Statistical Mechanics and Hydrodynamics of Bacterial Suspensions
Supplementary Material

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S1. DERIVATION OF HYDRODYNAMIC EQUATIONS

S1.1 Microdynamics

As outlined in the main text, the microdynamics of a collection of Stokesian swimmers is given by

$$\partial_t \mathbf{r}_{L\alpha} = \mathbf{u}(\mathbf{r}_{L\alpha}),$$
$$\partial_t \mathbf{r}_{S\alpha} = \mathbf{u}(\mathbf{r}_{S\alpha}), \tag{1}$$

where $\mathbf{r}_{L\alpha}$ and $\mathbf{r}_{S\alpha}$ denote the position of the large and small sphere with respect to a fixed coordinate system The flow velocity $\mathbf{u}(\mathbf{r})$ of the fluid at position $\mathbf{r}$ is determined by the solution of the Stokes equation

$$\eta\nabla^2 \mathbf{u}(\mathbf{r}) - \nabla p + \mathbf{F}_{active} - \mathbf{F}_{random} = 0\ , \tag{2}$$

where $\mathbf{F}_{active} = \sum_\alpha f\hat{\boldsymbol{\nu}}_\alpha \left[\delta\left(\mathbf{r}-\mathbf{r}_{L\alpha}\right) - \delta\left(\ \mathbf{r}-\mathbf{r}_{S\alpha}\right)\right]$ is the force density due to the active forces exerted by the swimmer and $\mathbf{F}_{random} = \sum_\alpha \left[\boldsymbol{\xi}^L_\alpha(t)\delta\left(\mathbf{r}-\mathbf{r}_{L\alpha}\right) + \boldsymbol{\xi}^S_\alpha(t)\delta\left(\mathbf{r}-\mathbf{r}_{S\alpha}\right)\right]$ is the random force density associated with the effect of fluid fluctuations on the swimmer. These random forces have zero mean and correlations $\langle \xi^\sigma_{\alpha i}(t)\xi^{\sigma'}_{\beta j}(t')\rangle = 2\zeta_\sigma\delta_{\sigma\sigma'}k_BT_a\delta_{ij}\delta_{\alpha\beta}\delta(t-t')$, with $\sigma,\sigma' = L,S$ and $\zeta_\sigma = 6\pi\eta a_\sigma$ the friction of a sphere of radius a_σ in a fluid of viscosity η . Further, we assume that the fluid is incompressible, i.e., $\nabla\cdot\mathbf{u} = 0$. From this set of equations, we eliminate the fluid and derive equations of motion for the hydrodynamic center and orientational coordinate of each swimmer (Eq.(2.8) and (2.9) in the main text). In this subsection we outline the key steps of this derivation.

Let us begin by considering one isolated swimmer. In this case, the Stokes equation is readily solved to obtain the flow field at the centers of the large and small spheres as

$$u_i\left(\mathbf{r}_L\right) = \frac{f\hat{\nu}_i}{\zeta_L} - \frac{f\hat{\nu}_i}{4\pi\eta\ell} + \frac{\xi^L_i}{\zeta_L} + \frac{1}{8\pi\eta\ell}\left(\delta_{ij} + \hat{\nu}_i\hat{\nu}_j\right)\xi^S_j(t)\ , \tag{3}$$

$$u_i\left(\mathbf{r}_S\right) = -\frac{f\hat{\nu}_i}{\zeta_S} + \frac{f\hat{\nu}_i}{4\pi\eta\ell} + \frac{\xi^S_i}{\zeta_S} + \frac{1}{8\pi\eta\ell}\left(\delta_{ij} + \hat{\nu}_i\hat{\nu}_j\right)\xi^L_j(t)\ . \tag{4}$$

The equations of motion for the hydrodynamic center $\mathbf{r}^C = \frac{\zeta_L\mathbf{r}_h + \zeta_S\mathbf{r}_t}{\zeta_L+\zeta_S}$ and the orientation $\hat{\boldsymbol{\nu}} = \left(\mathbf{r}_L - \mathbf{r}_S\right)/\ell$ of the swimmer are then given by

$$\partial_t \mathbf{r}^C = v_0\hat{\boldsymbol{\nu}} + \boldsymbol{\xi}^T(t)\ , \tag{5}$$

$$\partial_t \hat{\boldsymbol{\nu}} = \boldsymbol{\omega}\times\hat{\boldsymbol{\nu}}\ , \tag{6}$$

$$\boldsymbol{\omega} = \hat{\boldsymbol{\nu}}\times\boldsymbol{\xi}^R(t)\ , \tag{7}$$

where

$$v_0 = -\frac{f}{8\pi\eta\ell}\frac{\Delta a}{\overline{a}} \ , \tag{8}$$

is the self-propulsion velocity, with $\Delta a = a_L - a_S$ and $\overline{a} = (a_L + a_S)/2$, and

