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Supporting Material

Phase resetting curves allow for simple and accurate prediction of robust N:1 phase locking for strongly coupled neural oscillators

Carmen C. Canavier, Fatma Gurel Kazanci, and Astrid A. Prinz

Supplementary Material: Eigenvalue derivation

Periodicity criteria:

$ts_F[m] = tr_S[m]$; (1)

$P_F\, \phi_F\, [m] = P_S\, \{1 - \phi_{SN}\, [m] + f_{1S}\, (\phi_{SN}\, [m])\}$ (1a)

$tr_{F1}[m+1] = ts_{S1}[m+1]$; (2)

$P_F\, \{1 - \phi_F\, [m] + f_{1F}\, (\phi_F[m])\} = P_S\, \{\phi_{S1}\, [m+1] + f_{2S}\, (\phi_{SN}\, [m]\}$ (2a)

$tr_{F2}[m+1] = ts_{S2}[m+1]$; (3)

$$P_F \{N - 1 + f_{2F}\, (\phi_F[m])\} = P_S\, \{\phi_{SN}\, [m+1] - \phi_{S1}\, [m+1] + \sum_{j=1}^{N-1} f_{1S}(\phi_{Sj}\, [m+1])\, \} \quad (3a)$$

Note that 1, 2 and 3 state the periodicity criteria in terms of the intervals in Fig. 2A whereas 1a, 2a, and 3a rewrite them in terms of the dependence on phase under the pulsatile coupling assumption.

Solve for the phases in terms of ϕ_{SN} using (1a) and (2a):

$\phi_F[m] = (P_S/P_F)\, \{1 - \phi_{SN}[m] + f_{1S}\, (\phi_{SN}[m])\}$ (4)

$\phi_{S1}[m+1] = (P_F / P_S)\, \{1 - \phi_F[m] + f_{1F}\, (\phi_F[m]\,)\} - f_{2S}\, (\phi_{SN}[m]\,)$ (5)

Write the perturbations in terms of the perturbation in ϕ_{SN}^*

$\phi_F^* + \Delta\phi_F\, [m] = (P_S/P_F)\, \{1 - \phi_{SN}^* - \Delta\phi_{SN}\, [m] + f_{1S}\, (\phi_{SN}^*) + f'_{1S}\, (\phi_{SN}^*)\;\; \Delta\phi_{SN}\, [m]\, \}$ (6)

canceling, we have

$\Delta\phi_F\, [m] = (P_S/P_F)\, \{\, f'_{1S}\, (\phi_{SN}^*) - 1\}\Delta\phi_{SN}\, [m]$ *(7)*

$\phi_{S1}^* + \;\Delta\phi_{S1}\, [m+1] = (P_F / P_S)\, \{1 - \phi_F^* - \Delta\phi_F[m] + f_{1F}\, (\phi_F^*)$

$+ f'_{1F}\, (\phi_F^*)\;\; \Delta\phi_F\, [m]\} - f_{2S}\, (\phi_{SN}^*) - f'_{2S}\, (\phi_{SN}^*)\Delta\phi_{SN}\, [m]$ *(8)*

canceling, we have

$\Delta\phi_{S1}\, [m+1] = (P_F / P_S)\{f'_{1F}\, (\phi_F^*)\;\; - 1\}\Delta\, \phi_F\, [m] - f'_{2S}\, (\phi_{SN}^*)\Delta\phi_{SN}\, [m]$ *(9)*

substituting for $\Delta\, \phi_F\, [m]$ in the above, we have

$\Delta\phi_{S1}\, [m+1] = \{\, \{f'_{1F}\, (\phi_F^*)\;\; - 1\}\, \{\, f'_{1S}\, (\phi_{SN}^*) - 1\} - f'_{2S}\, (\phi_{SN}^*)\}\Delta\phi_{SN}\, [m]$ *(10)*

With the expressions for $\Delta\phi_{S1}\, [m+1]$ and $\Delta\phi_F\, [m]$ in hand, we move on to the $\Delta\phi_{Sj}\, [m+1]$

$\phi_{S2}[m+1] = \phi_{S1}[m+1] - f_{1S}(\phi_{S1}[m+1]) + P_F/P_S\{1 + f_{2F}(\phi_F[m])\}$ (11)

$\phi_{Sj}[m+1] = \phi_{S(j-1)}[m+1] - f_{1S}(\phi_{S1}[m+1]) + P_F/P_S$ *(12)*

The linearized expressions are:

$\phi_{S2}* + \Delta\phi_{S2}[m+1] = \phi_{S1}* + \Delta\phi_{S1}[m+1] - f_{1S}(\phi_{S1}*) - f'_{1S}(\phi_{S1}*)\Delta\phi_{S1}[m+1] +$

$P_F/P_S\{1 + f_{2F}(\phi_F*) + f'_{2F}(\phi_F*)\Delta\phi_F[m+1]\}$ (13)

cancelling we get

$\Delta\phi_{S2}[m+1] = \{1 - f'_{1S}(\phi_{S1}*)\}\Delta\phi_{S1}[m+1] + (P_F/P_S)f'_{2F}(\phi_F*)\Delta\phi_F[m]$ *(14)*

The recursive dependence of the $\Delta\phi_{Sj}$ on $\Delta\phi_{S1}[m+1]$ and $\Delta\phi_F[m]$ gives rise to the two summation terms containing Π in the expression for the eigenvalue.

$\phi_{Sj}[m+1] = \phi_{S,j-1}[m+1] - f_{1S}(\phi_{S,j-1}[m+1]) + P_F/P_S$ *(15)*

The linearized version is:

$\phi_{Sj}* + \Delta\phi_{Sj}[m+1] = \phi_{S,j-1}* + \Delta\phi_{S,j-1}[m+1] - f_{1S}(\phi_{S,j-1}*) - f_{1S}(\phi_{S,j-1}*)\Delta\phi_{S,j-1}[m+1] + P_F/P_S$ *(16)*

canceling, we get

$\Delta\phi_{Sj}[m+1] = \{1 - f_{1S}(\phi_{S,j-1}*)\}\Delta\phi_{S,j-1}[m+1]$ (17)

Since $\Delta\phi_{S2}[m+1] = \{1 - f'_{1S}(\phi_{S1}*)\}\Delta\phi_{S1}[m+1] + (P_F/P_S)f'_{2F}(\phi_F*)\Delta\phi_F[m]$ from (14)

then for j>2

$\Delta\phi_{Sj}[m+1] = -\prod_{k=1}^{j-1}\{1 - f'_{1S}(\phi^*_{Sk})\}\,\Delta\phi_{S1}[m+1] + \prod_{k=2}^{j-1}\{1 - f'_{1S}(\phi^*_{Sk})\}\,(P_F/P_S)f'_{2F}(\phi_F*)\Delta\phi_F[m]$ (18)

Substituting for $\Delta\phi_{S1}[m+1]$ *from (10) and* $\Delta\phi_F[m]$ from (7) we obtain

$\Delta\phi_{Sj}[m+1] =$

$\{-\{\{f'_{1F}(\phi_F^*) - 1\}\{f'_{1S}(\phi_{SN}^*) - 1\} - f'_{2S}(\phi_{SN}^*)\}\prod_{k=1}^{j-1}\{1 - f'_{1S}(\phi^*_{Sk})\} +$

$f'_{2F}(\phi_F*)\{f'_{1S}(\phi_{SN}^*) - 1\}\prod_{k=2}^{j-1}\{1 - f'_{1S}(\phi^*_{Sk})\}\}\Delta\phi_{SN}[m]$ (19)

We now solve for $\phi_{SN}[m+1]$ in (3)

$\phi_{SN}[m+1] = \phi_{S1}[m+1] - f_{1S}(\phi_{S1}[m+1]) - \sum_{j=2}^{N-1} f_{1S}(\phi_{Sj}[m+1]) + \{P_F/P_S\}\{N - 1 + f_{2F}(\phi_F[m])\}$

(20)

and linearize

$$\phi_{SN}^* + \Delta\phi_{SN}[m+1] = \phi_{S1}^* + \Delta\phi_{S1}[m+1] - f_{1S}(\phi_{S1}^*) - f'_{1S}(\phi_{S1}^*)\Delta\phi_{S1}[m+1]$$

$$-\sum_{j=2}^{N-1}\{f_{1S}(\phi_{Sj}^*) + f'_{1S}(\phi_{Sj}^*)\Delta\phi_{Sj}[m+1])\} + \{P_F/P_S\}\{N - 1 + f_{2F}(\phi_F^*) + f'_{2F}(\phi_F^*)\Delta\phi_F[m])\} \qquad (21)$$

