the vectors of Y_{ij} and C_{ij} for subject i. For notational simplicity, let $X_{ij} = [I(C_{i1} = c, Z_i = 0), \cdots, I(C_{i5} = n, Z_i = 1), \mathbf{A}_i]$ denote the row vector of the fixed effect, and $\mathbf{X}_i$ denote the design matrix of the fixed effect for subject i with 5 (number of follow-ups) rows. Let $\boldsymbol{\beta} = [\lambda_{1c0}, \cdots, \lambda_{5n1}, \boldsymbol{\gamma}]$ denote the vector of coefficients corresponding to the fixed effect.

We assume the conjugate priors $\beta \sim MVN(\mu_\beta, \Sigma_\beta)$ and $\sigma^2 \sim Inv - \chi^2(df = \nu_\sigma, \psi)$. We assume $\boldsymbol{\varphi}_i \sim MVN(\mathbf{0}, \Sigma_\varphi)$ for the subject-level random effects, and the hyperprior $\Sigma_\varphi \sim Inv - Wishart(df = \nu_\varphi, \Gamma)$. For compliance superclass and compliance class probabilities, we assume the priors $(p_1, \cdots, p_K) \sim Dirichlet(a_1, \cdots, a_K)$, $\boldsymbol{\alpha} \sim MVN(0, \Sigma_\alpha)$ where $\boldsymbol{\alpha} = [\alpha_{01c}, \cdots, \alpha_{0Kn}, \boldsymbol{\alpha}_{1c}, \boldsymbol{\alpha}_{1n}]$, and $(\pi_{kj\eta'c}, \pi_{kj\eta'n}) \sim Dirichlet(b_c, b_n)\ \forall k, j, \eta'$. Gibbs sampling (Geman and Geman, 1984; Gelfand and Smith, 1990; Imbens and Rubin, 1997) is used to obtain draws from the posterior distributions of the parameters. The posterior distributions from which the model parameters are drawn are presented in the Appendix. The posterior distribution $(\boldsymbol{\alpha}|\mathbf{C}, \mathbf{U}, \mathbf{Q})$ is not of a known parametric form. Therefore, we use the Metropolis-Hasting algorithm (Hastings, 1970) to draw the $\boldsymbol{\alpha}$ parameters.

The distributions from which parameters are drawn at each iteration in the Gibbs sampling are as follows:

$$\left(\boldsymbol{\beta}|\mathbf{X}, \mathbf{Y}, \mathbf{W}, \boldsymbol{\varphi}, \sigma^2, \boldsymbol{\mu}_\beta, \Sigma_\beta^{-1}\right) \sim MVN(\hat{\boldsymbol{\mu}}, \hat{\Sigma})$$
$$\hat{\boldsymbol{\mu}} = \frac{\sigma^{-2}\sum_{i=1}^N \mathbf{X}_i^T(\mathbf{Y}_i - \mathbf{W}_i^T\boldsymbol{\varphi}_i) + \Sigma_\beta^{-1}\boldsymbol{\mu}_\beta}{\sigma^{-2}\sum_{i=1}^N \mathbf{X}_i^T\mathbf{X}_i + \Sigma_\beta^{-1}}$$
$$\hat{\Sigma} = (\sigma^{-2}\sum_{i=1}^N \mathbf{X}_i^T\mathbf{X}_i + \Sigma_\beta^{-1})^{-1}$$

$$(\sigma^2|\mathbf{X}, \mathbf{Y}, \mathbf{W}, \boldsymbol{\varphi}, \boldsymbol{\beta}, \nu_\sigma, \psi) \sim Inv - \chi^2\left(df = 5N + \nu_\sigma, \frac{\sum_{i=1}^N F_i + \nu_\sigma\psi}{5N + \nu_\sigma}\right)$$
$$\text{where } F_i = \left(\mathbf{Y}_i - \mathbf{X}_i\boldsymbol{\beta} - \mathbf{W}_i^T\boldsymbol{\varphi}_i\right)^T\left(\mathbf{Y}_i - \mathbf{X}_i\boldsymbol{\beta} - \mathbf{W}_i^T\boldsymbol{\varphi}_i\right)$$

$$\left(\boldsymbol{\varphi}_i | \mathbf{X}_i, \mathbf{Y}_i, \mathbf{W}_i, \boldsymbol{\beta}, \sigma^2, \Sigma_\varphi, \right) \sim MVN\left(\hat{\boldsymbol{\varphi}}_i \hat{V}_i, \hat{V}_i\right)$$

$$\hat{\boldsymbol{\varphi}}_i = \frac{\mathbf{W}_i^T \left(\mathbf{Y}_i - \mathbf{X}_i\boldsymbol{\beta}\right)}{\sigma^2}$$

$$\hat{\mathbf{V}}_i = \left(\frac{\mathbf{W}_i^T \mathbf{W}_i}{\sigma^2} + \Sigma_\varphi^{-1}\right)^{-1}$$

$$(\Sigma_\varphi | \boldsymbol{\varphi}, \omega, \Gamma) \sim Inv-Wishart\left(df = \nu_\varphi + N, \sum_{i=1}^{N} \boldsymbol{\varphi}_i^T \boldsymbol{\varphi}_i + \Gamma\right)$$