$$\xi_i^T = \frac{1}{2\overline{\zeta}}\left(\xi_i^L + \xi_i^S\right) + \frac{1}{2\overline{a}\zeta_\ell}\left(\delta_{ij} + \ \hat{\nu}_i\hat{\nu}_j\right)\left[a_L\xi_j^S(t) + a_S\xi_j^L(t)\right] \ , \tag{9}$$

$$\boldsymbol{\xi}^R(t) = \frac{1}{\ell}\left[\left(\frac{1}{\zeta_L} - \frac{1}{\zeta_\ell}\right)\boldsymbol{\xi}^L - \left(\frac{1}{\zeta_S} - \frac{1}{\zeta_\ell}\right)\boldsymbol{\xi}^S\right] \ , \tag{10}$$

with $\zeta_\ell = 8\pi\eta\ell$ and $\overline{\zeta} = 6\pi\eta\overline{a} = (\zeta_L + \zeta_S)/2$. Also, from the variance of the noise at the large and small spheres, the two-point correlation functions of $\boldsymbol{\xi}^T$ and $\boldsymbol{\xi}^R$ are readily obtained as

$$\left\langle \xi_i^T(t)\xi_j^T(t')\right\rangle = \left[D_\parallel\hat{\nu}_i\hat{\nu}_j + D_\perp\left(\delta_{ij} - \ \hat{\nu}_i\hat{\nu}_j\right)\right]\delta\left(t - t'\right) \ , \tag{11}$$

$$\left\langle \xi_i^R\xi_j^R\right\rangle = D_R\delta_{ij}\delta\left(t - t'\right) \ , \tag{12}$$

with

$$D_\parallel = \frac{k_BT}{\overline{\zeta}}\left[1 + \frac{9}{4}\frac{a_La_S}{\ell^2}\right] + \frac{4k_BT}{\zeta_\ell}\frac{a_La_S}{\overline{a}^2} \ , \tag{13}$$

$$D_\perp = \frac{k_BT}{\overline{\zeta}}\left[1 + \frac{9}{16}\frac{a_La_S}{\ell^2}\right] + \frac{2k_BT}{\zeta_\ell}\frac{a_La_S}{\overline{a}^2} \ , \tag{14}$$

$$D_R = \frac{2k_BT}{\ell^2\zeta_\ell^2}\left[\frac{\left(\zeta_\ell - \zeta_L\right)^2}{\zeta_L} + \frac{\left(\zeta_\ell - \zeta_S\right)^2}{\zeta_S}\right] \ . \tag{15}$$

Thus an isolated swimmer in a fluid propels itself at velocity v_0 along its axis and undergoes translational and rotational diffusion due to fluid fluctuations. Note that hydrodynamic interactions due to the noise at the head and the tail of the swimmer render the diffusion process anisotropic. The effective noise on the swimmer corresponds to that of a rigid non-spherical Brownian particle.

Next, we repeat the same procedure for N swimmers in the fluid. To make the calculation analytically tractable, we introduce two approximations. First, we neglect the hydrodynamic interaction due to the noise on different swimmers, which is well characterized in the literature of interacting Brownian particles [2]. In addition, to leading order, these interactions will scale as $\frac{k_BT}{\overline{\zeta}}\left(\frac{\ell}{r_{12}^C}\right)^2$, while those due to the active forces scales as $\frac{v_0^2}{\overline{\zeta}}\left(\frac{\ell}{r_{12}^C}\right)^2$, and hence is subdominant in the regime of large Péclet number of interest here. Secondly, we carry out a multipole expansion of the hydrodynamic couplings among swimmers. To illustrate this, we consider the flow induced on swimmer 1 by the active forces generated in the fluid by swimmer 2. The flow field at the head and tail of swimmer 1 is given by

$$u_i\left(\mathbf{r}_{L1}\right) = f\left[\mathscr{O}_{ij}\left(\mathbf{r}_{12}^C + \frac{a_S\ell}{2\overline{a}}\left(\hat{\boldsymbol{\nu}}_1 - \hat{\boldsymbol{\nu}}_2\right)\right) - \mathscr{O}_{ij}\left(\mathbf{r}_{12}^C + \frac{\ell}{2\overline{a}}\left(a_S\hat{\boldsymbol{\nu}}_1 + a_L\,\hat{\boldsymbol{\nu}}_2\right)\right)\right]\hat{\nu}_{j2}\;, \tag{16}$$

$$u_i\left(\mathbf{r}_{S1}\right) = f\left[\mathscr{O}_{ij}\left(\mathbf{r}_{12}^C - \frac{\ell}{2\overline{a}}\left(a_L\hat{\boldsymbol{\nu}}_1 + a_{\hat{S}}\boldsymbol{\nu}_2\right)\right) - \mathscr{O}_{ij}\left(\mathbf{r}_{12}^C - \frac{a_L\ell}{2\overline{a}}\left(\hat{\boldsymbol{\nu}}_1 - \hat{\boldsymbol{\nu}}_2\right)\right)\right]\hat{\nu}_{j2}\;. \tag{17}$$

