Lévy laws and $1/f$ noises: A unified and universal explanation

Supporting Information

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Throughout the supplementary material the following notation will be used: $\mathbf{E}[\xi]$ is the expectation of the random variable ξ; $\mathbf{Cov}[\xi_1, \xi_2]$ is the covariance of the random variables ξ_1 and ξ_2.

1 The stationary superposition model: Proofs

In this Section we set:

$$\mathcal{P} = \{(a_k, \omega_k)\}_k \ ; \tag{1}$$

$$G_X(\theta) = 1 - \mathbf{E}[\exp(i\theta X(0))] \tag{2}$$

(θ real);

$$R_X(t) = \mathbf{Cov}[X(0), X(t)] \tag{3}$$

(t real);

$$\widehat{R}_X(\theta) = \int_{-\infty}^{\infty} R_X(t) \exp(i\theta t)\, dt \tag{4}$$

(θ real). Recall that $\phi(x) = \int_0^\infty \lambda(x, y) dy$ (x real) and $\psi(y) = \int_{-\infty}^\infty x^2 \lambda(x, y) dx$ ($y > 0$), and that the signal pattern X is a zero mean stationary process with short-range correlations.

1.1 Proof of amplitudal universality

Step 1. We compute the conditional Fourier transform of the random variable $Y(t)$ (t being an arbitrary time point), given the amplitude-frequency pairs $\mathcal{P}$:

$$\mathbf{E}[\exp(i\theta Y(t)) \ \mid \mathcal{P}] \tag{5}$$

(using the definition of the process Y)

$$= \mathbf{E}\left[\prod_k \exp\Big(i(\theta a_k) X_k(\omega_k t)\Big) \ \mid \mathcal{P}\right] \tag{6}$$

(using the independence of the transmission sources)

$$= \prod_k \mathbf{E}\left[\exp\Big(i(\theta a_k) X_k(\omega_k t)\Big) \ \mid \mathcal{P}\right] \tag{7}$$

(using the fact that the signal patterns transmitted by the sources are i.i.d. copies of the generic stationary signal pattern X)

$$= \prod_k \mathbf{E}\left[\exp\Big(i(\theta a_k) X(0)\Big)\right] \tag{8}$$

(using the definition of the function $G_X(\theta)$)

$$= \prod_k \Big(1 - G_X(\theta a_k)\Big) \ . \tag{9}$$

Step 2. We compute the Fourier transform of the random variable $Y(t)$ (t being an arbitrary time point):

$$\mathbf{E}\left[\exp\left(i\theta Y(t)\right)\right] \tag{10}$$

(using conditioning)

$$= \mathbf{E}\Big[\mathbf{E}\left[\exp\left(i\theta Y(t)\right) \mid \mathcal{P}\right]\Big] \tag{11}$$

(using Step 1)

$$= \mathbf{E}\left[\prod_k \Big(1 - G_X(\theta a_k)\Big)\right] \tag{12}$$

(using equation (3.35) in [1])

$$= \exp\left(-\int_{-\infty}^{\infty}\int_0^{\infty} G_X(\theta x)\,\lambda(x,y)\,dxdy\right) \tag{13}$$

(using the definition of the function $\phi(x)$)

$$= \exp\left(-\int_{-\infty}^{\infty} G_X(\theta x)\,\phi(x)\,dx\right) \tag{14}$$

(using the change of variables $u = \theta x$)

$$= \exp\left(-\frac{1}{|\theta|}\int_{-\infty}^{\infty} G_X(u)\,\phi\left(\frac{u}{\theta}\right)du\right) \ . \tag{15}$$

Step 3. Equation (15) implies that the random variable $Y(t)$ is independent of the signal pattern X – up to a scale factor – if and only if the function $\phi(x)$ is a power-law. Specifically, if $\phi(x) = c_1|x|^{-1-\alpha}$ then

$$\mathbf{E}\Big[\exp\Big(i\theta Y(t)\Big)\Big] = \exp\Big(-c_2|\theta|^{\alpha}\Big) \ , \tag{16}$$

where

$$c_2 = c_1\int_{-\infty}^{\infty}\frac{G_X(u)}{|u|^{1+\alpha}}du \ . \tag{17}$$

The function $G_X(u)$ satisfies $|G_X(u)| \le 2$ and $G_X(u) \sim \left(\frac{1}{2}R_X(0)\right)u^2$ as $u \to 0$. Hence, the integral appearing on the right hand side of equation (17) is convergent in the exponent range $0 < \alpha < 2$.

