Supplementary material

Studying the effect of cell division on expression patterns of the segment polarity genes

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1 Proof of Theorems 1 and 2

As mentioned in the main text, the proofs are similar to those given in [1], with appropriate index changes. For easier reference and completeness, we provide the proofs next. First, some observations regarding the solutions of system (1):

$$\frac{d\,\hat{X}_\ell}{dt} = \alpha_\ell(-\hat{X}_\ell + F_\ell(X_1, X_2, \ldots, X_N)), \quad \ell = 1, \ldots, N, \tag{1}$$

Let X denote any of the nodes in the network, and α its time rate. Since equations (1) are either of the form $d\hat{X}/dt = \alpha(-\hat{X}+1)$ or $d\hat{X}/dt = -\alpha\hat{X}$, their solutions are continous functions, piecewise combinations of:

$$\hat{X}^1(t) = 1 - (1 - \hat{X}^1(t_0))\, e^{-\alpha(t-t_0)} \tag{2}$$

$$\hat{X}^0(t) = \hat{X}^0(t_0)\, e^{-\alpha(t-t_0)} \tag{3}$$

$\hat{X}^1(t)$ (resp. $\hat{X}^0(t)$) is monotonically increasing (resp. decreasing). In addition, note that discrete variables X can only switch between 0 and 1 at those instants when $\hat{X}(t_{\text{switch}}) = \theta$, that is:

$$t^1_{\text{switch}} = t_0 + \frac{1}{\alpha}\, \ln\, \frac{(1 - \hat{X}(t_0))}{1 - \theta} \tag{4}$$

$$t^0_{\text{switch}} = t_0 + \frac{1}{\alpha}\, \ln\, \frac{\hat{X}(t_0)}{\theta} \tag{5}$$

From Proposition 6.1 of the main text we can immediately conclude:

$$\widehat{wg}_{1,\ldots,\text{FS}-1}(t) = \widehat{\text{WG}}_{1,2}(t) = 0, \tag{6}$$

$$\widehat{en}_{\text{FS},\ldots,\text{LS}}(t) = \widehat{\text{EN}}_{\text{FS},\ldots,\text{LS}}(t) = 0,$$

$$\widehat{hh}_{\text{FS},\ldots,\text{LS}}(t) = \widehat{\text{HH}}_{\text{FS},\ldots,\text{LS}}(t) = 0, \tag{7}$$

$$\widehat{ci}_{\text{FS},\ldots,\text{LS}}(t) = 1 \;\text{ and }\; \widehat{\text{CI}}_{\text{FS},\ldots,\text{LS}}(t) = 1 - e^{-\alpha_{\text{CI}_{\text{FS},\ldots,\text{LS}}}\, t}. \tag{8}$$

Lemma 1.1. Let $0 \le t_0 < t_3 \le t_1$ and $0 \le t_2 < t_3$. Define $\delta = \ln \frac{1}{1-\theta} / \max_{1,\ldots,N} \alpha_i$. Assume $\text{CIA}_{\text{FS}}(t) = 0$ for $t \in (t_2, t_3)$, and $wg_{\text{FS}}(t) = 0$ for $t \in [0, t_3)$. Then

(a) $wg_{\text{FS}}(t) = 0$ for $t \in [0, t_3 + \delta)$;

(b) $\text{WG}_{\text{FS}}(t) = 0$ for $t \in [0, t_3 + \delta)$;

(c) $en_{\text{FS}-1}(t) = \text{EN}_{\text{FS}-1}(t) = 0$ for $t \in [0, t_3 + \delta)$;

(d) $hh_{\text{FS}-1}(t) = \text{HH}_{\text{FS}-1}(t) = 0$ for $t \in [0, t_3 + \delta)$.

Assume further that $\text{PTC}_{\text{FS}}(t) = 1$ for $t \in (t_0, t_1)$. Then

(e) $\text{PTC}_{\text{FS}}(t) = 1$ for all $t \in (t_0, t_3 + \delta)$.

(f) $\text{CIA}_{\text{FS}}(t) = 0$ for all $t \in (t_2, t_3 + \delta)$.

Proof: Part (a) follows directly from the fact that $F_{wg_{\text{FS}}}(t) = 0$ on $[0, t_3)$, and from (4).