We carry out a multiple expansion of the flow field given in Eqs. (16) and (17) about the separation of the hydrodynamic centers of the two particles $\mathbf{r}_{12}^C$ to order $\left(\frac{\ell}{r_{12}}\right)^4$, i.e., to octupole order. Carrying out this procedure systematically gives us $6N$ coupled Langevin equations describing the "microdynamics" of the active particles given in the paper and repeated here for completeness,

$$\partial_t\mathbf{r}_1^C = v_0\hat{\boldsymbol{\nu}}_1 + \frac{1}{\zeta}\mathbf{F}_{12} + \boldsymbol{\xi}^T(t)\;, \tag{18}$$

$$\partial_t\hat{\boldsymbol{\nu}}_1 = \boldsymbol{\omega}_1 \times \hat{\boldsymbol{\nu}}_1\;, \tag{19}$$

$$\boldsymbol{\omega}_1 = \frac{1}{\zeta\ell^2}\boldsymbol{\tau}_{12} + \boldsymbol{\xi}^R(t)\;, \tag{20}$$

where the hydrodynamic forces and torques are given by

$$\mathbf{F}_{12} = \alpha_1\frac{\hat{\mathbf{r}}_{12}}{r_{12}^2}S_{jk}\left(\hat{\mathbf{r}}_{12}\right)\hat{\nu}_{2j}\hat{\nu}_{2k} + \beta_1\frac{1}{r_{12}^3}\left[\hat{\mathbf{r}}_{12}S_{jkl}\left(\hat{\mathbf{r}}_{12}\right)\hat{\nu}_{2j}\hat{\nu}_{2k}\hat{\nu}_{2l} - \hat{\boldsymbol{\nu}}_2S_{jk}\left(\hat{\mathbf{r}}_{12}\right)\hat{\nu}_{2j}\hat{\nu}_{2k}\right] + O\left(\frac{\ell^4}{r_{12}^4}\right) \tag{21}$$

and

$$\begin{aligned}\boldsymbol{\tau}_{12} \;=\; & -\alpha_2\left(\hat{\boldsymbol{\nu}}_1 \times \hat{\mathbf{r}}_{12}\right)\frac{1}{r_{12}^3}S_{jkl}\left(\hat{\mathbf{r}}_{12}\right)\hat{\nu}_{1j}\hat{\nu}_{2k}\hat{\nu}_{2l} + \beta_2\left(\hat{\boldsymbol{\nu}}_1 \times \hat{\boldsymbol{\nu}}_2\right)\frac{1}{r_{12}^4}S_{jkl}\left(\hat{\mathbf{r}}_{12}\right)\hat{\nu}_{2j}\hat{\nu}_{2k}\hat{\nu}_{2l} \\ & -5\beta_2(\hat{\boldsymbol{\nu}}_1 \times \hat{\mathbf{r}}_{12})\frac{1}{r_{12}^4}S_{jklm}\left(\hat{\mathbf{r}}_{12}\right)\hat{\nu}_{1j}(\hat{\nu}_{1k} + \hat{\nu}_{2k})\hat{\nu}_{2l}\hat{\nu}_{2m} \\ & +\beta_3\frac{1}{r_{12}^5}\Big[c_1(\hat{\boldsymbol{\nu}}_1 \times \hat{\boldsymbol{\nu}}_2)S_{klmn}\left(\hat{\mathbf{r}}_{12}\right)\hat{\nu}_{1k}\hat{\nu}_{1l}\hat{\nu}_{2m}\hat{\nu}_{2n} \\ & +c_2(\hat{\boldsymbol{\nu}}_1 \times \hat{\mathbf{r}}_{12})S_{klmns}\left(\hat{\mathbf{r}}_{12}\right)\hat{\nu}_{1k}\hat{\nu}_{1l}\hat{\nu}_{2m}\hat{\nu}_{2n}\hat{\nu}_{2s}\Big] + O\left(\frac{\ell^6}{r_{12}^6}\right)\end{aligned} \tag{22}$$

with c_1 and c_2 numerical constants and

$$\alpha_1 = \frac{9}{4}f\overline{a}\ell \tag{23}$$

$$\beta_1 = -\frac{9}{16}f\ell^2\left(a_L - a_S\right) \tag{24}$$

$$\alpha_2 = \frac{9}{4}f\ell^3\overline{a} \tag{25}$$

$$\beta_2 = -\frac{9}{16}f\ell^4\left(a_L - a_S\right) \tag{26}$$

$$\beta_3 = \frac{9f\ell^5}{16\overline{a}}\left(a_L - a_S\right)^2 \tag{27}$$

and

$$S_{kl}(\hat{\mathbf{r}}_{12}) = \left[\hat{r}_{12k}\hat{r}_{12k} - \frac{1}{3}\delta_{kl}\right], \tag{28}$$

