Supporting Information

M. E. Cates [*], D. Marenduzzo [*] , I. Pagonabarraga [†], J. Tailleur [*]

[*]SUPA, School of Physics and Astronomy, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, UK, and [†]Departament de Física Fonamental, Universitat de Barcelona - Carrer Martí Franqués 1, 08028-Barcelona, Spain

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We show here how to derive the amplitude equation [**8**] in main text. Let us start from the dimensionless equation of motion (Eq. [**6**] in main text)

$$\dot{u} = \nabla[Re^{-2\Phi u}(1-\Phi u)\nabla u] + u(1-u) - \nabla^4 u \qquad [\mathbf{1}]$$

and recall the two conditions for patterning (Eq. [**7**] in main text):

$$\Phi > 1; \qquad R\exp(-2\Phi)(\Phi-1) > 2 \qquad [\mathbf{2}]$$

To analyse precisely the transition, we derive below the steady-state limit of the amplitude equation in 1D. By inspection one sees that the unperturbed steady-state of [**1**] is given by $u = 1$. To characterise the amplitude of the perturbation around $u = 1$, we introduce $u = 1 + w/\Phi$ so that w evolves with

$$\dot{w} = -\partial_x[Re^{-2\Phi}(\Phi-1)(1+\frac{w}{\Phi-1})e^{-2w}\partial_x w] - w(1+\frac{w}{\Phi}) - \partial_x^4 w \qquad [\mathbf{3}]$$

We are interested by the vicinity of the transition where

$$Re^{-2\Phi}(\Phi-1) = 2(1+\epsilon) \qquad [\mathbf{4}]$$

for $\epsilon > 0$ and small. The dynamics now reads

$$\dot{w} = \mathcal{L}w - 2\epsilon\partial_x^2 w + g(w) \qquad [\mathbf{5}]$$

where $\mathcal{L} = -(1+\partial_x^2)^2$ is the linear part of the evolution operator *at the transition*, $2\epsilon\partial_x^2 w$ gives an extra linear part due to the perturbation ($\epsilon > 0$) and $g(w)$ is the non-linear part:

$$g(w) = -\frac{w^2}{\Phi} - \partial_x\Big[2(1+\epsilon)\big((1+\frac{w}{\Phi-1})e^{-2w} - 1\big)\partial_x w\Big] \qquad [\mathbf{6}]$$

Amplitude equation

As usual with the amplitude equation approach, we expand w in power series of the perturbation ϵ and study Eq. [**5**] order by order. As shown below (Eqs [**21-23**]), the correct expansion is

$$w = U_0\epsilon^{1/2} + U_1\epsilon + U_2\epsilon^{3/2} + \dots \qquad [\mathbf{7}]$$

Expanding [**6**] to the order $\epsilon^{3/2}$ and substituting in [**5**] yields order by order:

$$-\mathcal{L}U_0 = 0 \qquad [\mathbf{8}]$$

$$-\mathcal{L}U_1 = -\frac{U_0^2}{\Phi} - \frac{3-2\Phi}{\Phi-1}\partial_x^2 U_0^2 \qquad [\mathbf{9}]$$

$$-\mathcal{L}U_2 = -2\partial_x^2 U_0 - \frac{2U_0U_1}{\Phi} - \frac{3-2\Phi}{\Phi-1}2\partial_x^2 U_0U_1 - \frac{4}{3}\frac{\Phi-2}{\Phi-1}\partial_x^2 U_0^3 \qquad [\mathbf{10}]$$

Equation [**8**] can be easily solved and yields

$$U_0 = Ae^{ix} + A^*e^{-ix} \qquad [\mathbf{11}]$$

The amplitude of the perturbation we are trying to derive is thus $2|A|$. Equation [**9**] can also be solved directly:

$$U_1 = Be^{ix} + B^*e^{-ix} + C + De^{2ix} + D^*e^{-2ix} \qquad [\mathbf{12}]$$

where B can de determined from higher order equations (but does not interest us here), and C and D are given by

$$C = -\frac{2|A|^2}{\Phi}; \qquad D = \frac{A^2}{9}(4\frac{3-2\Phi}{\Phi-1} - \frac{1}{\Phi}) \qquad [\mathbf{13}]$$

as can be checked by direct substitution in Eq. [**9**]. Equation [**10**] does not always have a solution. Indeed, the application of $\mathcal{L}$ to any function U_2 cannot yield a multiple of e^{ix}, (since $\mathcal{L}e^{ix} = 0$ and $\mathcal{L}$ is linear). The r.h.s. however does contain a multiple of e^{ix} whose prefactor must thus vanish. This gives a condition for the expansion to provide a proper steady-state solution of the problem. Let us summarize the contributions of the different terms to the prefactor of e^{ix} in the r.h.s of Eq. [**10**]