To prove parts (b), (c), and (d), first note that initial conditions together with $wg_{\text{FS}}(t) = 0$ for $t \in [0, t_3)$ imply

$$\widehat{\text{WG}}_{\text{FS}}(t) = 0,\ \widehat{en}_{\text{FS}-1}(t) = \widehat{\text{EN}}_{\text{FS}-1}(t) = 0,\ \widehat{hh}_{\text{FS}-1}(t) = \widehat{\text{HH}}_{\text{FS}-1}(t) = 0,$$

for $t \in [0, t_3]$. Then, from equations (2) to (5) we conclude that the corresponding discrete variables cannot switch from 0 to 1 during an interval of the form $[0, t_3 + \frac{1}{\alpha_j} \ln \frac{1}{1-\theta})$. Taking the largest common interval yields the desired results.

To prove parts (e) and (f), assume also that $\text{PTC}_{\text{FS}}(t) = 1$ for $t \in (t_0, t_1)$. From (7) and part (d), it follows that function $F_{\text{PTC}_{\text{FS}}}$ does not switch in the interval $(t_0, t_3 + \delta)$ and in fact $\text{PTC}_{\text{FS}}(t) = 1$ for all t in this interval. This, together with (7) and part (d) yield $F_{\text{CIA}_{\text{FS}}}(t) = 0$ for $(t_0, t_3 + \delta)$, so that $\widehat{\text{CIA}}_{\text{FS}}$ cannot increase in this interval and the discrete level satisfies $\text{CIA}_{\text{FS}}(t) = 0$ for all $t \in (t_2, t_3 + \delta)$, as we wanted to show. ∎

Corollary 1.2. Let $0 \leq t_0 < t_3 \leq t_1$ and $0 \leq t_2 < t_3$. If $\text{PTC}_{\text{FS}}(t) = 1$ for $t \in (t_0, t_1)$, $\text{CIA}_{\text{FS}}(t) = 0$ for $t \in (t_2, t_3)$, and $wg_{\text{FS}}(t) = 0$ for $t \in [0, t_3)$, then $wg_{\text{FS}}(t) = 0$ for all t.

Proof: Applying Lemma 1.1 we conclude that, given any $k \geq 0$:

$$\begin{aligned} &\text{CIA}_{\text{FS}}(t) = 0, \ \text{ for } \ t \in (t_2, t_3 + k\delta) \\ &wg_{\text{FS}}(t) = 0, \ \text{ for } \ t \in [0, t_3 + k\delta) \\ &\text{PTC}_{\text{FS}}(t) = 1 \ \text{ for } \ t \in (t_0, t_3 + k\delta) \end{aligned}$$

imply

$$\begin{aligned} &\text{CIA}_{\text{FS}}(t) = 0, \ \text{ for } \ t \in (t_2, t_3 + (k+1)\delta) \\ &wg_{\text{FS}}(t) = 0, \ \text{ for } \ t \in [0, t_3 + (k+1)\delta) \\ &\text{PTC}_{\text{FS}}(t) = 1 \ \text{ for } \ t \in (t_0, t_3 + (k+1)\delta). \end{aligned}$$

Since δ is finite, we conclude by induction on k that $wg_{\text{FS}}(t) = 0$ for all t. ∎

Proof of Theorem 1: The rule for CIA_{FS} may be simplified to (by (7))

$$F_{\text{CIA}_{\text{FS}}} = \text{CI}_{\text{FS}} \text{ and } [\text{notPTC}_{\text{FS}} \text{ or } hh_{\text{FS}-1} \text{ or } \text{HH}_{\text{FS}-1}].$$

From equation (8), we have that

$$\text{CI}_{\text{FS}}(t) = 1, \ \text{ for all } t > \frac{1}{\alpha_{\text{CI}_{\text{FS}}}} \ln \frac{1}{1-\theta}. \tag{9}$$

On the other hand, since $ptc_{\text{FS}}(0) = 1$, by continuity of solutions $ptc_{\text{FS}}(t) = 1$ for all $t < \frac{1}{\alpha_{ptc_{\text{FS}}}} \ln \frac{1}{\theta}$. This implies that the Patched protein satisfies

$$\widehat{\text{PTC}}_{\text{FS}}(t) = 1 - e^{-\alpha_{\text{PTC}_{\text{FS}}} t}, \ \ 0 \leq t \leq \frac{1}{\alpha_{ptc_{\text{FS}}}} \ln \frac{1}{\theta}$$

and therefore

$$\text{PTC}_{\text{FS}}(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{\alpha_{\text{PTC}_{\text{FS}}}} \ln \frac{1}{1-\theta} \\ 1, & \frac{1}{\alpha_{\text{PTC}_{\text{FS}}}} \ln \frac{1}{1-\theta} < t < \frac{1}{\alpha_{ptc_{\text{FS}}}} \ln \frac{1}{\theta}. \end{cases} \tag{10}$$