$$S_{ijk}\left(\hat{\mathbf{r}}\right) = 5\hat{r}_i\hat{r}_j\hat{r}_k - \left(\delta_{jk}\hat{r}_i + \delta_{ik}\hat{r}_j + \delta_{ij}\hat{r}_k\right), \tag{29}$$

$$\begin{aligned} S_{ijkl}\left(\hat{\mathbf{r}}\right) &= 7\hat{r}_i\hat{r}_j\hat{r}_k\hat{r}_l - \left(\delta_{ij}\hat{r}_k\hat{r}_l + \delta_{kl}\hat{r}_i\hat{r}_j + \delta_{ik}\hat{r}_j\hat{r}_l + \delta_{jl}\hat{r}_i\hat{r}_k + \delta_{jk}\hat{r}_i\hat{r}_l + \delta_{il}\hat{r}_j\hat{r}_k\right) \\ &+ \frac{1}{5}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) . \end{aligned} \tag{30}$$

In the above expressions the parameters α_i scale as $(a_S + a_L)$, while the parameters β_i scale as $(a_L - a_S)$. So all the β_i terms vanish for a shaker. Further, each of the tensors S is a spherically symmetric traceless tensor generated from the vector $\hat{r}$.

In order to make analytical progress, we simplify the above expression for the hydrodynamic torque as follows. First note that the term with coefficient α_2 is long ranged and is the most important contribution in the long wavelength limit. This term is treated exactly below. Next, it will be shown below that the β_1 term in the force yields convective nonlinearities $\sim \mathbf{P}\boldsymbol{\nabla}\mathbf{P}$ in the hydrodynamic equation for the polarization. The coefficients of these terms will acquire corrections from the quadrupolar part of the torque (the terms proportional to β_2). For simplicity, we neglect this quadrupolar contribution to the torque in the following analysis. Finally, we need to go to octupole order in the multipole expansion of the torque to get the contribution of the form $\sim (\hat{\boldsymbol{\nu}}_1 \times \hat{\boldsymbol{\nu}}_2)$ that leads to long-ranged build up of polar order in the system. For tractability, we replace the angular kernel resulting from the derivatives of the Oseen tensor in this term by a lower order form that still preserves the critical properties of spherical symmetry and incompressibility contained in the full kernel. Implementing these approximations, we get the following expression for the hydrodynamic torque,

$$\begin{aligned} \tau_{12i} &= -\alpha_2\left(\hat{\boldsymbol{\nu}}_1 \times \hat{\mathbf{r}}_{12}\right)_i \frac{1}{r_{12}^3} S_{jkl}\left(\hat{\mathbf{r}}_{12}\right)\hat{\nu}_{1j}\hat{\nu}_{2k}\hat{\nu}_{2l} \\ &+ \beta_3 \frac{1}{r_{12}^5}\varepsilon_{ijk}\hat{\nu}_{1j} S_{kl}\left(\hat{\mathbf{r}}_{12}\right)\hat{\nu}_{2l} \end{aligned} \tag{31}$$

This simplified form of the torque is used in the remainder of the presentation.

S1.2 Statistical Mechanics

The nonequilibrium statistical mechanics of a system of overdamped Langevin equations can be reformulated as a Smoluchowski equation for an N-particle phase space density,

$C\left(r_1^C,...r_N^C;\hat{\nu}_1,...\hat{\nu}_N;t\right)$ [1]. At low density $((\ell/r_{12}) << 1)$, we can approximate the N-particle statistical mechanics as an effective theory that captures the interactions among the particles in mean field. In this limit the dynamics is described in terms of a one-particle distribution function, $\hat{c}(\mathbf{r}_1,\hat{\boldsymbol{\nu}}_1,t)$, that satisfies a one-body Smoluchowski equation, given by

$$\partial_t\hat{c}+v_0\hat{\boldsymbol{\nu}}_1\cdot\boldsymbol{\nabla}_{\mathbf{r}_1}\hat{c} = -\frac{1}{\bar{\bar{\zeta}}}\nabla_{\mathbf{r}_1}\cdot\left(\langle\mathbf{F}_{12}\rangle\hat{c}\right)-\frac{1}{\bar{\bar{\zeta}}\ell^2}\left(\hat{\boldsymbol{\nu}}_1\times\frac{\partial}{\partial\hat{\boldsymbol{\nu}}_1}\right)\cdot\langle\boldsymbol{\tau}_{12}\rangle\hat{c} + \left(D_\parallel\partial_\parallel^2\hat{c}+D_\perp\partial_\perp^2\hat{c}\right)+D_R\left(\hat{\boldsymbol{\nu}}_1\times\frac{\partial}{\partial\hat{\boldsymbol{\nu}}_1}\right)^2\hat{c}\,, \quad (32)$$

where

$$\langle X_{12}\rangle=\int d\mathbf{r}_2\int d\hat{\boldsymbol{\nu}}_2\;X_{12}\;\hat{c}\left(\mathbf{r}_2,\hat{\boldsymbol{\nu}}_2,t\right)\;. \quad (33)$$

