Eye movement statistics in humans are consistent with an optimal search strategy

(Auxiliary Material)

Jiri Najemnik and Wilson S. Geisler
Center for Perceptual Systems and Department of Psychology, University of Texas at Austin, Austin TX 78712

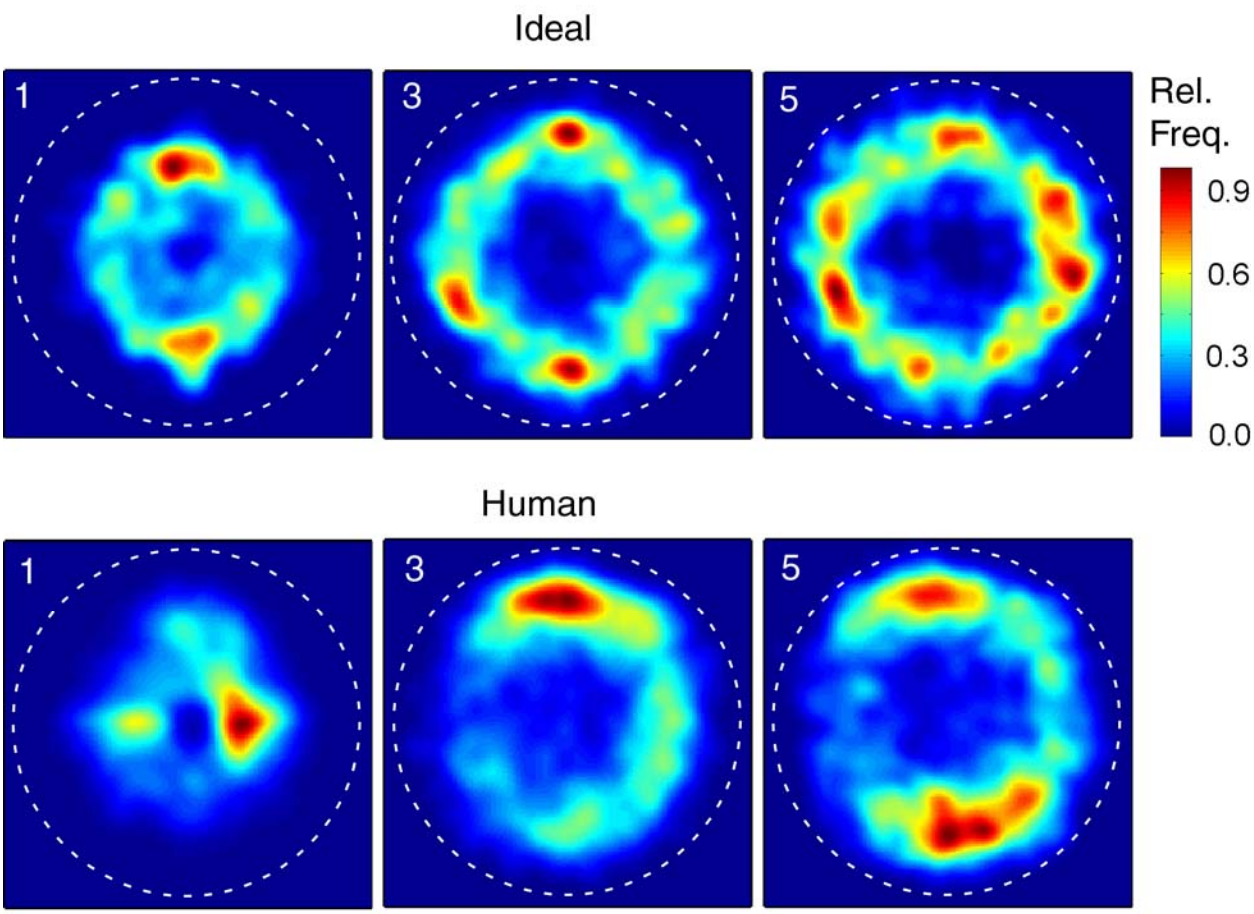

Figure A1 Average spatial distribution of fixation locations across the search area for the 1st, 3rd and 5th saccades. The color temperature indicates the relative proportion of fixations at each display location. The dashed circles (15 deg diameter) indicate the display region containing the 1/f noise texture. Humans and ideal start with relatively small saccade lengths. Humans have an initial bias for fixations along the horizontal meridian and a later bias for fixations along the vertical meridian, whereas the ideal has an initial bias for the vertical meridian and a later bias for the horizontal meridian.