By assumption, $\alpha_{\text{PTC}_{\text{FS}}} > \alpha_{ptc_{\text{FS}}}$ and also $\ln \frac{1}{1-\theta} \leq \ln \frac{1}{\theta}$, defining a nonempty interval where PTC_{FS} is expressed. Now let $t_c = \frac{1}{\alpha_{\text{CI}_{\text{FS}}}} \ln \frac{1}{1-\theta}$ and $t_p = \frac{1}{\alpha_{\text{PTC}_{\text{FS}}}} \ln \frac{1}{1-\theta}$. $\widehat{\text{CIA}}_{\text{FS}}(t)$ starts at zero and must remain so while $\text{CI}_{\text{FS}} = 0$, so that

$$\text{CIA}_{\text{FS}}(t) = 0 \;\text{ for } 0 < t < t_c.$$

In the case $t_c > t_p$, letting $t_0 = t_p$, $t_1 = \frac{1}{\alpha_{ptc_{\text{FS}}}} \ln \frac{1}{\theta}$, $t_2 = 0$, and $t_3 = t_c$ in Corollary 1.2, obtains $wg_{\text{FS}}(t) = 0$ for all t. This proves item (b) of the theorem, and part of (a).

To finish the proof of item (a), we assume that $(1-\theta)^2 < \theta$ and must now consider the case $t_c \leq t_p$. Then

$$\widehat{\text{CIA}}_{\text{FS}}(t) = \begin{cases} 0, & 0 \leq t \leq t_c \\ 1 - e^{-\alpha_{\text{CIA}_{\text{FS}}}(t-t_c)}, & t_c < t \leq t_p \\ \widehat{\text{CIA}}_{\text{FS}}(t_p)\, e^{-\alpha_{\text{CIA}_{\text{FS}}}(t-t_p)}, & t_p < t \leq \frac{1}{\alpha_{ptc_{\text{FS}}}} \ln \frac{1}{\theta}, \end{cases}$$

Following equation (4) with $t_0 = t_c$ and $\widehat{\text{CIA}}_{\text{FS}}(t_0) = 0$, CIA_{FS} might become expressed at time $t_c < t_a < t_p$:

$$t_a = t_c + \frac{1}{\alpha_{\text{CIA}_{\text{FS}}}} \ln \frac{1}{1-\theta},$$

but it would then become zero again at (equation (5) with $t_0 = t_p$)

$$t_b = t_p + \frac{1}{\alpha_{\text{CIA}_{\text{FS}}}} \ln \frac{\widehat{\text{CIA}}_{\text{FS}}(t_p)}{\theta}.$$

Finally, we show that, even if $\text{CIA}_{\text{FS}}(t) = 1$ for $t \in (t_a, t_b)$, wg_{FS} cannot become expressed in this interval. In this interval, $\widehat{wg}_{\text{FS}}$ evolves according to $\widehat{wg}_{\text{FS}}(t) = 1 - e^{-\alpha_{wg_{\text{FS}}}(t-t_a)}$, and wg_{FS} can switch to 1 at time

$$t_w = t_a + \frac{1}{\alpha_{wg_{\text{FS}}}} \ln \frac{1}{1-\theta}.$$

We will show that $t_w > t_b$, so $wg_{\text{FS}}(t) = 0$ in the interval $[0, t_b)$. Writing

$$\ln \frac{\widehat{\text{CIA}}_{\text{FS}}(t_p)}{\theta} = \ln \frac{\widehat{\text{CIA}}_{\text{FS}}(t_p)}{1-\theta}\frac{1-\theta}{\theta} = \ln \frac{\widehat{\text{CIA}}_{\text{FS}}(t_p)}{1-\theta} + \ln \frac{1-\theta}{\theta} \leq \ln \frac{1}{1-\theta} + \ln \frac{1}{1-\theta}$$

where we have used $\widehat{\text{CIA}}_{\text{FS}}(t_p) \leq 1$ and the assumption on θ: $\frac{1-\theta}{\theta} \leq \frac{1}{1-\theta}$. Therefore