and $\partial_\parallel=\hat{\boldsymbol{\nu}}_1\cdot\nabla_{\mathbf{r}_1}$ and $\partial_{\perp i}=\left(\delta_{ij}-\hat{\nu}_{1i}\hat{\nu}_{1j}\right)\partial_{r_{1j}}$. The mean field force and torque are given by

$$\langle F_{12i}\rangle=\alpha_1K_i^{F_1}\left(\mathbf{r}_1,t\right)-\beta_1K_i^{F_2}\left(\mathbf{r}_1,t\right)\;, \quad (34)$$

with

$$K_i^{F_1}\left(\mathbf{r}_1,t\right)=\int d\mathbf{r}_2\frac{\hat{r}_{12i}}{r_{12}^2}S_{jk}\left(\hat{\mathbf{r}}_{12}\right)c\left(\mathbf{r}_2,t\right)Q_{jk}\left(\mathbf{r}_2,t\right)\;, \quad (35)$$

$$K_i^{F_2}\left(\mathbf{r}_1,t\right)=\frac{2}{5}\int d\mathbf{r}_2\frac{1}{r_{12}^3}S_{ij}\left(\hat{\mathbf{r}}_{12}\right)c\left(\mathbf{r}_2,t\right)P_j\left(\mathbf{r}_2,t\right)\;, \quad (36)$$

and

$$\langle\tau_{12i}\rangle=-\varepsilon_{imn}\hat{\nu}_{1m}\hat{\nu}_{1k}\alpha_2K_{nk}^{\tau_1}\left(\mathbf{r}_1,t\right)+\beta_3\varepsilon_{imn}\hat{\nu}_{1m}K_n^{\tau_2}\left(\mathbf{r}_1,t\right)\;, \quad (37)$$

where

$$K_{nk}^{\tau_1}\left(\mathbf{r}_1,t\right)=\int d\mathbf{r}_2\frac{\hat{r}_{12n}}{r_{12}^3}S_{jkl}\left(\hat{\mathbf{r}}_{12}\right)c\left(\mathbf{r}_2,t\right)Q_{jl}\left(\mathbf{r}_2,t\right)\;, \quad (38)$$

and

$$K_n^{\tau_2}\left(\mathbf{r}_1,t\right)=\int d\mathbf{r}_2\frac{1}{r_{12}^5}S_{nk}\left(\hat{\mathbf{r}}_{12}\right)c\left(\mathbf{r}_2,t\right)P_k\left(\mathbf{r}_2,t\right) \quad (39)$$

In the above expressions, we have expressed the mean field forces and torques in terms of nonlocal kernels of the hydrodynamic fields of interest namely, the density, $c(\mathbf{r},t)$, the polarization, $\mathbf{P}(r,t)$, and the nematic order parameter, $\mathbf{Q}(r,t)$, defined as moments of the one particle distribution,

$$c\left(\mathbf{r},t\right)=\int d\hat{\boldsymbol{\nu}}\;\hat{c}\left(\mathbf{r},\hat{\boldsymbol{\nu}},t\right) \quad (40)$$

$$\mathbf{P}\left(\mathbf{r},t\right)=\frac{1}{c\left(\mathbf{r},t\right)}\int d\hat{\boldsymbol{\nu}}\;\hat{\boldsymbol{\nu}}\;\hat{c}\left(\mathbf{r},\hat{\boldsymbol{\nu}},t\right) \quad (41)$$

$$\mathbf{Q}\left(\mathbf{r},t\right)=\frac{1}{c\left(\mathbf{r},t\right)}\int d\hat{\boldsymbol{\nu}}\left(\hat{\boldsymbol{\nu}}\hat{\boldsymbol{\nu}}-\frac{1}{3}\boldsymbol{\delta}\right)\hat{c}\left(\mathbf{r},\hat{\boldsymbol{\nu}},t\right). \quad (42)$$

The context of such a representation is discussed in the next section.

S1.3 Hydrodynamics

The next step is to derive hydrodynamic equations that capture the dynamics of the system at large length and time scales. On hydrodynamic scales, the dynamics of the system can be described in term of conserved quantities and broken symmetry fields. The conserved quantities of a suspension in the Stokes regime are the total density of the suspension (assumed constant in the incompressible limit of interest here), the concentration c of swimmers, and the total momentum of the suspension. Since we have eliminated the fluid from our description by solving the Stokes equation and recasting the effect of the solvent in the form of hydrodynamic interactions among the swimmers, the only relevant conserved quantity is the density of swimmers, $c(\mathbf{r}, t)$, defined in Eq.(40). In addition, there are two symmetries that can be dynamically broken. First, the rotational symmetry of the system can be broken if the active particles align to form an ordered state that is invariant under the interchange of the head and the tail of all the particles. This results in nematic ordering characterized by the nematic order parameter $\mathbf{Q}$, defined in Eq.(42). Alternately, the rotational symmetry can be broken so that the system develops polar ordering, which also corresponds to spontaneous flow of the active particles in the broken symmetry direction. This is described by the polarization vector order parameter, $\mathbf{P}$, defined in Eq.(41). In the long time regime, we can construct hydrodynamic equations for the slow variables by taking the corresponding moments of the Smoluchowski equation and assuming that the dynamics is "normal", i.e.,