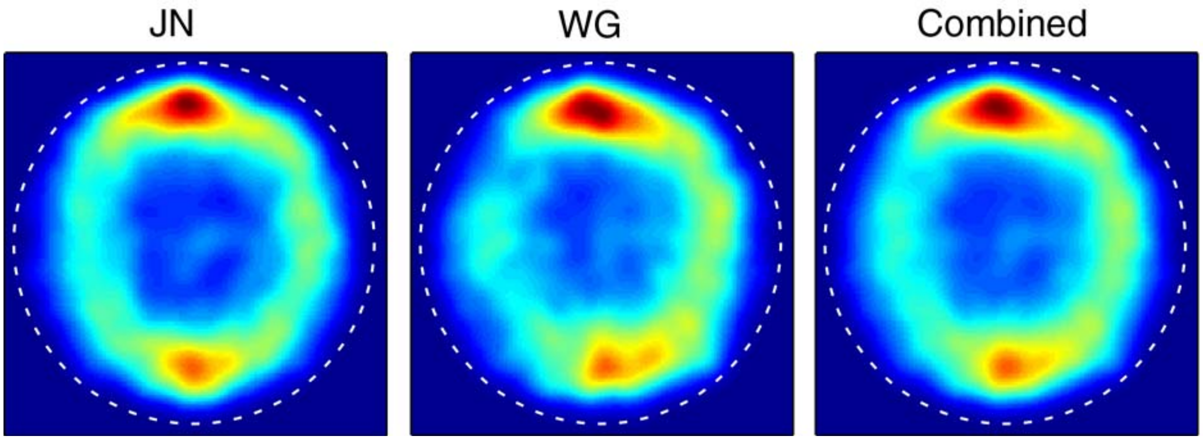

Figure A2. Average spatial distribution of fixation locations after the 2nd saccade for the human searchers. The dashed circles indicate the display region containing the 1/f noise texture.

Derivation of ideal visual searcher for static external noise

Here we derive the ideal searcher for the case of static external noise and dynamic (temporally uncorrelated) internal noise. In the supplementary material to Najemnik & Geisler (2005) we derive the ideal searcher for the easier case where both the external and the internal noise are dynamic. The formulas were derived via two separate methods: a direct derivation from the joint probability distributions (WG) and a derivation based upon the theory of Kalman filtering (JN).

Characterizing visual system

To derive the ideal searcher it is necessary to consider the meaning of the visibility map in more detail. To do this we describe the ideal detector for a known sine wave target presented at a known location in a single interval forced choice task.

First, consider an ideal detector with direct access to the retinal image; that is, an ideal detector for a non-foveated visual system. Each trial of a single interval forced choice task consists either of background noise alone or background noise plus the sine wave target. The ideal detector multiplies the retinal image with a template of the sine wave target and then integrates the product to obtain a template response W (i.e., the template response is the cross correlation of the target with the retinal stimulus). The magnitude of this template response is then compared to a criterion; the optimal behavior (to maximize detector's accuracy) is to respond "target present" if the template response exceeds the criterion and "target absent" otherwise. The accuracy of this ideal detector is determined by the signal-to-noise ratio, d', which is the average difference in the template response to the background plus target and background, divided by the standard deviation of the template response. The expected value of the template response to background plus target is proportional to the target contrast and the variance of the template response is proportional to the noise contrast power of the background, and hence

$$d'\left(c,e_n\right)^2 = \frac{Ac^2}{Be_n} \tag{A1}$$

where, c is the RMS contrast of the target, e_n is the contrast power of the noise background, and A and B are proportionality constants.

Now consider an ideal detector in a foveated visual system. In this case, the ideal detector does not have direct access to the retinal image, but instead to a representation that is degraded by variable spatial resolution and neural noise. Reduced spatial resolution due to spatial filtering will effectively reduce the contrast of a sine wave target, but will have little effect on the shape of the target, thus the same template can be used in regions of reduced resolution, although the responses to the target and background will be smaller. (We have verified that for our target and for eccentricities as large as the radius of our display, the appropriate template shape changes negligibly for transfer functions that match human contrast sensitivity functions.) Furthermore, the neural noise will add a term $C\left(\mathbf{p};c,e_n\right)$ to the variance of the template responses. In general, this neural noise term may depend on the target contrast, the background noise power, and the retinal position $\mathbf{p}=\left(x,y\right)$. Therefore, the visibility map of an ideal detector with a foveated visual system is given by

$$d'\left(\mathbf{p};c,e_n\right)^2 = \frac{A\left(\mathbf{p}\right)c^2}{B\left(\mathbf{p}\right)e_n + C\left(\mathbf{p};c,e_n\right)} \tag{A2}$$

where the proportionality factors on the template responses to the target and background now vary with retinal position. Without loss of generality this formula can be simplified by dividing numerator and denominator by $A\left(\mathbf{p}\right)$:

$$d'\left(\mathbf{p};c,e_n\right)^2 = \frac{c^2}{\alpha e_n + \beta\left(\mathbf{p};c,e_n\right)} \tag{A3}$$

where $\alpha = B\left(\mathbf{p}\right)/A\left(\mathbf{p}\right)$ and $\beta\left(\mathbf{p};c,e_n\right) = C\left(\mathbf{p};c,e_n\right)/A\left(\mathbf{p}\right)$. Because the same shaped template is used at different eccentricities, the value of α is a constant that does not change with position. Equation (A3) gives the visibility map of the ideal detector, once the values of α and β are specified for all positions.