1.2 Proof of temporal universality

Step 1. We compute the conditional mean of the random variable $Y(t)$ (t being an arbitrary time point), given the amplitude-frequency pairs $\mathcal{P}$:

$$\mathbf{E}\left[Y(t) \mid \mathcal{P}\right] \tag{18}$$

(using the definition of the process Y)

$$= \mathbf{E}\left[\sum_k a_k X_k\left(\omega_k t\right) \mid \mathcal{P}\right] \tag{19}$$

$$= \sum_k a_k \mathbf{E}\left[X_k\left(\omega_k t\right) \mid \mathcal{P}\right] \tag{20}$$

(using the fact that the signal patterns transmitted by the sources are i.i.d. copies of a generic stationary signal pattern X with zero mean)

$$= \sum_k a_k \mathbf{E}\left[X\left(0\right)\right] = 0 \ . \tag{21}$$

Step 2. We compute the conditional covariance of the random variables $Y(t)$ and $Y\left(t+\Delta\right)$(t being an arbitrary time point; Δ being an arbitrary lag), given the amplitude-frequency pairs $\mathcal{P}$:

$$\mathbf{Cov}\left[Y(t), Y\left(t+\Delta\right) \mid \mathcal{P}\right] \tag{22}$$

(using the definition of the process Y)

$$\begin{aligned} &= \mathbf{Cov}\left[\textstyle\sum_k a_k X_k\left(\omega_k t\right), \sum_j a_j X_j\left(\omega_j\left(t+\Delta\right)\right) \mid \mathcal{P}\right] \\ &= \textstyle\sum_k \sum_j a_k a_j \mathbf{Cov}\left[X_k\left(\omega_k t\right), X_j\left(\omega_j\left(t+\Delta\right)\right) \mid \mathcal{P}\right] \end{aligned} \tag{23}$$

(using the independence of the transmission sources)

$$= \sum_k a_k^2 \mathbf{Cov}\left[X_k\left(\omega_k t\right), X_k\left(\omega_k\left(t+\Delta\right)\right) \mid \mathcal{P}\right] \tag{24}$$

(using the fact that the signal patterns transmitted by the sources are i.i.d. copies of the generic stationary signal pattern X)

$$= \sum_k a_k^2 \mathbf{Cov}\left[X\left(0\right), X\left(\omega_k \Delta\right)\right] \tag{25}$$

(using the definition of the function $R_X\left(t\right)$)

$$= \sum_k a_k^2 R_X\left(\omega_k \Delta\right) \ . \tag{26}$$

Step 3. We compute the covariance of the of the random variables $Y(t)$ and $Y(t+\Delta)$(t being an arbitrary time point; Δ being an arbitrary lag):

$$\mathbf{Cov}\left[Y(t), Y(t+\Delta)\right] \tag{27}$$

(using conditioning)

$$= \mathbf{Cov}\Big[\mathbf{E}\left[Y(t) \mid \mathcal{P}\right], \mathbf{E}\left[Y(t+\Delta) \mid \mathcal{P}\right]\Big] + \mathbf{E}\Big[\mathbf{Cov}\left[Y(t), Y(t+\Delta) \mid \mathcal{P}\right]\Big] \tag{28}$$

(using Steps 1 and 2)

$$= \mathbf{Cov}\left[0,0\right] + \mathbf{E}\left[\sum_k a_k^2 R_X\left(\omega_k \Delta\right)\right] \tag{29}$$

(using equation (3.9) in [1])

$$= \int_{-\infty}^{\infty}\int_0^{\infty} x^2 R_X\left(y\Delta\right)\lambda\left(x,y\right) dxdy \tag{30}$$

(using the definition of the function $\psi\left(y\right)$)

$$= \int_0^{\infty} R_X\left(y\Delta\right)\psi\left(y\right) dy\ . \tag{31}$$

Step 4. We compute the power spectrum of the output process Y:

$$\int_{-\infty}^{\infty}\mathbf{Cov}\Big[Y\left(0\right), Y\left(\Delta\right)\Big]\exp\left(if\Delta\right) d\Delta \tag{32}$$