Cancelling we get

$$\Delta\phi_{SN}[m+1] = \{1 - f_{1S}(\phi_{S1}^*)\}\Delta\phi_{S1}[m+1] - \sum_{j=2}^{N-1} f'_{1S}(\phi_{Sj}^*)\Delta\phi_{Sj}[m+1] + \{P_F/P_S\}f'_{2F}(\phi_F^*)\Delta\phi_F[m]) \qquad (22)$$

Substituting for $\Delta\phi_{S1}[m+1]$ from (10), $\Delta\phi_{Sj}[m+1]$ from (19)and for $\Delta\phi_F[m]$from (7) into (22)

$$\Delta\phi_{SN}[m+1] = \{\{1 - f_{1S}(\phi_{S1}^*)\}\{\{f'_{1F}(\phi_F^*) - 1\}\{f'_{1S}(\phi_{SN}^*) - 1\} - f'_{2S}(\phi_{SN}^*)\} -$$

$$\sum_{j=2}^{N-1} f'_{1S}(\phi_{Sj}^*) \{\{f'_{1F}(\phi_F^*) - 1\}\{f'_{1S}(\phi_{SN}^*) - 1\} - f'_{2S}(\phi_{SN}^*)\}\prod_{k=1}^{j-1}\{1 - f'_{1S}(\phi_{Sk}^*)\} +$$

$$\sum_{j=2}^{N-1}\{f'_{1S}(\phi_{Sj}^*) f'_{2F}(\phi_F^*)\{f'_{1S}(\phi_{SN}^*) - 1\}\prod_{k=2}^{j-1}\{1 - f'_{1S}(\phi_{Sk}^*)\}\}$$

$$+ f'_{2F}(\phi_F^*)\{f'_{1S}(\phi_{SN}^*) - 1\}\}\ \Delta\phi_{SN}[m] \qquad (23)$$

we obtain

Simpliflying

$$\Delta\phi_{SN}[m+1] = \lambda\Delta\phi_{SN}[m] \qquad (24)$$

where

$$\lambda = f'_{2F}(\phi_F^*)\{f'_{1S}(\phi_{SN}^*) - 1\} + \sum_{j=2}^{N-1}\{f'_{1S}(\phi_{Sj}^*) f'_{2F}(\phi_F^*)\{f'_{1S}(\phi_{SN}^*) - 1\}\prod_{k=2}^{j-1}\{1 - f'_{1S}(\phi_{Sk}^*)\}\} +$$

$$\{\{f'_{1F}(\phi_F^*) - 1\}\{f'_{1S}(\phi_{SN}^*) - 1\} - f'_{2S}(\phi_{SN}^*)\}\{1 - f'_{1S}(\phi_{S1}^*) - \sum_{j=2}^{N-1} f'_{1S}(\phi_{Sj}^*)\prod_{k=1}^{j-1}\{1 - f'_{1S}(\phi_{Sk}^*)\}\} \qquad (25)$$

for N=3, the term $\prod_{k=2}^{j-1}\{1 - f'_{1S}(\phi_{Sk}^*)\}$ drops out.

Supplementary Material XPP Notes

In order to compute the coupling functions for the coupled oscillators using XPPAUT (15), we follow the following procedure:

1. First, we compute the voltage and synaptic activation traces for the uncoupled Wang and Buzsáki neurons using a code written for a single neuron. The equations are integrated for a certain time for both slow and fast neurons. By identifying the peak of a spike in the slow neuron, we integrate for its period. The fast neuron is also integrated for the period of the slow neuron. Once we have a full period of the values of voltage and synaptic activation variables, we can compute the adjoint (iPRC). XPPAUT has a built-in code that takes care of the calculation with a few key strokes (starting with the traces for the slow neuron's period, hit Numerics, then Averaging and New Adjoint, after a few seconds, the adjoint can be plotted in the active view window.) Before exiting, we need to save all the traces in a tabular format for use in the next step. This can be done by going to Data Viewer and saving V, s and adjoint traces for a normalized period of length 1.
2. We use another XPPAUT code to compute the coupling functions. This code basically takes all the information from the last step and computes the convolution of the synaptic current perturbation with the iPRC. The perturbation needs to be applied at uniform phase values ranging between 0 and 1. We have used 200 points in our calculations. The coupling functions can be seen in the Data Viewer for each individual neuron and depending on the system and the coupling strength the effective coupling function can be calculated and plotted against the relative phase.
3. The last step is the prediction step where we adjust the applied current to one of the neurons (in our case the fast neuron) and see if the effective coupling function has zero crossings or not. The predictions about stability of the modes are made accordingly by using the slope at the zero crossings.

The codes used in this calculation can be requested from the authors.