We now define the signal-to-noise ratio for an ideal detector limited only by external noise as

$$d_E'\left(c,e_n\right)^2 = c^2/\alpha e_n \tag{A4}$$

and the signal-to-noise ratio of an ideal detector limited only by the internal inefficiencies as

$$d_I'\left(\mathbf{p};c,e_n\right)^2 = c^2/\beta\left(\mathbf{p};c,e_n\right) \tag{A5}$$

Substituting equations (A4) and (A5) into (A3) we find that the detectability of the target at all retinal locations can also be written as

$$d'(\mathbf{p};c,e_n)^2 = \frac{d'_E(c,e_n)^2\, d'_I(\mathbf{p};c,e_n)^2}{d'_E(c,e_n)^2 + d'_I(\mathbf{p};c,e_n)^2} \quad \text{(A6)}$$

Representing the search signals and noise

During a fixation interval the ideal searcher captures template responses from all n potential target locations. The template responses obtained from all n potential target locations at time step t (on t^{th} fixation) while searcher fixated the display position $\left(x_{k(t)}, y_{k(t)}\right)$ can be represented as n element indexed vector $\mathbf{W}_{k(t)} = \left(W_{1k(t)},\cdots,W_{nk(t)}\right)$, where k is an index into a list of possible fixation locations. Given all the template responses collected thus far, it then computes the posterior probability $p_i(t)$ that the target is located at each of the potential target locations, indexed by i. If the maximum of the posterior probabilities exceeds a criterion (the value of the criterion determines the error rate), then the search is stopped and the location with largest posterior probability is reported as the location of the target. If the criterion is not exceeded, then the ideal searcher determines the location to move the eyes that will maximize the probability of finding the target after the eye movement is made and the posterior probabilities computed. It then moves its eyes to that location, and the process repeats. Formally, the template response when the searcher fixates location k on t^{th} fixation is $W_{ik(t)} = \mathrm{X}_{\mathrm{i}} + \mathrm{N}_{ik(t)} + 0.5$ at the actual target location $\left(x_i, y_i\right)$, and is $W_{jk(t)} = \mathrm{X}_{\mathrm{j}} + \mathrm{N}_{jk(t)} - 0.5$ at all other locations $\left(x_j, y_j\right)$. The X_{i} are statistical independent samples from a Gaussian distribution with mean zero and variance $1/d'^2_E$, and represent the template responses to the external $1/f$ noise. These static noise samples remain the same throughout the search trial, and hence d'_E does not depend on the fixation position $k(t)$ or time t. The $N_{ik(t)}$ are independent Gaussian noise samples that are generated for each potential target location on each fixation t and added to the static noise sample. These noise samples account for observers internal sources of inefficiency, and have mean zero and variance $1/d'_I\left[j,k(t)\right]^2$. (Note that we have left it implicit that d'_I and d'_E also depend on the target contrast and background noise power, since those values are fixed in a given search trial.)

Optimal computation of posterior probabilities

The ideal searcher computes posterior probability in a way that achieves the optimal integration of template responses across fixations. To do this, the ideal searcher uses the Bayes rule:

$$p_i(T) = p\left(i\left|\mathbf{W}_{k(1)},\cdots,\mathbf{W}_{k(T)}\right.\right) = \frac{p_i(0)\,p\left(\mathbf{W}_{k(1)},\cdots,\mathbf{W}_{k(T)}\left|i\right.\right)}{\sum_{j=1}^{n} p_j(0)\,p\left(\mathbf{W}_{k(1)},\cdots,\mathbf{W}_{k(T)}\left|j\right.\right)} \tag{A7}$$

where $p_j(0)$ is the prior probability of the target being at location *j*.

To evaluate this expression, we will make use of the following formula for the definite integral of a product of *n* univariate Gaussians, which we leave to be verified by the reader:

$$\int_{-\infty}^{\infty}\prod_{k=1}^{n} g(x;\mu_k,\sigma_k)dx = \frac{\sqrt{\prod_{k=1}^{n}\frac{1}{\sigma_k^2}}}{(2\pi)^{\frac{n-1}{2}}\sqrt{\sum_{k=1}^{n}\frac{1}{\sigma_k^2}}}\exp\left[-\frac{1}{2}\left(\left(\sum_{k=1}^{n}\frac{\mu_k^2}{\sigma_k^2}\right)-\frac{\left(\sum_{k=1}^{n}\frac{\mu_k}{\sigma_k^2}\right)^2}{\sum_{k=1}^{n}\frac{1}{\sigma_k^2}}\right)\right] \tag{A8}$$

where $g(x;\mu_k,\sigma_k)$ is a Gaussian function of variable x with mean μ_k and standard deviation σ_k; that is,

$$g(x;\mu_k,\sigma_k) = \frac{1}{\sqrt{2\pi}\sigma_k}\exp\left[-\frac{1}{2}\frac{(x-\mu_k)^2}{\sigma_k^2}\right] \tag{A9}$$