(using Step 3)

$$\begin{aligned} &= \textstyle\int_{-\infty}^{\infty}\left(\int_0^{\infty} R_X\left(y\Delta\right)\psi\left(y\right) dy\right)\exp\left(if\Delta\right) d\Delta \\ &= \textstyle\int_0^{\infty}\left(\int_{-\infty}^{\infty} R_X\left(y\Delta\right)\exp\left(if\Delta\right) d\Delta\right)\psi\left(y\right) dy \end{aligned} \tag{33}$$

(using the change of variables $t = y\Delta$)

$$= \int_0^{\infty}\left(\frac{1}{y}\int_{-\infty}^{\infty} R_X\left(t\right)\exp\left(i\frac{f}{y}t\right) d\Delta\right)\psi\left(y\right) dy \tag{34}$$

(using the definition of the function $\widehat{R}_X\left(\theta\right)$)

$$= \int_0^{\infty}\left(\frac{1}{y}\widehat{R}_X\left(\frac{f}{y}\right)\right)\psi\left(y\right) dy \tag{35}$$

(using the change of variables $u = f/y$)

$$= \int_0^{\infty}\widehat{R}_X\left(u\right)\left(\frac{1}{u}\psi\left(\frac{|f|}{u}\right)\right) du\ . \tag{36}$$

Step 5. Equation (36) implies that the power spectrum of the output process Y is independent of the signal pattern X – up to a scale factor – if and only if the function $\psi(y)$ is a power-law. Specifically, if $\psi(y) = c_3 y^{-\beta}$ then

$$\int_{-\infty}^{\infty} \mathbf{Cov}\left[Y(0), Y(t)\right] \exp(ift)\, dt = \frac{c_4}{|f|^{\beta}} \ , \tag{37}$$

where

$$c_4 = c_3 \int_0^{\infty} \frac{\widehat{R}_X(u)}{u^{1-\beta}} du \ . \tag{38}$$

And, plugging $\psi(y) = c_3 y^{-\beta}$ into equation (31) yields

$$\mathbf{Cov}\left[Y(t), Y(t+\Delta)\right] = \frac{c_6}{\Delta^{1-\beta}} \ , \tag{39}$$

where

$$c_6 = c_3 \int_0^{\infty} \frac{R_X(u)}{u^{\beta}} du \ . \tag{40}$$

Since the signal pattern X has short-range correlations, the integral $\int_0^{\infty} R_X(u)\, du$ is finite, and the function $\widehat{R}_X(u)$ satisfies $\widehat{R}_X(0) = \int_{-\infty}^{\infty} R_X(t)\, dt$ (finite) and $\int_0^{\infty} \widehat{R}_X(u)\, du = \pi R_X(0)$ (finite). Hence, the integral appearing on the right hand side of equation (38) is convergent in the exponent range $0 < \beta \leq 1$, and the integral appearing on the right hand side of equation (40) is convergent in the exponent range $0 < \beta < 1$. The admissible exponent range is thus $0 < \beta < 1$.

2 From $1/f$ noise to super diffusion: Proof

In this Section we set:

$$R_Y(t) = \mathbf{Cov}\left[Y(0), Y(t)\right] \tag{41}$$

(t real);

$$\widehat{R}_Y(\theta) = \int_{-\infty}^{\infty} R_Y(t) \exp(i\theta t)\, dt \tag{42}$$

(θ real). We prove that if the stationary process Y has a $1/f$ power spectrum then the integrated process Z – given by $Z(t) = \int_0^t Y(t')dt'$ ($t \geq 0$) – is super-diffusive.