$$-2\partial_x^2 U_0 \quad \text{yields} \quad 2A \qquad [\mathbf{14}]$$

$$(\frac{3-2\Phi}{\Phi-1}2\partial_x - \frac{2}{\Phi})U_0U_1 \quad \text{yields} \quad 2A|A|^2\Big(\frac{3-2\Phi}{\Phi-1} - \frac{1}{\Phi}\Big) \times\Big(\frac{1}{9}(4\frac{3-2\Phi}{\Phi-1} - \frac{1}{\Phi}) - \frac{2}{\Phi}\Big) \qquad [\mathbf{15}]$$

$$-\frac{4}{3}\frac{\Phi-2}{\Phi-1}\partial_x^2 U_0^3 \quad \text{yields} \quad 4\frac{\Phi-2}{\Phi-1}A|A|^2 \qquad [\mathbf{16}]$$

The sum of these terms vanishes only if

$$A\Big(9\Phi^2(\Phi-1)^2 + 2|A|^2(34\Phi^4 - 56\Phi^3 - 24\Phi^2 + 31\Phi + 19)\Big) = 0 \qquad [\mathbf{17}]$$

and thus either

$$A = 0; \qquad \text{or} \qquad |A|^2 = \frac{9\Phi^2(1-\Phi)^2}{2(34\Phi^4 - 56\Phi^3 - 24\Phi^2 + 31\Phi + 19)} \qquad [\mathbf{18}]$$

Finally, the first order in the amplitude equation yields

$$w(x) = 2|A|\sqrt{\epsilon}\cos(x - x_0) \qquad [\mathbf{19}]$$

where x_0 is a constant. Note that by construction $|A|^2 > 0$ and a non-zero solution only exists for $\Phi \in [1.08439, 1.59237]$. For these values of Φ, Eq. [**18**] and [**19**] work very well, as can be checked in figure 1. Outside this range the transition becomes subcritical and the standard approach does not work anymore. Alternative treatments have been proposed but are not as reliable (see ref [**20**] in main text for more details). Interestingly, we see that the order of the transition

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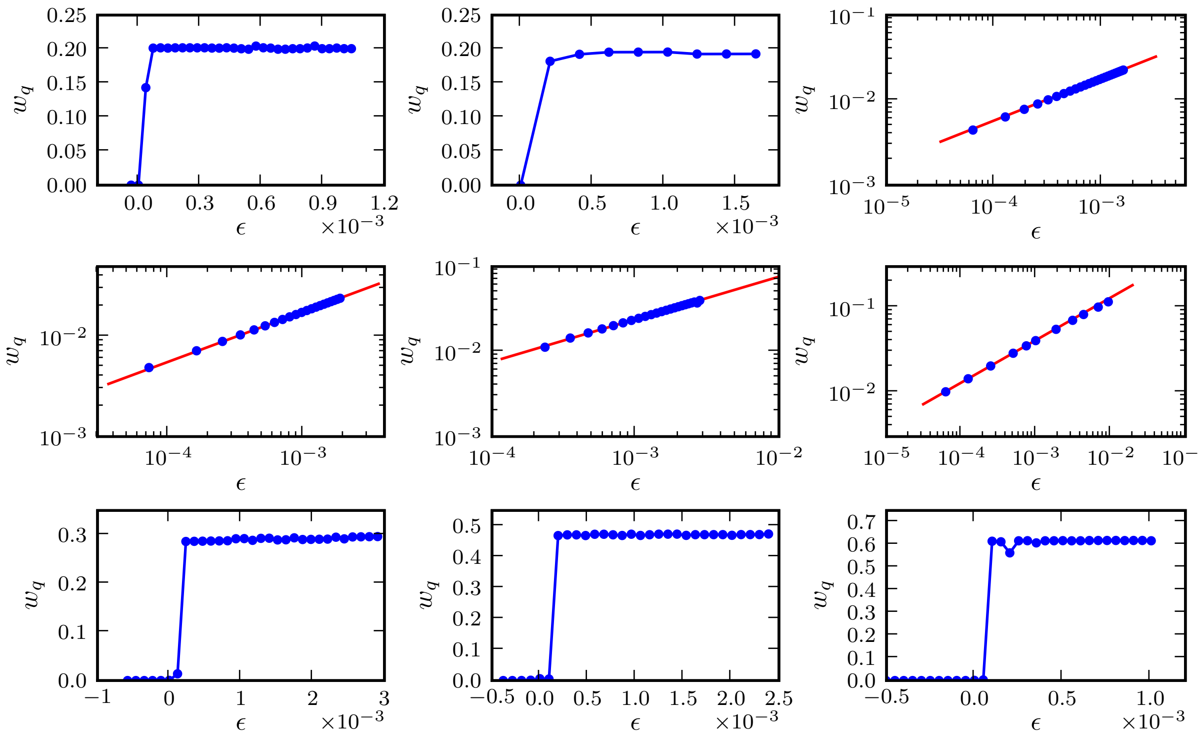