We start by evaluating the likelihood portion of equation (A7). Conditioning on the external noise $\mathbf{x} = (x_1,\cdots,x_n)$, we have

$$p\left(\mathbf{W}_{k(1)},...,\mathbf{W}_{k(T)}\left|i\right.\right) = \int_{x_1}\cdots\int_{x_n} dx_1\cdots dx_n\, p\left(\mathbf{W}_{k(1)},...,\mathbf{W}_{k(T)} \mid i,\mathbf{x}\right) p(x_1,\cdots,x_n) \tag{A10}$$

Once we condition on the external noise the responses at each template location become statistically independent over time. Furthermore, we are assuming statistical independence of both the external and internal noise over space. Thus,

$$p\left(\mathbf{W}_{k(1)},...,\mathbf{W}_{k(T)}\left|i\right.\right) = \int_{x_1}\cdots\int_{x_n} dx_1\cdots dx_n \prod_{t=1}^{T} p\left(\mathbf{W}_{k(t)}\left|i,\mathbf{x}\right.\right)\prod_{q=1}^{n} p(x_q) \tag{A11}$$

$$p\left(\mathbf{W}_{k(1)},...,\mathbf{W}_{k(T)}\left|i\right.\right) = \int_{x_1}\cdots\int_{x_n} dx_1\cdots dx_n \prod_{t=1}^{T}\prod_{q=1}^{n} p\left(W_{qk(t)}\left|i,x_q\right.\right) p(x_q) \tag{A12}$$

$$p\left(\mathbf{W}_{k(1)},...,\mathbf{W}_{k(T)}\left|i\right.\right) = \prod_{q=1}^{n}\int_{x_q} p(x_q)\prod_{t=1}^{T} p\left(W_{qk(t)}\left|i,x_q\right.\right)dx_q \tag{A13}$$

Now we plug in the appropriate Gaussian distributions for $p\left(x_q\right)$ and $p\left(W_{qk(T)} \mid i, x_q\right)$. The static portion of performance-limiting noise on the template response is due to the external noise. This portion of noise is referred to as static because it remains unchanged throughout the duration of each search trial:

$$p\left(x_q\right) = g\left(x_q; 0, \frac{1}{d'_E}\right)$$

$$p\left(W_{qk(t)} \mid i, x_q\right) = g\left(W_{qk(t)}; x_q + 0.5, \frac{1}{d'_I\left[q, k(t)\right]}\right) \text{ when } q = i$$

and

$$p\left(W_{qk(t)} \mid i, x_q\right) = g\left(W_{qk(t)}; x_q - 0.5, \frac{1}{d'_I\left[q, k(t)\right]}\right) \text{ when } q \neq i$$

Thus,

$$p\left(\mathbf{W}_{k(1)}, \ldots, \mathbf{W}_{k(T)} \middle| i\right) = \prod_{q=1}^{n} \int_{-\infty}^{\infty} g\left(x_q, 0, \frac{1}{d'_E}\right) \prod_{t=1}^{T} g\left(W_{qk(t)}; x_q + 0.5\lambda_{q|i}, \frac{1}{d'_I\left[q, k(t)\right]}\right) dx_q \qquad \text{(A14)}$$

where $\lambda_{q|i} = 1$ when $q = i$ and $\lambda_{q|i} = -1$ when $q \neq i$. We can now use formula (A8) to obtain

$$p\left(\mathbf{W}_{k(1)}, \ldots, \mathbf{W}_{k(T)} \middle| i\right) = \prod_{q=1}^{n} \frac{\sqrt{d_E'^2 \prod_{t=1}^{T} d'_I\left[q, k(t)\right]^2}}{\left(2\pi\right)^{\frac{T}{2}} \sqrt{B_{qk(T)}}} \exp\left[A_{q|ik(T)}\right] \qquad \text{(A15)}$$

where $B_{qk(T)}$ is defined as

$$B_{qk(T)} = d_E'^2 + \sum_{t=1}^{T} d'_I\left[q, k(t)\right]^2 \qquad \text{(A16)}$$

and

$$A_{q|i}\left(T\right) = -\frac{1}{2} \frac{B_{qk(T)}\left(\sum_{t=1}^{T} d'_I\left[q, k(t)\right]^2 \left(W_{qk(t)} - 0.5\lambda_{q|i}\right)^2\right) - \left(\sum_{t=1}^{T} d'_I\left[q, k(t)\right]^2 \left(W_{qk(t)} - 0.5\lambda_{q|i}\right)\right)^2}{B_{qk(T)}} \qquad \text{(A17)}$$