Step 1. We compute the mean square displacement of the integrated process Z, in terms of the power spectrum $\widehat{R}_Y(\theta)$ of the process Y:

$$\mathbf{E}\left[Z(t)^2\right] \tag{43}$$

(using the definition of the integrated process Z)

$$\begin{aligned} &= \mathbf{E}\left[\left(\int_0^t Y(t_1)dt_1\right)\left(\int_0^t Y(t_2)dt_2\right)\right] \\ &= \int_0^t \int_0^t \mathbf{E}\left[Y(t_1)Y(t_2)\right] dt_1 dt_2 \end{aligned} \tag{44}$$

(using the fact that the stationary process Y has zero mean)

$$= \int_0^t \int_0^t \mathbf{Cov}\left[Y(t_1), Y(t_2)\right] dt_1 dt_2 \tag{45}$$

(using the definition of the function $R_Y(t)$)

$$= \int_0^t \int_0^t R_Y\left(t_2 - t_1\right)) dt_1 dt_2 \tag{46}$$

(using Fourier inversion)

$$\begin{aligned}
&= \textstyle\int_0^t \int_0^t \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{R}_Y(\theta) \exp\left(-i\left(t_2 - t_1\right)\theta\right) d\theta\right) dt_1 dt_2 \\
&= \textstyle\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{R}_Y(\theta) \left(\int_0^t \exp\left(it_1\theta\right) dt_1\right) \left(\int_0^t \exp\left(-it_2\theta\right) dt_2\right) d\theta \\
&= \textstyle\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{R}_Y(\theta) \left(\frac{\exp(it\theta)-1}{i\theta}\right) \left(\frac{\exp(-it\theta)-1}{-i\theta}\right) d\theta \\
&= \textstyle\frac{1}{\pi}\int_{-\infty}^{\infty} \widehat{R}_Y(\theta) \frac{1-\cos(t\theta)}{\theta^2} d\theta
\end{aligned} \tag{47}$$

(using the change of variables $u = t\theta$)

$$= \frac{t}{\pi}\int_{-\infty}^{\infty} \widehat{R}_Y\left(\frac{u}{t}\right) \frac{1-\cos(u)}{u^2} du\ . \tag{48}$$

Step 2. If the stationary process Y is a $1/f$ noise then

$$\widehat{R}_Y(\theta) = \frac{c_4}{|\theta|^\beta}\ . \tag{49}$$

Plugging the $1/f$ power spectrum of equation (49) into equation (48) yields

$$\mathbf{E}\left[Z(t)^2\right] = c_5 t^{1+\beta}\ , \tag{50}$$

where

$$c_5 = c_4 \frac{2}{\pi}\int_0^{\infty} \frac{1-\cos(u)}{u^{2+\beta}} du\ . \tag{51}$$

The integral appearing on the right hand side of equation (51) is convergent in the exponent range $0 < \beta < 1$.

3 The dissipative superposition model: Proofs

In this Section we set:

$$\mathcal{P} = \left\{\left(\tau_k, a_k, \omega_k\right)\right\}_k\ ; \tag{52}$$

$$G_X(\theta) = \int_0^{\infty} \left(1 - \mathbf{E}\left[\exp\left(i\theta X(t)\right)\right]\right) dt \tag{53}$$

(θ real);

$$R_X(t) = \int_0^\infty \mathbf{Cov}\left[X(u), X(u+|t|)\right] du \tag{54}$$

(t real);

$$\widehat{R}_X(\theta) = \int_{-\infty}^\infty R_X(t) \exp(i\theta t)\, dt \tag{55}$$

(θ real). Recall that $\phi(x) = \int_0^\infty [\eta(x,y)/y]\, dy$ (x real) and $\psi(y) = \int_{-\infty}^\infty x^2 [\eta(x,y)/y]\, dx$ ($y > 0$), and that signal pattern X is a zero mean dissipative process. We require the following conditions regarding the stochastic decay of the process X to zero:

Condition 1 *The function $G_X(\theta)$ is bounded.*

Condition 2 *The function $R_X(t)$ is finite at the origin ($t = 0$), and is integrable over the real line.*

An important case for which the function $G_X(\theta)$ is bounded is the following: There exists a random time T_X – with finite mean – above which the dissipative signal pattern X vanishes. Indeed, in the aforementioned case we have ($I\{E\}$ denoting the indicator function of the event E):

$$\begin{aligned} &|G_X(\theta)| \le \int_0^\infty \mathbf{E}\left[|1 - \exp(i\theta X(t))|\right] dt \\ &= \int_0^\infty \mathbf{E}\left[|1 - \exp(i\theta X(t))| \cdot I\{t < T_X\}\right] dt \\ &\le \int_0^\infty \mathbf{E}\left[2 \cdot I\{t < T_X\}\right] dt = 2\int_0^\infty \mathbf{P}(T_X > t) dt \\ &= 2\mathbf{E}[T_X] < \infty\ . \end{aligned} \tag{56}$$