$$(p_1, \cdots, p_K | \mathbf{U}, a_1, \cdots, a_k) \sim Dirichlet(r_1, \cdots, r_K)$$

$$r_1 = \sum_{i=1}^{N} I(U_i = 1) + a_1$$

$$r_K = \sum_{i=1}^{N} I(U_i = K) + a_K$$

$$(\pi_{kj\eta' c}, \pi_{kj\eta' n} | \mathbf{C}, b_c, b_n) \sim Dirichlet(s_c, s_n)$$

$$s_c = \sum_{i=1}^{N} I(U_i = k, C_{i,j-1} = \eta', C_{ij} = c) + b_c$$

$$s_n = \sum_{i=1}^{N} I(U_i = k, C_{i,j-1} = \eta', C_{ij} = n) + b_n$$

$$P(U_i = k | \mathbf{C}_i, \mathbf{Q}_i, \boldsymbol{\alpha}, p_1, \cdots, p_k)$$

$$\propto \quad p_k \times \left[\prod_{\eta} \omega_{k\eta}(\mathbf{Q}_i)^{I(U_i=k, C_{i1}=\eta)}\right] \left[\prod_{j=2}^{5} \prod_{\eta'} \prod_{\eta} \pi_{kj\eta'\eta}^{I(U_i=k, C_{i,j-1}=\eta', C_{ij}=\eta)}\right]$$

$$P(C_{ij}=c|Y_{ij},Z_i,D_{ij},U_i,\boldsymbol{\lambda},\mathbf{A}_i,\boldsymbol{\gamma},\mathbf{W}_i,\boldsymbol{\varphi}_i,\mathbf{Q}_i,\boldsymbol{\alpha},\sigma^2)$$
$$=\begin{cases} \frac{\pi^{**}_{ijc}\times\phi\left(\frac{Y_{ij}-(\lambda_{jc0}+\mathbf{A}_i^T\boldsymbol{\gamma}+\mathbf{W}_i^T\boldsymbol{\varphi}_i)}{\sigma}\right)}{\sum_\eta\left[\pi^{**}_{ij\eta}\times\phi\left(\frac{Y_{ij}-(\lambda_{j\eta0}+\mathbf{A}_i^T\boldsymbol{\gamma}+\mathbf{W}_i^T\boldsymbol{\varphi}_i)}{\sigma}\right)\right]} & \text{if} \in Z_i=0,D_{ij}=0,U_i=k \\ 0 & \text{if} \in Z_i=1,D_{ij}=0,U_i=k \\ 1 & \text{if} \in Z_i=1,D_{ij}=1,U_i=k \end{cases}$$

$$P(C_{ij}=n|Y_{ij},Z_i,D_{ij},U_i,\boldsymbol{\lambda},\mathbf{A}_i,\boldsymbol{\gamma},\mathbf{W}_i,\boldsymbol{\varphi}_i,\mathbf{Q}_i,\boldsymbol{\alpha},\sigma^2)$$
$$=\begin{cases} \frac{\pi^{**}_{ijn}\times\phi\left(\frac{Y_{ij}-(\lambda_{jn0}+\mathbf{A}_i^T\boldsymbol{\gamma}+\mathbf{W}_i^T\boldsymbol{\varphi}_i)}{\sigma}\right)}{\sum_\eta\left[\pi^{**}_{ij\eta}\times\phi\left(\frac{Y_{ij}-(\lambda_{j\eta0}+\mathbf{A}_i^T\boldsymbol{\gamma}+\mathbf{W}_i^T\boldsymbol{\varphi}_i)}{\sigma}\right)\right]} & \text{if} \in Z_i=0,D_{ij}=0,U_i=k \\ 1 & \text{if} \in Z_i=1,D_{ij}=0,U_i=k \\ 0 & \text{if} \in Z_i=1,D_{ij}=1,U_i=k \end{cases}$$

$$\text{where } \pi^{**}_{ij\eta}=\begin{cases} \prod_{\eta'}\left[\omega_{k\eta}(\mathbf{Q}_i)\pi_{k2\eta\eta'}\right]^{I(U_i=k,C_{i1}=\eta,C_{i2}=\eta')} & \text{if } j=1 \\ \prod_{\eta'}\prod_{\eta''}\left[\pi_{kj\eta'\eta}\pi_{k,j+1,\eta\eta''}\right]^{I(U_i=k,C_{i,j-1}=\eta',C_{ij}=\eta,C_{i,j+1}=\eta'')} & \text{if } 1<j<5 \\ \prod_{\eta'}\pi_{k5\eta'\eta}{}^{I(U_i=k,C_{i4}=\eta',C_{i5}=\eta)} & \text{if } j=5 \end{cases}$$

and $\phi(S)$ is the pdf for standard normal distribution evaluated at S

REFERENCES

Geman, S., and Geman, D. (1984). Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **6**, 721-741.

Gelfand, A. E., and Smith, A. F. M. (1990). Sampling-based approaches to calculating marginal densities. *Journal of the American Statistical Association* **85**, 398-409.

Hastings, W. K. (1970). Monte Carlo sampling methods using Markov chains and their applications. *Biometrika* **57**, 97-109.

Imbens, G. W., and Rubin, D. B. (1997). Bayesian inference for causal effects in randomized experiments with noncompliance. em The Annals of Statistics **25**, 305-327.