Substituting equation (A15) into equation (A7), the posterior probability at location i is given by

$$p_i(T) = \frac{p_i(0)\prod_{q=1}^{n}\left(\frac{\sqrt{d_E'^2\prod_{t=1}^{T} d_I'\left[q,k(t)\right]^2}}{(2\pi)^{\frac{T}{2}}\sqrt{B_{qk(T)}}}\exp\left[A_{q|ik(T)}\right]\right)}{\sum_{j=1}^{n} p_j(0)\prod_{q=1}^{n}\left(\frac{\sqrt{d_E'^2\prod_{t=1}^{T} d_I'\left[q,k(t)\right]^2}}{(2\pi)^{\frac{T}{2}}\sqrt{B_{qk(T)}}}\exp\left[A_{q|jk(T)}\right]\right)} \tag{A18}$$

We can simplify by dividing out the multiplying constants in the top and bottom to obtain

$$p_i(T) = \frac{p_i(0)\prod_{q=1}^{n}\exp\left[A_{q|ik(T)}\right]}{\sum_{j=1}^{n} p_j(0)\prod_{q=1}^{n}\exp\left[A_{q|jk(T)}\right]} \tag{A19}$$

Dividing the top and bottom of this equation by the top gives,

$$p_i(T) = \frac{1}{\sum_{j=1}^{n}\frac{p_j(0)}{p_i(0)}\exp\left[\sum_{q=1}^{n}\left(A_{q|jk(T)} - A_{q|ik(T)}\right)\right]}$$

When $q \neq i$ and $q \neq j$ then $A_{q|jk(T)} = A_{q|ik(T)}$, and thus,

$$p_i(T) = \frac{1}{\sum_{j=1}^{n}\frac{p_j(0)}{p_i(0)}\exp\left[\left(A_{j|jk(T)} - A_{j|ik(T)}\right) - \left(A_{i|ik(T)} - A_{i|jk(T)}\right)\right]} \tag{A20}$$

Now, we simplify the quantities inside the exponential:

$$\left(A_{i|ik(T)} - A_{i|jk(T)}\right) =$$

$$-\frac{1}{2}\left(\sum_{t=1}^{T} d'_I\left[i,k(t)\right]^2 \left(W_{ik(t)} - 0.5\right)^2\right) + \frac{1}{2}\frac{\left(\sum_{t=1}^{T} d'_I\left[i,k(t)\right]^2 \left(W_{ik(t)} - 0.5\right)\right)^2}{B_{ik(T)}} \tag{A21}$$

$$+\frac{1}{2}\left(\sum_{t=1}^{T} d'_I\left[i,k(t)\right]^2 \left(W_{ik(t)} + 0.5\right)^2\right) - \frac{1}{2}\frac{\left(\sum_{t=1}^{T} d'_I\left[i,k(t)\right]^2 \left(W_{ik(t)} + 0.5\right)\right)^2}{B_{ik(T)}}$$

$$= \sum_{t=1}^{T} d'_I\left[i,k(t)\right]^2 W_{ik(t)} - \frac{\sum_{t=1}^{T} d'_I\left[i,k(t)\right]^2 \sum_{t=1}^{T} d'_I\left[i,k(t)\right]^2 W_{ik(t)}}{B_{ik(T)}}$$

Substituting for $B_{ik(T)}$ using equation (A16) gives,

$$\left(A_{i|ik(T)} - A_{i|jk(T)}\right) = \frac{d_E'^2\left(\sum_{t=1}^{T} d'_I\left[i,k(t)\right]^2 W_{ik(t)}\right)}{d_E'^2 + \sum_{t=1}^{T} d'_I\left[i,k(t)\right]^2} \tag{A22}$$

Finally, substituting equation (A22) (and the equivalent equation for location *j*) into equation (A20) gives:

$$p_i(T) = \frac{p_i(0)\exp\left[\sum_{t=1}^{T} g_T\left[i,k(t)\right] W_{ik(t)}\right]}{\sum_{j=1}^{n} p_j(0)\exp\left[\sum_{t=1}^{T} g_T\left[j,k(t)\right] W_{jk(t)}\right]} \tag{A23}$$

where,

$$g_T\left[i,k(t)\right] = \frac{d_E'^2 d'_I\left[i,k(t)\right]^2}{d_E'^2 + \sum_{x=1}^{T} d'_I\left[i,k(x)\right]^2} \tag{A24}$$

This formula is identical to the one given (but not derived) in the supplementary materials for Najemnik & Geisler (2005); however, here it is expressed in terms of the external and internal d-primes.

Optimal selection of the next location to fixate

To compute the optimal next fixation point, $k_{opt}(T+1)$, the ideal searcher considers each possible next fixation and picks the location that, given its knowledge of the current posterior probabilities and the visibility map, will maximize the probability of correctly identifying the location of the target after the next fixation is made and the posterior probabilities computed:

$$k_{opt}(T+1) = \underset{k(T+1)}{\arg\max}\left\{ p\left[C \middle| k(T+1)\right]\right\}$$

Conditioning on the target location gives the following equation:

$$k_{opt}(T+1) = \underset{k(T+1)}{\arg\max}\left\{ \sum_{i=1}^{n} p_i(T)\, p\left[C \middle| i, k(T+1)\right]\right\} \quad \text{(A25)}$$

Here we derive a version of this equation that is practical to evaluate in computer simulations.