3.1 Proof of amplitudal universality

Step 1. We compute the conditional Fourier transform of the random variable $Y(t)$ (t being an arbitrary time point), given the initiation-amplitude-frequency triplets $\mathcal{P}$:

$$\mathbf{E}\left[\exp(i\theta Y(t)) \mid \mathcal{P}\right] \tag{57}$$

(using the definition of the process Y)

$$= \mathbf{E}\left[\prod_{\tau_k \le t} \exp\Big(i(\theta a_k) X_k(\omega_k(t - \tau_k))\Big) \mid \mathcal{P}\right] \tag{58}$$

(using the independence of the dissipative signals)

$$= \prod_{\tau_k \le t} \mathbf{E}\Big[\exp\Big(i(\theta a_k) X_k(\omega_k(t - \tau_k))\Big) \mid \mathcal{P}\Big] \tag{59}$$

(using the fact that the dissipative signals X_k are i.i.d. copies of the generic signal pattern X)

$$= \prod_{\tau_k \le t} \mathbf{E}\left[\exp\left(i\left(\theta a_k\right) X\left(\omega_k\left(t-\tau_k\right)\right)\right)\right] \ . \tag{60}$$

Step 2. We compute the Fourier transform of the random variable $Y(t)$ (t being an arbitrary time point):

$$\mathbf{E}\left[\exp\left(i\theta Y(t)\right)\right] \tag{61}$$

(using conditioning)

$$= \mathbf{E}\left[\mathbf{E}\left[\exp\left(i\theta Y(t)\right) \mid \mathcal{P}\right]\right] \tag{62}$$

(using Step 1)

$$= \mathbf{E}\left[\prod_{\tau_k \le t} \mathbf{E}\left[\exp\left(i\left(\theta a_k\right) X\left(\omega_k\left(t-\tau_k\right)\right)\right)\right]\right] \tag{63}$$

(using equation (3.35) in [1])

$$= \exp\left(-\int_{-\infty}^{t}\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(1-\mathbf{E}\left[\exp\left(i\left(\theta x\right) X\left(y\left(t-s\right)\right)\right)\right]\right)\eta\left(x,y\right)dsdxdy\right)$$

$$= \exp\left(-\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\int_{-\infty}^{t}\left(1-\mathbf{E}\left[\exp\left(i\left(\theta x\right) X\left(y\left(t-s\right)\right)\right)\right]\right)ds\right)\eta\left(x,y\right)dxdy\right) \tag{64}$$

(using the change of variables $u = y\left(t-s\right)$)

$$= \exp\left(-\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{1}{y}\int_{0}^{\infty}\left(1-\mathbf{E}\left[\exp\left(i\left(\theta x\right) X(u)\right)\right]\right)du\right)\eta\left(x,y\right)dxdy\right) \tag{65}$$

(using the definition of the function $G_X\left(\theta\right)$)

$$= \exp\left(-\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{1}{y}G_X\left(\theta x\right)\right)\eta\left(x,y\right)dxdy\right) \tag{66}$$

(using the definition of the function $\phi\left(x\right)$)

$$= \exp\left(-\int_{-\infty}^{\infty} G_X\left(\theta x\right)\phi\left(x\right)dydx\right) \tag{67}$$

(using the change of variables $u = \theta x$)

$$= \exp\left(-\frac{1}{|\theta|}\int_{-\infty}^{\infty} G_X\left(u\right)\phi\left(\frac{u}{\theta}\right)du\right) \ . \tag{68}$$

Step 3. Equation (68) implies that the random variable $Y\left(t\right)$ is independent of the signal pattern X – up to a scale factor – if and only if the function $\phi\left(x\right)$ is a power-law. Specifically, if $\phi(x) = c_1|x|^{-1-\alpha}$ then

$$\mathbf{E}\left[\exp\left(i\theta Y(t)\right)\right] = \exp\left(-c_2|\theta|^{\alpha}\right) \ , \tag{69}$$

where

$$c_2 = c_1 \int_{-\infty}^{\infty} \frac{G_X(u)}{|u|^{1+\alpha}} du \ . \tag{70}$$

The function $G_X(u)$ is bounded (Condition 1), and satisfies $G_X(u) \sim \left(\frac{1}{2} R_X(0)\right) u^2$ as $u \to 0$ (Condition 2 assures that $R_X(0)$ is finite). Hence, the integral appearing on the right hand side of equation (69) is convergent in the exponent range $0 < \alpha < 2$.