$$t_b \leq t_p + \frac{2}{\alpha_{\text{CIA}_{\text{FS}}}} \ln \frac{1}{1-\theta} < \frac{1}{\alpha_{wg_{\text{FS}}}} \ln \frac{1}{1-\theta} + \frac{1}{\alpha_{\text{CIA}_{\text{FS}}}} \ln \frac{1}{1-\theta} < t_w$$

where we have used the timescale separation assumption (A1). Letting $t_0 = t_p$, $t_2 = 0$, and $t_1 = t_3 = \min\{t_b, \alpha_{ptc_{\text{FS}}}^{-1} \ln \frac{1}{\theta}\}$ in the Corollary, obtains $wg_{\text{FS}}(t) = 0$ for all t. ∎

We will next show that if $wg_{\text{LS}}(t) = 1$ in a given interval $[0, T)$, then in fact $wg_{\text{LS}}(t)$ remains expressed for a longer time, up to $T + \delta$, with $\delta > 0$. This is mainly due to assumption (A1), which says that mRNAs take longer than proteins to update their discrete values, because they have longer half-lives: $\alpha_{mRNA}^{-1} > \alpha_{Prot}^{-1}$. This allows the initial signal "$wg_{\text{LS}} = 1$" to travel down the network, sequencially affecting the wingless protein, *engrailed*, *hedgehog* and CIA, and feed back into *wingless* allowing wg_{LS} to remain expressed for a further time interval.

Lemma 1.3. Let $T \geq \frac{1}{\alpha_{wg_{\text{LS}}}} \ln \frac{1}{\theta}$ and define

$$\delta = \frac{1}{\alpha_{\text{WG}_{\text{LS}}}} \ln \frac{(1 - e^{-\frac{\alpha_{\text{WG}_{\text{LS}}}}{\alpha_{wg_{\text{LS}}}} \ln \frac{1}{\theta}})}{\theta}. \tag{11}$$

If $wg_{\text{LS}}(t) = 1$ for $0 \leq t < T$, then

(a) $\mathrm{WG}_{\mathrm{LS}}(t)=1$ for $t\in(\frac{1}{\alpha_{\mathrm{WG}_{\mathrm{LS}}}}\ln\frac{1}{1-\theta},T+\delta)$;

(b) $en_1(t)=1$ for $t\in[0,T+\delta)$;

(c) $\widehat{\mathrm{EN}}_1(t)=1-e^{-\alpha_{\mathrm{EN}_1}t}$ for $t\in[0,T+\delta)$, and $\mathrm{EN}_1(t)=1$ for $(\frac{1}{\alpha_{\mathrm{EN}_1}}\ln\frac{1}{1-\theta},T+\delta)$;

(d) $ci_1(t)=0$, $\mathrm{CI}_1(t)=0$, $\mathrm{CIA}_1(t)=0$, and $\mathrm{CIR}_1(t)=0$ for $t\in[0,T+\delta)$;

(e) $hh_1(t)=1$, for $t\in[0,T+\delta)$;

(f) $\mathrm{CIA}_{\mathrm{LS}}(t)=1$, for $t\in(\frac{1}{\alpha_{\mathrm{CI}_{\mathrm{LS}}}}\ln\frac{1}{1-\theta}+\frac{1}{\alpha_{\mathrm{CIA}_{\mathrm{LS}}}}\ln\frac{1}{1-\theta},T+\delta)$, and $\mathrm{CIR}_{\mathrm{LS}}(t)=0$, for $t\in[0,T+\delta)$;

(g) $wg_{\mathrm{LS}}(t)=1$ for $t\in[0,T+\delta)$.

Proof: Let $T\geq\frac{1}{\alpha_{wg_{\mathrm{LS}}}}\ln\frac{1}{\theta}$, and assume that $wg_{\mathrm{LS}}(t)=1$ for $0\leq t<T$. To prove part (a), note that $\widehat{\mathrm{WG}}_{\mathrm{LS}}(t)$ is of the form (2) (with $t_0=0$, and $\widehat{\mathrm{WG}}_{\mathrm{LS}}(0)=0$) and the corresponding discrete variable is $\mathrm{WG}_{\mathrm{LS}}(t)=1$, for $t\in(\frac{1}{\alpha_{\mathrm{WG}_{\mathrm{LS}}}}\ln\frac{1}{1-\theta},T)$. Moreover, suppose that $wg_{\mathrm{LS}}(t)=0$ for $t>T$, then

$$\widehat{\mathrm{WG}}_{\mathrm{LS}}(t)=(1-e^{-\alpha_{\mathrm{WG}_{\mathrm{LS}}}T})e^{-\alpha_{\mathrm{WG}_{\mathrm{LS}}}(t-T)},\quad t>T.$$