The posterior probability of each possible target location is given by equations (A23) and (A24), thus our job is to derive an expression for $p\left(C \middle| i, k(T+1)\right)$, the probability of being correct given that the true target location is i and the location of the next fixation is $k(T+1)$.

After making the next fixation, the decision rule that would maximize accuracy would be to pick the location with the maximum posterior probability. If one uses that decision rule, then the percent correct is equal to the probability that the posterior probability at location i (now regarded as a random variable) will be greater than that at all other locations:

$$p\left[C \middle| i, k(T+1)\right] = p\left[P_i(T+1) > P_1(T+1), \cdots, P_i(T+1) > P_n(T+1) \middle| i, k(T+1)\right]$$

or equivalently,

$$p\left[C \middle| i, k(T+1)\right] = p\left[Z_i(T+1) > Z_1(T+1), \cdots, Z_i(T+1) > Z_n(T+1) \middle| i, k(T+1)\right] \quad \text{(A26)}$$

where,

$$Z_j(T+1) = p_j(0)\exp\left(\sum_{t=1}^{T+1} g_{T+1}\left[j, k(t)\right] W_{jk(t)} \right) \quad \text{(A27)}$$

To evaluate equation (A26) we condition on the value of the template response observed at the target location. Note that once we condition on the template response, the events $Z_i(T+1) > Z_j(T+1)$ become statistically independent. Thus,

$$p\left(C\middle|i,k(T+1)\right) = \int_{-\infty}^{\infty}\left[\prod_{j\neq i} p\left(Z_i(T+1) > Z_j(T+1)\middle|W_{ik(T+1)} = z\right)\right] p\left[z\middle|i,k(T+1)\right]dz \tag{A28}$$

Consider an arbitrary term in the product:

$$\begin{aligned} & p\left(Z_i(T+1) > Z_j(T+1)\middle|W_{ik(T+1)} = z\right) \\ &= p\left[p_j(0)\exp\left(\sum_{t=1}^{T+1} g_{T+1}\left[j,k(t)\right]W_{jk(t)}\right) < p_i(0)\exp\left(\sum_{t=1}^{T+1} g_{T+1}\left[i,k(t)\right]W_{ik(t)}\right)\right] \\ &= p\left[\sum_{t=1}^{T+1} g_{T+1}\left[j,k(t)\right]W_{jk(t)} < \ln\left(\frac{p_i(0)}{p_j(0)}\right) + \sum_{t=1}^{T+1} g_{T+1}\left[i,k(t)\right]W_{ik(t)}\right] \end{aligned} \tag{A29}$$

Now define $h_{ij}(T+1)$ to be the quantity:

$$h_{ij}(T+1) = \ln\left(\frac{p_i(0)}{p_j(0)}\right) + \sum_{t=1}^{T}\left\{g_{T+1}\left[i,k(t)\right]W_{ik(t)} - g_{T+1}\left[j,k(t)\right]W_{jk(t)}\right\} \tag{A30}$$

Then,

$$\begin{aligned} & p\left(Z_i(T+1) > Z_j(T+1)\middle|W_{ik(T+1)} = z\right) \\ &= p\left(g_{T+1}\left[j,k(T+1)\right]W_{jk(T+1)} < g_{T+1}\left[i,k(T+1)\right]z + h_{ij}(T+1)\right) \\ &= p\left(W_{jk(T+1)} < \frac{g_{T+1}\left[i,k(T+1)\right]z + h_{ij}(T+1)}{g_{T+1}\left[j,k(T+1)\right]}\right) \end{aligned} \tag{A31}$$

The probability distribution of $W_{jk(T+1)}$ is Gaussian with some mean and standard deviation: $u_{jk(T+1)}$ and $\sigma_{jk(T+1)}$ which we will derive below. However, we can now substitute into equation (A28):

$$p\left(C\middle|i,k(T+1)\right)$$
$$=\int_{-\infty}^{\infty}\frac{1}{\sigma_{ik(T+1)}}\phi\left(\frac{z-u_{ik(T+1)}}{\sigma_{ik(T+1)}}\right)\prod_{j\neq i}\Phi\left(\frac{g_{T+1}\left[i,k(T+1)\right]z+h_{ij}(T+1)}{g_{T+1}\left[j,k(T+1)\right]\sigma_{jk(T+1)}}-\frac{\mu_{jk(T+1)}}{\sigma_{jk(T+1)}}\right)dz$$

where $\phi(x)$ is the standard normal density function and $\Phi(x)$ is the standard normal integral function: $\phi(x)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)$, $\Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y^2\right)dy$.