3.2 Proof of temporal universality

Step 1. We compute the conditional mean of the random variable $Y(t)$ (t being an arbitrary time point), given the initiation-amplitude-frequency triplets $\mathcal{P}$:

$$\mathbf{E}[Y(t) \mid \mathcal{P}] \tag{71}$$

(using the definition of the process Y)

$$\begin{aligned} &= \mathbf{E}\left[\textstyle\sum_{\tau_k \le t} a_k X_k(\omega_k(t-\tau_k)) \mid \mathcal{P}\right] \\ &= \textstyle\sum_{\tau_k \le t} a_k \mathbf{E}[X_k(\omega_k(t-\tau_k)) \mid \mathcal{P}] \end{aligned} \tag{72}$$

(using the fact that the dissipative signals X_k are i.i.d. copies of the generic signal pattern X with zero mean)

$$= \sum_{\tau_k \le t} a_k \mathbf{E}[X(\omega_k(t-\tau_k))] = 0 \ . \tag{73}$$

Step 2. We compute the conditional covariance of the random variables $Y(t)$ and $Y(t+\Delta)$(t being an arbitrary time point; Δ being an arbitrary lag), given the initiation-amplitude-frequency triplets $\mathcal{P}$:

$$\mathbf{Cov}[Y(t), Y(t+\Delta) \mid \mathcal{P}] \tag{74}$$

(using the definition of the process Y)

$$\begin{aligned} &= \mathbf{Cov}\left[\textstyle\sum_{\tau_k \le t} a_k X_k(\omega_k(t-\tau_k)), \sum_{\tau_j \le t+\Delta} a_j X_j(\omega_j(t+\Delta-\tau_j)) \mid \mathcal{P}\right] \\ &= \textstyle\sum_{\tau_k \le t} \sum_{\tau_j \le t+\Delta} a_k a_j \mathbf{Cov}[X_k(\omega_k(t-\tau_k)), X_j(\omega_j(t+\Delta-\tau_j)) \mid \mathcal{P}] \end{aligned} \tag{75}$$

(using the independence of the dissipative signals)

$$= \sum_{\tau_k \le t} a_k^2 \mathbf{Cov}\left[X_k(\omega_k(t-\tau_k)), X_k(\omega_k(t+\Delta-\tau_k)) \mid \mathcal{P}\right] \tag{76}$$

(using the fact that the dissipative signals are i.i.d. copies of the dissipative signal pattern X)

$$= \sum_{\tau_k \le t} a_k^2 \mathbf{Cov}\left[X(\omega_k(t-\tau_k)), X(\omega_k(t+\Delta-\tau_k))\right] \ . \tag{77}$$

Step 3. We compute the covariance of the of the random variables $Y(t)$ and $Y(t+\Delta)$(t being an arbitrary time point; Δ being an arbitrary lag):

$$\mathbf{Cov}\left[Y(t), Y(t+\Delta)\right] \tag{78}$$

(using conditioning)

$$= \mathbf{Cov}\Big[\mathbf{E}\left[Y(t) \mid \mathcal{P}\right], \mathbf{E}\left[Y(t+\Delta) \mid \mathcal{P}\right]\Big] + \mathbf{E}\Big[\mathbf{Cov}\left[Y(t), Y(t+\Delta) \mid \mathcal{P}\right]\Big] \tag{79}$$

(using Steps 1 and 2)

$$= \mathbf{Cov}\left[0,0\right] + \mathbf{E}\left[\sum_{\tau_k \le t} a_k^2 \mathbf{Cov}\Big[X\left(\omega_k\left(t-\tau_k\right)\right), X\left(\omega_k\left(t+\Delta-\tau_k\right)\right)\Big]\right] \tag{80}$$

