Web supplementary materials for

"Semiparametric regression in size-biased sampling" by Chen

1. An illustrative example of invariance property

To better understand the invariance property, consider the example when ξ are standard Exponential, $\beta = -\log 2$, and $Z = 0/1$ in the model

$$-\log X = \beta Z + \varepsilon. \tag{1}$$

In Figure 1, we plot $f(x \mid Z)$ and $F(x \mid Z)$, and their respective size-biased $f(x \mid Z, S = 1)$ and $F(x \mid Z, S = 1)$.

As shown in the figures, the size-biased $f(x \mid Z, S = 1)$ appears very different from $f(x \mid Z)$. But the shapes of $F(x \mid Z)$ and $F(x \mid Z, S = 1)$ remain similar, except for an apparent scale-change along the horizontal axis. This reflects the fact that the size-biased sampling can alter individual distribution functions, but not the relative comparison measured by the regression parameter β in model (1).

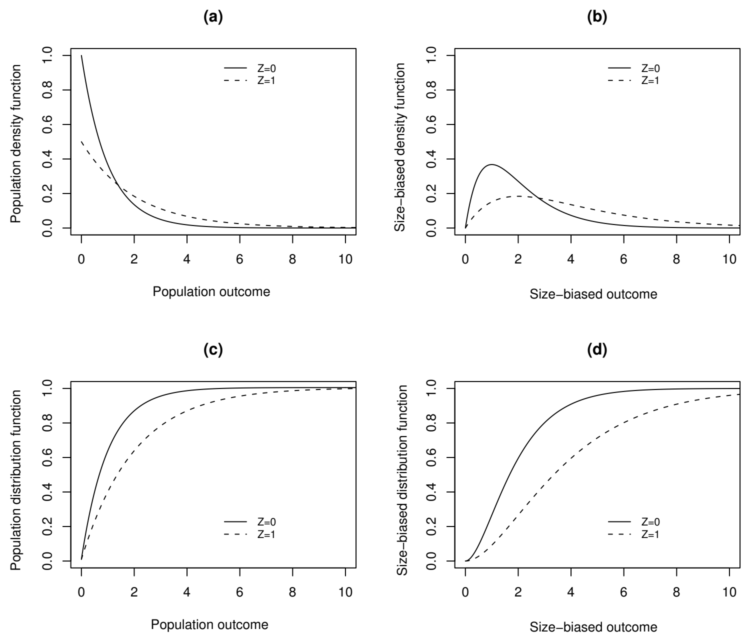
2. A variance calculation algorithm

In actual numerical studies, a less computer-intensive sample-based method can be used to directly estimate the variance of $n^{1/2}(\widehat{\beta}_n - \beta_0)$, as recommended in Kalbfleisch and Prentice (2002, p. 238). That is, first use a recursive bisection algorithm to solve for b_j in

$$n^{1/2} U_n(b_j) = v_j,$$

$j = 1, 2, \ldots, p$, where v_j are the p-vectors such that $v = (v_1, v_2, \ldots, v_p)^{\mathrm{T}}$ and $v^{\otimes 2} = \widehat{V}$. Then a consistent variance estimator of $n^{1/2}(\widehat{\beta}_n - \beta_0)$ is given by $(b_1 - \widehat{\beta}_n, b_2 - \widehat{\beta}_n, \ldots, b_p - \widehat{\beta}_n)^{\otimes 2}$. Our finite-sample simulation shows that this method generally performs well. Details on more justification of this method can be found in Chen and Jewell (2000).

Figure 1: Plots of density and distribution functions under model (1) of $\log X = -\beta^{\mathrm{T}} Z + \varepsilon$, where $\beta = -\log 2$, $Z = 0/1$ and $\xi = \exp(\varepsilon) \sim$ Exponential(1): (a) density functions $f(x \mid Z)$; (b) size-biased density functions $f(x \mid Z, S = 1)$; (c) distribution functions $F(x \mid Z)$; (d) size-biased distribution functions $F(x \mid Z, S = 1)$.



3. Proof of Theorem 2

We assume the following regularity conditions similar to those in Ying (1993):

1. The covariates are uniformly bounded.

2. The parameter space $\mathscr{B}$ of β is compact.

3. The density function of $f_\eta(\cdot)$ and its derivative $f'_\eta(\cdot)$ are bounded, and satisfying that

$$\int_0^y \left\{ \frac{f'_\eta(y)}{f_\eta(y)} \right\}^2 f_\eta(y) dy < \infty.$$

Denote $\mathscr{R}_n(y) = \mathscr{E}^{(2)}(y) - \mathscr{E}^{(1)}(y)^{\otimes 2}/\mathscr{E}^{(0)}(y)$. Let $D_n = \int_0^\infty \lambda'_\eta(y)/\lambda_\eta(y)\mathscr{R}_n(y)f_\eta(y)dy$, and $V_n = \int_0^\infty \mathscr{R}_n(y)f_\eta(y)dy$. Since

$$\mathscr{R}_n(y) = n^{-1}\sum_{i=1}^n \left\{ Z_i - \frac{\sum_j Z_j \overline{F}_j(ye^{-\beta^\mathrm{T} Z_j})}{\sum_j Z_j \overline{F}_j(ye^{-\beta^\mathrm{T} Z_j})} \right\}^{\otimes 2} \overline{F}_i(ye^{-\beta^\mathrm{T} Z_i}) \geq 0,$$