Letting $w=\frac{z-u_{ik(T+1)}}{\sigma_{ik(T+1)}}$ we have

$$p\left(C\middle|i,k(T+1)\right)$$
$$=\int_{-\infty}^{\infty}\phi(w)\prod_{j\neq i}\Phi\left(\frac{g_{T+1}\left[i,k(T+1)\right]\left[\sigma_{ik(T+1)}w+u_{ik(T+1)}\right]+h_{ij}(T+1)}{g_{T+1}\left[j,k(T+1)\right]\sigma_{jk(T+1)}}-\frac{u_{jk(T+1)}}{\sigma_{jk(T+1)}}\right)dw \qquad \text{(A32)}$$

Equation (A32) is the computational formula used in equation (A25) for the selecting the optimal next fixation. Although this formula contains an integral it can be evaluated rapidly using numerical integration, because the standard normal density function approaches zero rapidly away from the origin.

The last step is to determine the values of $u_{jk(T+1)}$ and $\sigma_{jk(T+1)}$. The probability distribution of $W_{jk(T+1)}$ depends upon the prior history of template responses and fixations and the location of the target.

$$p\left(W_{jk(T+1)}=z\middle|W_{jk(1)},\cdots,W_{jk(T)},i\right)=\frac{p\left(W_{jk(1)},\cdots,W_{jk(T)},W_{jk(T+1)}=z\middle|i\right)}{p\left(W_{jk(1)},\cdots,W_{jk(T)}\middle|i\right)} \qquad \text{(A33)}$$

It follows from equation (A15) that

$$p\left(W_{jk(T+1)}=z\left|W_{jk(1)},\cdots,W_{jk(T)},i\right.\right)=\frac{\dfrac{\sqrt{d_E'^2\prod_{t=1}^{T+1}d_I'\left[j,k(t)\right]^2}}{(2\pi)^{\frac{T+1}{2}}\sqrt{B_{jk(T+1)}}}\exp\left[A_{j|ik(T+1)}\right]}{\dfrac{\sqrt{d_E'^2\prod_{t=1}^{T}d_I'\left[j,k(t)\right]^2}}{(2\pi)^{\frac{T}{2}}\sqrt{B_{jk(T)}}}\exp\left[A_{j|ik(T)}\right]} \tag{A34}$$

$$p\left(W_{jk(T+1)}=z\left|W_{jk(1)},\cdots,W_{jk(T)},i\right.\right)=\frac{d_I'\left[j,k(T+1)\right]\sqrt{B_{jk(T)}}}{(2\pi)^{\frac{1}{2}}\sqrt{B_{jk(T+1)}}}\exp\left[A_{j|ik(T+1)}-A_{j|ik(T)}\right] \tag{A35}$$

We now evaluate the term inside the exponent

$$A_{j|ik(T+1)}-A_{j|ik(T)}=$$

$$-\frac{1}{2}\left(\sum_{t=1}^{T+1}d_I'\left[j,k(t)\right]^2\left(W_{jk(t)}-0.5\lambda_{j|i}\right)^2\right)+\frac{1}{2}\frac{\left(\sum_{t=1}^{T+1}d_I'\left[j,k(t)\right]^2\left(W_{jk(t)}-0.5\lambda_{j|i}\right)\right)^2}{B_{jk(T+1)}}$$
$$+\frac{1}{2}\left(\sum_{t=1}^{T}d_I'\left[j,k(t)\right]^2\left(W_{jk(t)}-0.5\lambda_{j|i}\right)^2\right)-\frac{1}{2}\frac{\left(\sum_{t=1}^{T}d_I'\left[j,k(t)\right]^2\left(W_{jk(t)}-0.5\lambda_{j|i}\right)\right)^2}{B_{jk(T)}} \tag{A36}$$

To simplify our notation we define the following quantities:

$$V_{j|i}(t)=d_I'\left[j,k(t)\right]^2\left(W_{jk(t)}-0.5\lambda_{j|i}\right) \tag{A37}$$

$$S_{j|i}(T)=\sum_{t=1}^{T}d_I'\left[j,k(t)\right]^2\left(W_{jk(t)}-0.5\lambda_{j|i}\right)=\sum_{t=1}^{T}V_{j|i}(t) \tag{A38}$$

$$A_{i|ik(T+1)}-A_{i|ik(T)}=$$

$$=-\frac{1}{2}\left[\sum_{t=1}^{T+1}d_I'\left[j,k(t)\right]^2\left(W_{jk(t)}-0.5\lambda_{j|i}\right)^2-\frac{S_{j|i}(T+1)^2}{B_{jk(T+1)}}-\sum_{t=1}^{T}d_I'\left[j,k(t)\right]^2\left(W_{jk(t)}-0.5\lambda_{j|i}\right)^2+\frac{S_{j|i}(T)^2}{B_{jk(T)}}\right]$$

$$= -\frac{1}{2}\left[d_I'\left[j,k\left(T+1\right)\right]^2\left(W_{jk(T+1)} - 0.5\lambda_{j|i}\right)^2 - \frac{S_{j|i}\left(T+1\right)^2}{B_{jk(T+1)}} + \frac{S_{j|i}\left(T\right)^2}{B_{jk(T)}}\right]$$