But $\mathrm{WG}_{\mathrm{LS}}$ remains 1 until the switching threshold is attained, that is up to time

$$T+\frac{1}{\alpha_{\mathrm{WG}_{\mathrm{LS}}}}\ln\frac{(1-e^{-\alpha_{\mathrm{WG}_{\mathrm{LS}}}T})}{\theta}\geq T+\frac{1}{\alpha_{\mathrm{WG}_{\mathrm{LS}}}}\ln\frac{(1-e^{-\alpha_{\mathrm{WG}_{\mathrm{LS}}}\frac{1}{\alpha_{wg_{\mathrm{LS}}}}\ln\frac{1}{\theta}})}{\theta}\equiv T+\delta.$$

Thus we conclude that $\mathrm{WG}_{\mathrm{LS}}(t)=1$ in the desired interval.

To prove part (b), observe that $F_{en_1}(t)=\mathrm{WG}_{\mathrm{LS}}(t)$ for all t, from (6), and recall that $en_1(0)=1$. From part (a), $F_{en_1}(t)=1$ for $t\in(\frac{1}{\alpha_{\mathrm{WG}_{\mathrm{LS}}}}\ln\frac{1}{1-\theta},T+\delta)$. On the other hand, en_1 can only switch from 1 to 0 at $t=\alpha_{en_1}^{-1}\ln\frac{1}{\theta}$ which is larger than $\alpha_{\mathrm{WG}_{\mathrm{LS}}}^{-1}\ln\frac{1}{1-\theta}$. So, in fact, $en_1(t)=1$ for all $0\leq t<T+\delta$.

Part (c) follows immediately by integration of the $\widehat{\mathrm{EN}}_1$ equation.

To prove part (d), first recall $F_{ci_1}=$ not EN_1 and the initial conditions $ci_1(0)=0=\mathrm{CI}_1(0)=\mathrm{CIA}_1(0)=\mathrm{CIR}_1(0)$. Therefore $\widehat{ci}_1(t)$ increases up to $t=\frac{1}{\alpha_{\mathrm{EN}_1}}\ln\frac{1}{1-\theta}$ and then decreases in $\alpha_{\mathrm{EN}_1}^{-1}\ln\frac{1}{1-\theta}<t<T+\delta$. Now note that the discrete variable $ci_1(t)$ remains 0 in the whole interval $[0,T+\delta)$. This is because $\widehat{ci}_1$ never reaches the θ threshold: this would be attained at some $t\geq\alpha_{ci_1}^{-1}\ln\frac{1}{1-\theta}$ but, since $\alpha_{ci_1}^{-1}\ln\frac{1}{1-\theta}>\alpha_{\mathrm{EN}_1}^{-1}\ln\frac{1}{1-\theta}$, the function $\widehat{ci}_1$ starts decreasing before it could reach the value θ. Finally, from the rules of the Cubitus proteins it is immediate to see that $\mathrm{CI}_1(t)=\mathrm{CIA}_1(t)=\mathrm{CIR}_1(t)=0$ for $t\in[0,T+\delta)$.

To prove part (e), recall that $F_{hh_1}=\mathrm{EN}_1$ and not CIR_1. From part (a), it follows that $F_{hh_1}(t)=0$ in the interval $[0,\alpha_{\mathrm{EN}_1}^{-1}\ln\frac{1}{1-\theta})$ and $F_{hh_1}(t)=1$ in the interval $(\alpha_{\mathrm{EN}_1}^{-1}\ln\frac{1}{1-\theta},T+\delta)$. Since $hh_1(0)=1$, $\widehat{hh}_1(t)$ decreases in the interval $[0,\alpha_{\mathrm{EN}_1}^{-1}\ln\frac{1}{1-\theta})$ but increases in $(\alpha_{\mathrm{EN}_1}^{-1}\ln\frac{1}{1-\theta},T+\delta)$. The discrete value is $hh_1(t)=1$ in the whole interval, since $\widehat{hh}_1(t)$ remains above the θ threshold. (The justification is similar to the case of $ci_1(t)$ in part (d).)