where $\overline{F}_i(y) = 1 - F_i(y) = 1 - F(y \mid Z_i)$, we know that $\mathscr{R}_n(y)$ is nonnegative-definite. In addition, for $y_1 \leq y_2$,

$$\begin{aligned}\mathscr{R}_n(y_1) &\geq n^{-1}\sum_{i=1}^n \left\{ Z_i - \frac{\sum_j Z_j \overline{F}_j(y_1 e^{-\beta^\mathrm{T} Z_j})}{\sum_j Z_j \overline{F}_j(y_1 e^{-\beta^\mathrm{T} Z_j})} \right\}^{\otimes 2} \overline{F}_i(y_2 e^{-\beta^\mathrm{T} Z_i}) \\ &\geq n^{-1}\sum_{i=1}^n \left\{ Z_i - \frac{\sum_j Z_j \overline{F}_j(y_2 e^{-\beta^\mathrm{T} Z_j})}{\sum_j Z_j \overline{F}_j(y_2 e^{-\beta^\mathrm{T} Z_j})} \right\}^{\otimes 2} \overline{F}_i(y_2 e^{-\beta^\mathrm{T} Z_i}) = \mathscr{R}_n(y_2).\end{aligned}$$

We hence know that $\mathscr{R}_n(y)$ is non-increasing. Since

$$D_n = \int_0^\infty \mathscr{R}_n(y) \left\{ f'_\eta(y)) + \frac{f_\eta^2(y)}{\overline{F}_\eta(y)} \right\} dy = \int_0^\infty \mathscr{R}_n(y) \frac{f^2(y)}{\overline{F}_\eta(y)} dy + \int_0^\infty f(y) d\left\{ -\mathscr{R}_n(y) \right\},$$

D_n are also nonnegative-definite. By Theorem 1 in Ying (1993), $U_n(\beta)$ is thus asymptotically linear such that

$$\sup_{\|\beta - \beta_0\| \leq d_n} \left\{ \frac{\|U_n(\beta) - U_n(\beta_0) - D_n n(\beta - \beta_0)\|}{n^{1/2} + n\|\beta - \beta_0\|} \right\} = o_p(1),$$

as $d_n > 0$ and $d_n \to_p 0$.

Assume that all the eigenvalues of D_n are bounded away from zero for sufficiently large n. Then the eigenvalues of V_n would be eventually bounded away from zero as well. According to Corollary 1 in Ying (1993), there exists a closed neighborhood containing β_0 as an interior point such that $\widehat{\beta}_n$ is strongly consistent, and $n^{1/2}V_n^{-1/2}D_n(\widehat{\beta}_n - \beta_0) \to_{\mathscr{D}} N(0, I_{p\times p})$. In addition, since $\lim_{n\to\infty} \mathscr{E}^{(2)}(y;\beta) = e^{(k)}(y;\beta)$, $k = 0, 1, 2$, we further have $D_n \to_p D$ and $V_n \to_p V$. As a result, we have

$$n^{1/2}(\widehat{\beta}_n - \beta_0) \to_{\mathscr{D}} \mathscr{N}\{0, D^{-1}V(D^{-1})^\mathrm{T}\}$$

as the asymptotic properties stated in Theorem 2.

For the weighted estimating equations $U_n^w(\beta)$, in addition to the regularity conditions for Theorem 2, we further assume that $w_n(y;\beta)$ are left-continuous in y and $\sigma\{N_i(ue^{-\beta_0^{\mathrm{T}} Z_i}), Z_i, u \leq y, i = 1, 2, \ldots, n\}$−measurable, as well as

1. $\limsup_n \sup_{\|\beta\| \leq b} \{|w_n(0;\beta)| + \int_0^\infty |dw_n(y;\beta)|\} < \infty$ a.s., for any $b > 0$.

2. $\sup_{y \leq y_0, \|\beta-\beta_0\| \leq n^{-1/3}} |w_n(y;\beta) - w_n(y;\beta_0) - \delta_w(y;\beta_0)^{\mathrm{T}}(\beta - \beta_0)| = o(n^{-1/2})$ a.s., for some $y_0 > 0$, $\delta_0 > 0$ and $\delta_w(y;\beta)$ such that $\int_0^{y_0} |d\delta_w(y)| = o(n^{1/3-\delta_0})$.

Thus, the weighted estimating functions are also asymptotically linear such that

$$\sup_{\|\beta-\beta_0\| \leq d_n} \left\{ \frac{\|U_n^w(\beta) - U_n^w(\beta_0) - D_{w,n} n(\beta - \beta_0)\|}{n^{1/2} + n\|\beta - \beta_0\|} \right\} = o_p(1)$$

as $d_n \to_p 0$, where $D_{w,n} = \int_0^\infty w(y;\beta) dD_n(y;\beta) \to D_w$. Assume that all the eigenvalues of $D_{w,n}$ are bounded away from 0. Then according to Corollary 2 in Ying (1993), $\widehat{\beta}_n^w$ is strongly consistent, and

$$n^{1/2}(\widehat{\beta}_n^w - \widehat{\beta}_0) \to_{\mathscr{D}} \mathscr{N}\{0, D_w^{-1} V_w (D_w^{-1})^{\mathrm{T}}\}.$$

It is straightforward to verify that the Gehan-type of weight functions satisfy the above-mentioned regularity conditions.

References

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