$$= -\frac{1}{2}\left[\frac{V_{j|i}\left(T+1\right)^2}{d_I'\left[j,k\left(T+1\right)\right]^2} - \frac{V_{j|i}\left(T+1\right)^2 + 2V_{j|i}\left(T+1\right)S_{j|i}\left(T\right) + S_{j|i}\left(T\right)^2}{B_{jk(T+1)}} + \frac{S_{j|i}\left(T\right)^2}{B_{jk(T)}}\right]$$

$$= -\frac{1}{2}\left[\frac{B_{jk(T+1)} - d_I'\left[j,k\left(T+1\right)\right]^2}{d_I'\left[j,k\left(T+1\right)\right]^2 B_{jk(T+1)}} V_{j|i}\left(T+1\right)^2 - \frac{2S_{j|i}\left(T\right)}{B_{jk(T+1)}} V_{j|i}\left(T+1\right) + \frac{B_{jk(T+1)} - B_{jk(T)}}{B_{jk(T)}B_{jk(T+1)}} S_{j|i}\left(T\right)^2 \right]$$

$$= -\frac{1}{2}\left[\frac{B_{jk(T)}}{d_I'\left[j,k\left(T+1\right)\right]^2 B_{jk(T+1)}} V_{j|i}\left(T+1\right)^2 - \frac{2S_{j|i}\left(T\right)}{B_{jk(T+1)}} V_{q|i}\left(T+1\right) + \frac{d_I'\left[j,k\left(T+1\right)\right]^2}{B_{jk(T)}B_{jk(T+1)}} S_{j|i}\left(T\right)^2 \right]$$

$$= -\frac{1}{2}\left[\sqrt{\frac{B_{jk(T)}}{d_I'\left[j,k\left(T+1\right)\right]^2 B_{jk(T+1)}}} V_{j|i}\left(T+1\right) - \sqrt{\frac{d_I'\left[j,k\left(T+1\right)\right]^2}{B_{jk(T)}B_{jk(T+1)}}} S_{j|i}\left(T\right) \right]^2$$

$$= -\frac{1}{2}\left[\sqrt{\frac{B_{jk(T)}}{d_I'\left[j,k\left(T+1\right)\right]^2 B_{jk(T+1)}}} \left(d_I'\left[j,k\left(T+1\right)\right]^2\left(W_{jk(T+1)} - 0.5\lambda_{j|i}\right) - \frac{d_I'\left[j,k\left(T+1\right)\right]^2}{B_{jk(T)}} S_{j|i}\left(T\right)\right)\right]^2$$

$$= -\frac{1}{2}\frac{d_I'\left[j,k\left(T+1\right)\right]^2 B_{jk(T)}}{B_{jk(T+1)}}\left[W_{jk(T+1)} - 0.5\lambda_{j|i} - \frac{S_{j|i}\left(T\right)}{B_{jk(T)}}\right]^2 \tag{A39}$$

Substituting into equation (A35):

$$p\left(W_{jk(T+1)} = z \middle| W_{jk(1)}, \cdots, W_{jk(T)}, i\right) =$$

$$\frac{d_I'\left[j,k\left(T+1\right)\right]\sqrt{B_{jk(T)}}}{\left(2\pi\right)^{\frac{1}{2}}\sqrt{B_{jk(T+1)}}} \exp\left[-\frac{1}{2} \frac{\left[z - 0.5\lambda_{j|i} - \frac{S_{j|i}\left(T\right)}{B_{jk(T)}}\right]^2}{\frac{B_{jk(T+1)}}{d_I'\left[j,k\left(T+1\right)\right]^2 B_{jk(T)}}}\right] \tag{A40}$$

Thus,

$$u_{jk(T+1)} = \frac{\sum_{t=1}^{T} d_I'\left[j,k(t)\right]^2 \left(W_{jk(t)} - 0.5\right)}{d_E'^{\,2} + \sum_{t=1}^{T} d_I'\left[j,k(t)\right]^2} + 0.5 \quad j = i \tag{A41}$$

$$u_{jk(T+1)} = \frac{\sum_{t=1}^{T} d_I'\left[j,k(t)\right]^2 \left(W_{jk(t)} + 0.5\right)}{d_E'^{\,2} + \sum_{t=1}^{T} d_I'\left[j,k(t)\right]^2} - 0.5 \quad j \neq i \tag{A42}$$

$$\sigma_{jk(T+1)} = \sqrt{\frac{d_E'^{\,2} + \sum_{t=1}^{T+1} d_I'\left[j,k(t)\right]^2}{d_I'\left[j,k(T+1)\right]^2 \left(d_E'^{\,2} + \sum_{t=1}^{T} d_I'\left[j,k(t)\right]^2\right)}} \tag{A43}$$

In summary, to select the optimal next fixation location we combine equations (A25), (A30), (A32), (A41), (A42) and (A43).

We note that there are typos in equations (S15) and (S16) in the supplementary material of Najemnik & Geisler (2005); the correct equations are (A41) and (A42) above. All predictions reported in that study and in the current study were obtained using the correct equations.