$$\hat{c}(\mathbf{r}, \hat{\boldsymbol{\nu}}, t) \rightarrow \hat{c}\Big(\mathbf{r},\ \hat{\boldsymbol{\nu}} | c(\mathbf{r}, t), \mathbf{P}(\ \mathbf{r}, t), \mathbf{Q}\left(\mathbf{r}, t\right)\Big)\ . \tag{43}$$

In other words the one particle distribution depends on time only through the hydrodynamic variables. This is traditionally called the functional assumption underlying hydrodynamics [3]. There are several well developed techniques to construct such a functional. For analytical tractability, we use the simplest closure traditionally used for anisotropic fluids, namely

$$\hat{c}(\mathbf{r}, \hat{\boldsymbol{\nu}}, t) = \frac{c(\mathbf{r}, t)}{4\pi}\left[1 + 3\hat{\boldsymbol{\nu}} \cdot \mathbf{P}(\mathbf{r}, t) + \frac{15}{2}\left(\hat{\boldsymbol{\nu}}\hat{\boldsymbol{\nu}} - \frac{1}{3}\boldsymbol{\delta}\right) : \mathbf{Q}(\mathbf{r}, t)\right], \tag{44}$$

that expresses the one-particle distribution as a linear functional of the hydrodynamic variables. Using this closure, we obtain hydrodynamic equations of for each of these variables.

Before giving the explicit expression for each of these equations, it is useful to first discuss their structure. Generically these equations have the form

$$\begin{aligned}\partial_t y_\alpha &= \left[\int d\mathbf{r}' K^{(\alpha)}_{\beta\gamma}\left(\mathbf{r}-\mathbf{r}'\right) y_\beta\left(\mathbf{r}'\right)\right] y_\gamma\left(\mathbf{r}\right) + v_0 \nabla_{r_\gamma} C^{(\alpha)}_{\gamma\sigma} y_\sigma \\ &+ D_{\alpha\sigma}\nabla^2 y_\sigma - D_R\left(1-\delta_{\alpha 1}\right) y_\alpha\left(\mathbf{r}\right) \end{aligned} \tag{45}$$

where $y_\alpha \to \{c, \mathbf{P}, \mathbf{Q}\}$ are the hydrodynamic fields, $K^{(\alpha)}_{\beta\gamma} \sim \frac{1}{r^m}$, with $m \geq 2$, are nonlocal kernels that capture the effects of hydrodynamic interactions, and $C^{(\alpha)}_{\gamma\sigma}$ and $D_{\alpha\sigma}$ are parameters that govern convective couplings from self propulsion and diffusive effect due to noise, respectively. The hydrodynamic equation are nonlinear and nonlocal, due the power-law dependence of the hydrodynamic interactions on the swimmers separation. It is precisely these nonlocal terms that give rise to the peculiar nature of fluctuations in a suspension of active particles.

In detail, the hydrodynamic equations are

$$\begin{aligned}\partial_t c\left(\mathbf{r},t\right) + \quad & v_0 \nabla\cdot\left(c\mathbf{P}\left(\mathbf{r},t\right)\right) = -\nabla_{\mathbf{r}}\cdot\left[\left(\frac{\alpha_1}{\bar{\zeta}}\mathbf{K}^{F_1}\left(\mathbf{r},t\right) - \frac{\beta_1}{\bar{\zeta}}\mathbf{K}^{F_2}\left(\mathbf{r},t\right)\right) c\left(\mathbf{r},t\right)\right] \\ & + \frac{1}{3}\left(D_\parallel + 2D_\perp\right)\nabla^2_{\mathbf{r}} c\left(\mathbf{r},t\right) + (D_\parallel - D_\perp)\partial_i\partial_j c Q_{ij}\left(\mathbf{r},t\right)\end{aligned} \tag{46}$$