(using equation (3.9) in [1])

$$\begin{aligned} &= \textstyle\int_{-\infty}^{t}\int_{-\infty}^{\infty}\int_{0}^{\infty} x^2 \mathbf{Cov}\left[X\left(y\left(t-s\right)\right), X\left(y\left(t+\Delta-s\right)\right)\right]\lambda\left(x,y\right) dsdxdy \\ &= \textstyle\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\int_{-\infty}^{t} \mathbf{Cov}\left[X\left(y\left(t-s\right)\right), X\left(y\left(t+\Delta-s\right)\right)\right] ds\right) x^2\lambda\left(x,y\right) dxdy \end{aligned} \tag{81}$$

(using the change of variables $u = y\left(t-s\right)$)

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{1}{y}\int_{0}^{\infty} \mathbf{Cov}\left[X\left(u\right), X\left(u+y\Delta\right)\right] du\right) x^2\lambda\left(x,y\right) dxdy \tag{82}$$

(using the definition of the function $R_X\left(t\right)$)

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{1}{y}R_X\left(y\Delta\right)\right) x^2\lambda\left(x,y\right) dxdy \tag{83}$$

(using the definition of the function $\psi\left(y\right)$)

$$= \int_{0}^{\infty} R_X\left(\Delta y\right)\psi\left(y\right) dy \ . \tag{84}$$

Step 4. We compute the power spectrum of the output process Y:

$$\int_{-\infty}^{\infty} \mathbf{Cov}\Big[Y\left(0\right), Y\left(\Delta\right)\Big] \exp\left(if\Delta\right) d\Delta \tag{85}$$

(using Step 3)

$$\begin{aligned} &= \textstyle\int_{-\infty}^{\infty}\left(\int_{0}^{\infty} R_X\left(y\Delta\right)\psi\left(y\right) dy\right)\exp\left(if\Delta\right) d\Delta \\ &= \textstyle\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} R_X\left(y\Delta\right)\exp\left(if\Delta\right) d\Delta\right)\psi\left(y\right) dy \end{aligned} \tag{86}$$

(using the change of variables $t = y\Delta$)

$$= \int_0^\infty \left(\frac{1}{y} \int_{-\infty}^\infty R_X(t) \exp\left(i \frac{f}{y} t \right) d\Delta \right) \psi(y)\, dy \tag{87}$$

(using the definition of the function $\widehat{R}_X(\theta)$)

$$= \int_0^\infty \left(\frac{1}{y} \widehat{R}_X \left(\frac{f}{y} \right) \right) \psi(y)\, dy \tag{88}$$

(using the change of variables $u = f/y$)

$$= \int_0^\infty \widehat{R}_X(u) \left(\frac{1}{u} \psi \left(\frac{|f|}{u} \right) \right) du \ . \tag{89}$$

Step 5. Equation (89) implies that the power spectrum of the output process Y is independent of the signal pattern X – up to a scale factor – if and only if the function $\psi(y)$ is a power-law. Specifically, if $\psi(y) = c_3 y^{-\beta}$ then

$$\int_{-\infty}^\infty \mathbf{Cov}\left[Y(0), Y(t) \right] \exp(ift)\, dt = \frac{c_4}{|f|^\beta} \ , \tag{90}$$

where

$$c_4 = c_3 \int_0^\infty \frac{\widehat{R}_X(u)}{u^{1-\beta}} du \ . \tag{91}$$

And, plugging $\psi(y) = c_3 y^{-\beta}$ into equation (84) yields

$$\mathbf{Cov}\left[Y(t), Y(t+\Delta) \right] = \frac{c_6}{\Delta^{1-\beta}} \ , \tag{92}$$

where

$$c_6 = c_3 \int_0^\infty \frac{R_X(u)}{u^\beta} du \tag{93}$$

Condition 2 implies that: The integral $\int_0^\infty R_X(u)\, du$ is finite; The function $\widehat{R}_X(u)$ satisfies $\widehat{R}_X(0) = \int_{-\infty}^\infty R_X(t)\, dt$ (finite) and $\int_0^\infty \widehat{R}_X(u)\, du = \pi R_X(0)$ (finite). Hence, the integral appearing on the right hand side of equation (91) is convergent in the exponent range $0 < \beta \le 1$, and the integral appearing on the right hand side of equation (93) is convergent in the exponent range $0 < \beta < 1$. The admissible exponent range is thus $0 < \beta < 1$.

References

[1] Kingman J-F-C (1993) *Poisson processes* (Oxford University Press, Oxford).