Fig. 1. We simulated equation [**1**] for systems of size $L = 400$ with periodic boundary conditions, using several values of Φ (From left to right, top to bottom, $\Phi = 1.06;\ 1.07;\ 1.12;\ 1.2;\ 1.35;\ 1.5;\ 1.7;\ 1.95;\ 2.5$). The steady-state $w(x)$ was then decomposed in Fourier series $w(x) = a_0 + \sum_n^{N/2} a_n \cos(2\pi nx/L) + b_n \sin(2\pi nx/L)$, where N is the number of data points and the cut-off when $n \to \infty$ is given by the Nyquist frequency. The blue points correspond to the amplitude of the largest mode: $w_q = \max_n \sqrt{a_n^2 + b_n^2}$. When the transition is continuous, we compare these points with the results of the amplitude equation $w_q = 2\sqrt{\epsilon}|A|$, where $|A|$ is solution of [**18**] (red lines) and the agreement is excellent. For $\Phi > 1.58$ or $\Phi < 1.08$, the transition is clearly discontinuous.

and the amplitude of the perturbation depends on how non-linear terms in $g(w)$ balance the linear growth term $-2\epsilon\partial_x^2 w$ in [**5**]. Since the former depends on the non-linear relation $v(\rho)$, we do not expect equation [**18**] to be generic, as opposed to the stability analysis which can be expressed solely in terms of $\mathcal{D}_e(\rho_0)$ and its derivative.

What is the correct expansion?

In [**7**], we expanded w in power series of $\sqrt{\epsilon}$, thus assuming that the amplitude is an analytic function of $\sqrt{\epsilon}$. One could look for a more general expansion:

$$w = U_0\epsilon^\alpha + U_1\epsilon^{2\alpha} + U_2\epsilon^{3\alpha} \qquad [20]$$

In this case, the expansion of equation [**5**] yields two power series: $\sum R_k\epsilon^{\alpha k}$ and $\sum W_k\epsilon^{\alpha k+1}$. For the two series to give terms that can balance each-other, one needs $\alpha + 1 = k\alpha$ for $k \geq 2$ and thus

$$\alpha = \frac{1}{k-1} \qquad [21]$$

The candidates for α are thus $1;\ 1/2;\ 1/3;\ \ldots$. Note that $\alpha \leq 1$ implies $2\alpha + 1 \geq 3\alpha$. We can therefore stop the expansion at 3α and $2\alpha + 1$ to get the first three terms in the expansion of equation [**5**]

Let us first try $\alpha = 1$. The order by order the expansion yields

$$L^2 U_0 = 0, \quad \mathcal{O}(\epsilon) \qquad [22]$$

$$L^2 U_1 = -\frac{U_0^2}{\Phi} - \frac{3 - 2\Phi}{\Phi - 1}\partial_x^2 U_0^2 - 2\partial_x^2 U_0, \quad \mathcal{O}(2\epsilon) \qquad [23]$$

Equation [**22**] yields $U_0 = Ae^{ikx} + A^* e^{-ikx}$ but equation [**23**] cannot be solved since there is a non-zero multiple of e^{ikx} on the r.h.s. $(-2\partial_x^2 U_0)$ which cannot result from the application of L^2 to any function. Thus $\alpha = 1$ is not an option.

For $\alpha \leq 1/3$, then $\alpha + 1 > 1 \geq 3\alpha$. There is thus no contribution of $-2\epsilon\partial_x^2 w$ to the first three orders in the expansion of [**5**]. In particular, the two first order are still given by [**8**] and [**9**], whereas the third order is given by [**10**] without the term linear in U_0. This means that the contribution [**14**] is not present and the prefactor of e^{ikx} in the r.h.s. of [**10**] only contains multiples of $|A|^2 A$. The resolvability condition [**17**] is thus of the form $A|A|^2 f(\Phi) = 0$ which implies $|A| = 0$. The only expansion which yields a result is thus for $\alpha = 1/2$.