$$\begin{aligned}\partial_t c P_i\left(\mathbf{r},t\right) + \quad & v_0 \partial_j\left(cQ_{ij}\right) + \frac{v_0}{3}\partial_i c = -\nabla_{\mathbf{r}}\cdot\left[\left(\frac{\alpha_1}{\bar{\zeta}}\mathbf{K}^{F_1}\left(\mathbf{r},t\right) - \frac{\beta_1}{\bar{\zeta}}\mathbf{K}^{F_2}\left(\mathbf{r},t\right)\right) cP_i\left(\mathbf{r},t\right)\right] \\ & + \frac{2}{5}(D_\parallel - D_\perp)\partial_i\left(\nabla\cdot c\mathbf{P}\right) + \frac{1}{5}\left(D_\parallel + 4D_\perp\right)\nabla^2 cP_i - D_R cP_i \\ & - \frac{\alpha_2}{5\bar{\zeta}\ell^2}\left[4K^{\tau_1}_{ij}\left(\mathbf{r},t\right) - K^{\tau_1}_{ji}\left(\mathbf{r},t\right) - \delta_{ij}K^{\tau_1}_{ss}\left(\mathbf{r},t\right)\right] cP_j\left(\mathbf{r},t\right) \\ & + \frac{\beta_3}{\bar{\zeta}\ell^2}K^{\tau_2}_j\left(\mathbf{r},t\right) c\left(Q_{ij}\left(\mathbf{r},t\right) + \frac{2}{3}\delta_{ij}\right)\end{aligned} \tag{47}$$

$$\begin{aligned}\partial_t cQ_{ij} + \quad & \frac{2v_0}{5}\left[\partial_i cP_j\right]^{ST} = -\nabla_{\mathbf{r}}\cdot\left[\left(\frac{\alpha_1}{\bar{\zeta}}\mathbf{K}^{F_1}\left(\mathbf{r},t\right) - \frac{\beta_1}{\bar{\zeta}}\mathbf{K}^{F_2}\left(\mathbf{r},t\right)\right) cQ_{ij}\left(\mathbf{r},t\right)\right] \\ & + \frac{2}{15}\left(D_\parallel - D_\perp\right)\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right) c + \frac{2}{7}\left(D_\parallel - D_\perp\right)\partial_k\left[\partial_i cQ_{jk} + \partial_j cQ_{ik} - \frac{2}{3}\delta_{ij}\partial_l cQ_{kl}\right] \\ & + \frac{1}{7}\left(D_\parallel + 6D_\perp\right)\nabla^2 cQ_{ij} - 4D_R cQ_{ij} - \frac{2\alpha_2}{5\bar{\zeta}\ell^2}\left[K^{\tau_1}_{ij}\left(\mathbf{r},t\right)\right]^{ST} c - \frac{\beta_3}{\bar{\zeta}\ell^2}\left(\mathbf{K}^{\tau_2}\cdot c\mathbf{P}\right)^{ST}_{ij}\end{aligned} \tag{48}$$

where $\left[Y_{ij}\right]^{ST}$ denotes the symmetric traceless contraction of the tensor Y_{ij}, i.e., $\left[Y_{ij}\right]^{ST} = (1/2)(Y_{ij} + Y_{ji}) - (1/3)\delta_{ij}Y_{kk}$. Finally, we have neglected $\mathscr{O}(\mathbf{Q}^2)$ terms in the contribution form the nematic part of the torque to the equation for the alignment tensor. These terms have a very complicated angular dependence; at long wavelength they give convective nonlinearities quadratic in $\mathbf{Q}$ and linear in gradients in the hydrodynamic equations.

In the main body of the paper we focus on the linearized form of the hydrodynamic equations, obtained by expanding the hydrodynamic fields y_α about their homogeneous values y_α^0, with $\delta y_\alpha = y_\alpha - y_\alpha^0$. Introducing a Fourier representation $\delta \widetilde{y}_\alpha(\mathbf{k}) = \int_{\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}} \delta y_\alpha(\mathbf{r})$, the nonlocal terms take the form $\left[\widetilde{K}_{\beta\gamma}(-\mathbf{k})\,\delta\widetilde{y}_\beta(\mathbf{k})\right] y_\gamma^0$. Since we are interested in the long wavelength behavior, we expand the kernels as $\widetilde{K} \sim K^{(0)} + ikK^{(1)} + ...$ and retain terms up to quadratic order in k. This procedure, when carried out in each of the homogeneous states, yields the linearized hydrodynamic equations discussed in the paper. Finally, one can radily verify that for the long ranged part of the force and torque, the kernels K^{F_1} and K^{τ_1} go as k^0 in the wavevector and hence are solely responsible for the scale free instabilities identified in the main text of the paper.

The hydrodynamic equations linearized about the isotropic state are

$$\partial_t \delta\widetilde{c} = \overline{\alpha}_1 c_0 \delta\widetilde{Q}_{\|\|\|} + ikv_0 c_0 \delta\widetilde{P}_\| - k^2 D\delta\widetilde{c} \tag{49}$$

$$\partial_t \delta\widetilde{P}_\| = -D_R \delta P_\| + ik\frac{v_0}{3c_0}\delta\widetilde{c} - k^2 D_p \delta\widetilde{P}_\| \tag{50}$$

and

$$\partial_t \delta Q_{\|\|\|} = -4D_R \delta Q_{\|\|\|} - k^2 D_{sp}\frac{\delta\widetilde{c}}{c_0} \tag{51}$$

$$\partial_t \delta Q_{\|\perp} = -4D_R \delta Q_{\|\perp} + \overline{\alpha}_2 Q_{\|\perp} \tag{52}$$

