We can express the binomial random variable x_{ij} as the sum $x_{ij} = \sum_{k=1}^{n_{ij}} x_{ijk}$ of independent and identically distributed Bernoulli variable $x_{ijk} \sim Ber(\alpha_{ij})$. We have to derive the expression of the first two moments of x_{ij}. We have

$$E(x_{ij}) = n_{ij} E(x_{ijk}) \quad \text{(II.1)}$$

and

$$Var(x_{ij}) = \sum_k Var(x_{ijk}) + 2\sum_{k<k'} Cov(x_{ijk}, x_{ijk'}) \quad \text{(II.2)}$$

Using the conditioning-deconditioning theorem,

$$E(x_{ijk}) = E_{\alpha_{ij}}(x_{ijk} \mid \alpha_{ij}) = E(\alpha_{ij}) \quad \text{(II.3)}$$

$$Var(x_{ijk}) = E_{\alpha_{ij}}\left[Var(x_{ijk} \mid \alpha_{ij})\right] + Var_{\alpha_{ij}}\left[E(x_{ijk} \mid \alpha_{ij})\right]$$

Now, $Var(x_{ijk} \mid \alpha_{ij}) = \alpha_{ij}(1-\alpha_{ij})$ so that

$$Var(x_{ijk}) = E(\alpha_{ij}) - E(\alpha_{ij}^2) + Var(\alpha_{ij}) \quad \text{(II.4)}$$

Similarly

$$Cov(x_{ijk}, x_{ijk'}) = E_{\alpha_{ij}}\left[Cov(x_{ijk}, x_{ijk'} \mid \alpha_{ij})\right] + Cov_{\alpha_{ij}}\left[E(x_{ijk} \mid \alpha_{ij}), E(x_{ijk'} \mid \alpha_{ij})\right].$$

x_{ijk} and $x_{ijk'}$ being conditionally independent, this expression reduces to

$$Cov(x_{ijk}, x_{ijk'}) = Var(\alpha_{ij}) \quad \text{(II.5)}$$

Moreover

$$E(\alpha_{ij}) = \pi_i \, , \; Var(\alpha_{ij}) = c_j \pi_i (1-\pi_i) . \quad \text{(II.6)}$$

These expressions are exact under model 2 and approximate under model 1 due to the truncation. Then substituting (II.6) into (II.4), (II.5) and (II.1), (II.2) gives

$$E\left(x_{ij}\right) = n_{ij}\pi_i$$

$$Var\left(x_{ij}\right) = n_{ij}\pi_i\left(1-\pi_i\right)\left[1+\left(n_{ij}-1\right)c_j\right]$$

from which we can immediately deduce the homologous expressions for the frequency $f_{ij} = x_{ij}/n_{ij}$.