To prove part (f), note that part (e) and then the use of (8), allows us to simplify $F_{\mathrm{CIA}_{\mathrm{LS}}}$:

$$F_{\mathrm{CIA}_{\mathrm{LS}}}(t)=\mathrm{CI}_{\mathrm{LS}}(t)\ \text{ and }\ hh_1(t)=1,\quad t\in\left(\frac{1}{\alpha_{\mathrm{CI}_{\mathrm{LS}}}}\ln\frac{1}{1-\theta},T+\delta\right).$$

Thus

$$\widehat{\mathrm{CIA}}_{\mathrm{LS}}(t)=\begin{cases}0, & 0\leq t\leq\frac{1}{\alpha_{\mathrm{CI}_{\mathrm{LS}}}}\ln\frac{1}{1-\theta}\\ 1-e^{-\alpha_{\mathrm{CIA}_{\mathrm{LS}}}\left(t-\frac{1}{\alpha_{\mathrm{CI}_{\mathrm{LS}}}}\ln\frac{1}{1-\theta}\right)}, & \frac{1}{\alpha_{\mathrm{CI}_{\mathrm{LS}}}}\ln\frac{1}{1-\theta}<t\leq T+\delta,\end{cases}$$

and $\text{CIA}_{\text{LS}}(t) = 1$ for $t \in [\frac{1}{\alpha_{\text{CI}_{\text{LS}}}} \ln \frac{1}{1-\theta} + \frac{1}{\alpha_{\text{CIA}_{\text{LS}}}} \ln \frac{1}{1-\theta}, T+\delta)$. Observe that this interval is indeed nonempty, by assumption (A1). Finally, $F_{\text{CIR}_{\text{LS}}}(t) = \text{CI}_{\text{LS}}(t)$ and not*hh*$_1(t) = 0$, and hence $\text{CIR}_{\text{LS}}(t) = 0$ for $t \in [0, T+\delta)$.

To prove part (g), we note that (from part (f))

$$F_{wg_{\text{LS}}}(t) = 1, \quad t \in \left(\frac{1}{\alpha_{\text{CI}_{\text{LS}}}} \ln \frac{1}{1-\theta} + \frac{1}{\alpha_{\text{CIA}_{\text{LS}}}} \ln \frac{1}{1-\theta}, T+\delta \right),$$

implying that $\widehat{wg}_{\text{LS}}(t)$ increases in this interval. On the other hand, we know that $\widehat{wg}_{\text{LS}}(t) \geq \theta$ and $wg_{\text{LS}}(t) = 1$ up to at least $t = \frac{1}{\alpha_{wg_{\text{LS}}}} \ln \frac{1}{\theta} > \frac{1}{\alpha_{\text{CI}_{\text{LS}}}} \ln \frac{1}{1-\theta} + \frac{1}{\alpha_{\text{CIA}_{\text{LS}}}} \ln \frac{1}{1-\theta}$. This shows that in fact $wg_{\text{LS}}(t) = 1$ for all $t \in [0, T+\delta)$. ∎

Proof of Theorem 2: Since $wg_{\text{LS}}(0) = 1$, from equations (4), (5), we know that the earliest possible switching time from 1 to 0 is $\alpha_{wg_{\text{LS}}}^{-1} \ln \frac{1}{\theta}$. Applying Lemma 1.3 with $T = \alpha_{wg_{\text{LS}}}^{-1} \ln \frac{1}{\theta}$ establishes that $wg_{\text{LS}}(t) = 1$ for $t \in [0, T+\delta)$, with δ given by (11). Next, applying Lemma 1.3 with $T = \alpha_{wg_{\text{LS}}}^{-1} \ln \frac{1}{\theta} + k\delta$, $k \in \mathbb{N}$, shows that $wg_{\text{LS}}(t) = 1$ for $t \in [0, T+(k+1)\delta)$. Since δ is finite, we can conclude by induction that $wg_{\text{LS}}(t) = 1$ for all $t \geq 0$.

To prove that $\text{PTC}_1(t) \equiv 0$, note that $\text{CIA}_1(t) \equiv 0$ (Lemma 1.3, with $T = +\infty$) implies $ptc_1(t) \equiv 0$. Since $\text{PTC}_1(0) = 0$ and PTC_1 cannot become expressed unless ptc_1 is first expressed, the desired result follows. ∎

References

[1] M. Chaves, E.D. Sontag, and R. Albert. Methods of robustness analysis for boolean models of gene control networks. *IEE Proc. Syst. Biol.*, 153:154–167, 2006.