where

$$\overline{\alpha}_1 = \frac{4\pi}{3}\frac{\alpha_1 c_0}{\zeta} = \frac{3\pi}{2} f\left(a_L + a_S\right)\ell\frac{c_0}{\zeta} \tag{53}$$

$$D = \frac{D_\| + 2D_\perp}{3} \tag{54}$$

$$D_p = \frac{2}{5}(D_\| - D_\perp) + \frac{D_\| + 4D_\perp}{5} \tag{55}$$

$$D_{sp} = \frac{4}{45}\left(D_\| - D_\perp\right) \tag{56}$$

$$\overline{\alpha}_2 = \frac{4\pi c_0}{15}\frac{2\alpha_1}{5\zeta} = \frac{9\pi c_0}{75\zeta} f\left(a_S + a_L\right)\ell \tag{57}$$

The hydrodynamic equations linearized about the an orientationally ordered state are

$$\begin{aligned}\partial_t \delta\widetilde{\mathbf{n}} - iv_0 ck_\|\delta\widetilde{\mathbf{n}} \;=\; & -\frac{\overline{\alpha}_2}{2}\left(\left[15\frac{k_\|^2}{k^2} - 12\frac{k_\|}{k} + 1\right]\delta\widetilde{c} + 2\left[15\frac{k_\|^2}{k^2} + 1\right]\left(\hat{k}_\perp \cdot \delta\widetilde{\mathbf{n}}\right)\right)\hat{\mathbf{k}}_\perp + \frac{\overline{\alpha}_2}{4}\frac{k_\|^2}{k^2}\delta\widetilde{\mathbf{n}} \\ & -D_{nc}\mathbf{k}_\perp k_\| \delta\widetilde{c} - \left(D_s - D_b\right)\mathbf{k}_\perp \cdot \delta\widetilde{\mathbf{n}} - D_b k^2 \delta\widetilde{\mathbf{n}}\end{aligned} \tag{58}$$

where

$$D_{nc} = D_s - D_b = \frac{2}{5}(D_\| - D_\perp) \tag{59}$$

and

$$D_b = \frac{D_\parallel + 4D_\perp}{5} \tag{60}$$

The stability analysis of these equations is reported in the main text of the paper.

S3. RELATIONSHIP TO PHENOMENOLOGICAL HYDRODYNAMICS

In this section, we show that the hydrodynamic equations obtained here have the same formal structure as those proposed in the literature on a pure phenomenological basis. For simplicity, we restrict ourselves to the equations for the density and polarization fields, although it is straightforward to carry out the derivation for the general case.

Phenomenological hydrodynamic equations for active suspensions are generically written in the form

$$\partial_t c + \nabla \cdot (c\mathbf{u} - \beta c\mathbf{P}) = D_1 \nabla^2 c \tag{61}$$

$$\partial_t \mathbf{P} + \mathbf{u} \cdot \nabla \mathbf{P} = -\omega_{ij} P_j + \lambda_p u_{ij} P_j + + \frac{K_1 - K_3}{\zeta} \nabla_i (\nabla \cdot \mathbf{P}) + \frac{K_3}{\zeta} \nabla^2 P_i \tag{62}$$

and

$$\eta \nabla^2 u_i - \nabla_i p - \alpha \partial_j \frac{P_i P_j}{c} + \beta' \partial_j \partial_i P_j + \beta'' \partial^2 P_i \tag{63}$$

with $\nabla \cdot \mathbf{u} = 0$, $\omega_{ij} = \frac{1}{2} (\partial_i u_j - \partial_j u_i)$ and $u_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i)$. All the coefficients in the equations are phenomenological constants.

The Stokes Eq. (63) can be formally solved (with the condition $\nabla \cdot \mathbf{u} = 0$). In the absence of external forces in an infinite system, the solution is given by

$$u_i(\mathbf{r}, t) = \int d\mathbf{r}' \mathscr{O}_{ij} (\quad \mathbf{r} - \mathbf{r}') \left[-\alpha \partial'_k \frac{P_j (\mathbf{r}', t) P_k (\mathbf{r}', t)}{c(\mathbf{r}', t)} + \beta' \partial'_k \partial'_j P_k (\mathbf{r}', t) + \beta'' \partial'^2 P_j (\mathbf{r}', t) \right] \tag{64}$$

where $\mathscr{O}_{ij} (\mathbf{r}) = \frac{1}{8\pi\eta r} (\delta_{ij} + \hat{r}_i \hat{r}_j)$ is the Oseen tensor and ∂' denotes gradients with respect to $\mathbf{r}'$. We then eliminate the flow velocity from the hydrodynamic equations by inserting Eq. (64) in Eqs. (61) and (62). This yields a set of nonlocal hydrodynamic equations that have precisely the structure of those obtained here by eliminating fluid flow from the outset in term of pairwise hydrodynamic interactions. It is then easy to verify that the linearized form of the equation is identical to that given in the main body of the paper for each homogeneous state.

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