Comparing treatments in the presence of crossing survival curves: an application to bone marrow transplantation

Web Appendix A: Variance of the Weighted Kaplan-Meier Statistic

The weighted Kaplan-Meier test under H_0 is equivalent to

$$W_{\text{WKM}}(t_0) = \int_{t_0}^{t_m} \hat{w}(t)\left[\left\{\hat{S}_1(t) - S_1(t)\right\} - \left\{\hat{S}_0(t) - S_0(t)\right\}\right] dt = W_1 - W_0,$$

where

$$W_k = \int_0^{t_m} \hat{w}(t)\left\{\hat{S}_k(t) - S_k(t)\right\} I(t > t_0) dt.$$

Suppose that $n_k/n \to p_k < 1$ for $k = 0, 1$, and that there exists a non-negative function w such that $\hat{w}$ converges in probability uniformly on $[0, t_m]$. Since $\hat{S}_k(t) - S_k(t)$ is asymptotically equivalent to $-S_k(t)\int_0^t dM_k(u)$, where M_k is a zero mean Gaussian martingale with independent increments, then W_k is asymptotically equivalent to

$$\begin{aligned}
X_k &= -\int_0^{t_m} w(t)S_k(t)I(t > t_0)\left\{\int_0^t dM_k(u)\right\} dt \\
&= -\int_0^{t_m}\left\{\int_u^{t_m} w(t)S_k(t)I(t > t_0)dt\right\} dM_k(u) \\
&= -\int_0^{t_0}\left\{\int_{t_0}^{t_m} w(t)S_k(t)dt\right\} dM_k(u) - \int_{t_0^+}^{t_m}\left\{\int_t^{t_m} w(t)S_k(t)dt\right\} dM_k(u) \\
&= -\int_0^{t_m}\left\{I(u \le t_0)\int_{t_0}^{t_m} w(t)S_k(t)dt + I(u > t_0)\int_t^{t_m} w(t)S_k(t)dt\right\} dM_k(u).
\end{aligned}$$

Then X_k has predictable variation process

$$\begin{aligned}
\langle X_k\rangle_{t_m} &= \int_0^{t_m}\left\{I(u \le t_0)\int_{t_0}^{t_m} w(t)S_k(t)dt + I(u > t_0)\int_t^{t_m} w(t)S_k(t)dt\right\}^2 d\langle M_k\rangle(u) \\
&= \left\{\int_{t_0}^{t_m} w(t)S_k(t)dt\right\}^2 \int_0^{t_0}\frac{\alpha_k(u)}{Y_k(u)}du + \int_{t_0^+}^{t_m}\left\{\int_t^{t_m} w(t)S_k(t)dt\right\}^2 \frac{\alpha_k(u)}{Y_k(u)}du.
\end{aligned}$$

The variance of X_k can be estimated by

$$\begin{aligned}
\widehat{\text{Var}}(X_k) &= \left\{\int_{t_0}^{t_m} \hat{w}(t)\hat{S}_k(t)dt\right\}^2 \sum_{j=1}^{\ell-1}\frac{d_{kj}}{Y_{kj}^2} + \sum_{j=\ell}^{m-1}\left\{\int_{t_j}^{t_m} \hat{w}(t)\hat{S}_k(t)dt\right\}^2 \frac{d_{kj}}{Y_{kj}^2} \\
&= A_{k0}^2\sum_{j=1}^{\ell-1}\frac{d_{kj}}{Y_{kj}^2} + \sum_{j=\ell}^{m-1} A_{kj}^2\frac{d_{kj}}{Y_{kj}^2}.
\end{aligned}$$

and under independent samples the variance of $W_{\text{WKM}}(t_0)$ is $\widehat{\text{Var}}_{\text{WKM}} = \sum_{k=0}^{1}\widehat{\text{Var}}(X_k)$.